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CONTENTS

No. 1. JANUARY

Coulomb Interactions and the Diamagnetism of Free Electrons, II .... Hideo Kanazawa 1

The Functional Integrals in Quantum Theory .... Takeshi Kanki, Koichi Murata and Shigenobu Sunakawa 7

A Lattice Model of Liquid Helium, II .... Hirotsugu Matsuda and Takeo Matsubara 19

A Method of Renormalization for Unstable Particles .... Takeo Okabayashi and Shigeo Sato 30

Relation between the Nuclear Collective Model and the Individual Particle Model with Configurational Mixing .... Satio Hayakawa and Toshio Marumori 43

On the Bethe-Salpeter Equation and the Excited States of the Hydrogen Atom .... Hideji Kita and Masami Wakano 63

Electromagnetic Properties of Deuteron, II .... Kiyomi Itabashi 80

On the Spins and Decay Interactions of $K_{\mu 3}$ and K_{e3} .... Susumu Furuichi, Tatsuro Kodama, Shuzo Ogawa, Yoshiko Sugahara, Akira Wakasa and Minoru Yonezawa 89

On the Universality of the Weak Interactions .... Suteo Gotô 107

Letters to the Editor:

Elastic Scattering of 24.6 Mev Alpha Particles by Au .... S. Igarasi and T. Takano 124

On the Effective Normal Fluid Density of Solution of He^3 in He^4 .... S. K. Trikha and V. S. Nanda 126

Errata 129

No. 2. FEBRUARY

On a Solution of Field Equations in Einstein's Unified Field Theory, II .... N. N. Ghosh 131

Note on the Isotopic Spin of the Antiparticle .... Shinzo Nakai 139

A Method of Calculation of Electrical Conductivity .... Huzio Nakano 145

One-Center Expansion of Molecular Wave Function, II .... ——*Lower Excited States of* H_2—— .... Sigeru Huzinaga 162

One-Center Expansion of Molecular Wave Function, III .....
——*He-He Repulsive Potential*—— ..... Sigeru Huzinaga 169

Theory of the Magnetic Properties of Ferrous and Cobaltous Oxides, I
..... Junjiro Kanamori 177

Theory of the Magnetic Properties of Ferrous and Cobaltous Oxides, II
..... Junjiro Kanamori 197

Antiferromagnetism of Zn-Ferrite.....Masashi Tachiki and Kei Yosida 223

Hydrodynamical Treatment of Multiple Meson Production in High Energy Nucleon-Nucleus Collisions .....
..... Saburo Amai, Hiroshi Fukuda, Chikashi Iso and Masatomo Sato 241

Note on Models of Multiple Meson Production at Extremely High Energy..... Ziro Koba 288

Letters to the Editor:

The Surface Tension of Liquid He^3 ..... S. K. Trikha and O. P. Rustgi 303
Note on Bloch-Nordsieck Transformation and Electron-Lattice Interaction ..... H. Kanazawa 304
On the Exchange Effect in the Stripping Reactions ..T. Kammuri, R. Nakasima and S. Takagi 306
A Model for the Inelastic Scattering of Nucleons .....S. Okai 308

No. 3. MARCH

An Attempt of Generalizing the Hypothesis of Charge Independence, I
——*Selection Rule for the Decay Process*—— ..Osamu Hara and Yasunori Fujii 313

Effect of Hard Cores on the Binding Energies of H^3 and He^3, II .....
..... Takashi Ohmura, Masato Morita and Masami Yamada 326

On the Origin of the Magnetic Anisotropy Energy of Ferrites .....
..... Kei Yosida and Masashi Tachiki 331

Pion Theory of Three-Body Forces....Jun-ichi Fujita and Hironari Miyazawa 360

Spin-Orbit Coupling in Heavy Nuclei..Jun-ichi Fujita and Hironari Miyazawa 366

Effects of the Central Spin on Planetary Motion.....Anadijiban Das 373

Low's Scattering Equation and S-Matrix .....Hideo Fukutome 383

General Integral Formula of Perturbation Term in the Quantized Field Theory..... Noboru Nakanishi 401

Quantum Field Theory of Unstable Particles .....
.... Huzihiro Araki, Yasuo Munakata, Masaaki Kawaguchi and Tetsuo Gotô 419

The Magnetic Octupole Moments of Nuclei .....
..... Shôta Suekane and Yoshio Yamaguchi 443

The Hexadecapole Moments of Atomic Nuclei ..... Shôta Suekane and Yoshio Yamaguchi 449

On the Wave Propagation in the Non-Linear Fields .....Tosiya Taniuti 461

Theory of the Susceptibility and Specific Heat of He^3.....Sigeo Yomosa 487

Letters to the Editor:

Antiproton-Nucleus and Antiproton-Proton Interactions ..... M. Nozawa, T. Gotô, N. Yajima and R. Nakasima 504

Remarks on the Elastic Scattering of Deuterons by Heavy Nuclei .....Y. Nishida 506

K-π Interactions and K^+-N Scattering .....S. Minami 508

Quantum Numbers for Lepton Processes.....Y. Katayama 510

He-He Repulsive Potential .....S. Huzinaga 512

Equivalence between the Line Width and the Rate of Energy Relaxation in the Inversion Spectrum of Ammonia .....K. Tomita 513

The Coupling Constants of Strong Interactions .....S. Miyachi 515

A Remark on the Lepton Processes .....S. Gotô 517

On a Classification of Light Particles .....K. Senba 519

Some Identities for Renormalization Constants in S(S) and PS(PS) Meson Theories..K. Hiida 520

Errata 522

No. 4. APRIL

On the Vibration of Disordered Linear Lattice ..... Jun-ichi Hori and Takashi Asahi 523

Cluster Development Method in the Quantum Mechanics of Many Particle System, I.....Fumiaki Iwamoto and Masami Yamada 543

On the Elastic Scattering of Charged Particles by Nuclei in the Intermediate Energy Region.....S. N. Ghoshal and B. B. Baliga 556

Saturation of Nuclear Forces·····Neutron Proton Scattering..... A. B. Bhatia and S. M. Shah 561

The l-Forbidden Magnetic Dipole Transitions ..... Akito Arima, Hisashi Horie and Mitsuo Sano 567

Pion-Nucleon Scattering and Structure of Fourth-Order Matrix Elements in Ps(Ps) Meson Theory..... Kugao Nakabayasi and Kazu Hasegawa 581

Mathematical Formulation of the Gell-Mann-Nishijima Scheme for New Particles .....Sigeo Hanawa 592

Letters to the Editor:

On the Air Shower Core.....S. Hasegawa 603

A Kinematical Test for the Relation between the Coupling Constants in Meson Theory..K. Hiida 605
Electron-Neutron Interaction .....F. Ando and H. Miyazawa 607
Viscosity and Heat Conductivity of Liquid He^3 .....S. K. Trikha 609
On the Annihilation Cross Section of the Antinucleon and the Nucleon .....Y. Yamaguchi 612
Interaction with the Sound Quanta in a System consisting of Two Kinds of Charged Particles.. ——*The Method of Supplementary Variables*—— ..... Z. Galasiewicz 613
The Catalysis of Nuclear Fusion Reactions by μ-Mesons..... ..... C. Hayashi, T. Nakano, M. Nishida, S. Suekane and Y. Yamaguchi 615
Errata 618

No. 5. MAY

Quantum-Electrodynamical Corrections to Energy-Levels of Helium .... Huzihiro Araki 619
The Penetration of the Diffuse Nuclear Potential Barrier....Ken Kikuchi 643
The Photon-Nucleon Scattering .....Shigeo Minami and Yoshio Yamaguchi 651
Note on the Ferromagnetic Resonance Absorption .....Takehiko Oguchi 659
On Solutions of He^3 and He^4 .....S. K. Trikha and V. S. Nanda 679
On Collective and Internal Motions in a Bose System .... Toshiyuki Nishiyama 711

Letters to the Editor:

On the Systematization of the Strongly Interacting Particles .....M. Ida 733
Errata 734
Contents of Journ. of Phys. Soc. of Japan, Vol. 12, No. 3, 4 735

No. 6. JUNE

Giant Stars with Shell Source of C-N and p-p Reactions..Chushiro Hayashi 737
Selection Rule Imposed by Mass Reversal ..... Tadashi Ouchi 743
Investigation of Chew-Low Equation by Iteration..... ..... Haruyuki Fujino and Jun'ichi Osada 751
On the Asymptotic Conditions in Quantum Field Theory..... .....Kazuhiko Nishijima 765
Mass Difference of Σ^{1+}, Σ^{1-} and Σ^{10} Hyperons.....Hiroshi Katsumori 803
A Remark on High Energy Phenomena ....Saburo Amai, Toshiyuki Murota and Minoru Nishida 807

Letters to the Editor:

Effect of Long-Range Correlations on Free Electron Diamagnetism ..... B. Donovan, H. Kanazawa and N. H. March 813

Acceleration of Cosmic Rays by Hydromagnetic Wave ..Y. Terashima, K. Kitao and K. Ogawa 814

Interpretation on the Ratio of the Emission Probabilities for Protons and Neutrons ..... R. Nakasima and K. Kikuchi 816

Some Remarks on the Charged Mesic Decay of $^3H_\Lambda$.....T. Ogimoto and Y. Yamaguchi 817

An Attempt of Generalizing the Invariance of the Theory under the Charge Conjugation..... O. Hara, Y. Fujij and Y. Ohnuki 819

Pion-Nucleon Interaction according to the Previous Theory ..O. Hara, Y. Fujii and Y. Ohnuki 820

The Annihilation of Antinucleon according to the Previous Theory..... O. Hara, Y. Fujii and Y. Ohnuki 822

The Production Ratio of K^+ to K^- Mesons ..... Y. Kakudo and K. Kobayakawa 823

Errata 825

Contents of Journ. of Phys. Soc. of Japan, Vol. 12, No. 5 826

Contents and Authors Index to Vol. 17 ..... i～viii

AUTHORS INDEX TO VOLUME 17

Amai-S, Fukuda-H, Iso-C & Sato-M, Hydrodynamical Treatment of Multiple Meson Production in High Energy Nucleon-Nucleus Collisions 241
Amai-S, Murota-T & Nishida-M, A Remark on High Energy Phenomena 807
Ando-F & Miyazawa-H, Electron-Neutron Interaction .....607(L)
Araki-H, Munakata-Y, Kawaguchi-M & Goto-T, Quantum Field Theory of Unstable Particles 419
Araki-H, Quantum-Electrodynamical Corrections to Energy-Levels of Helium..... 619
Arima-A, Horie-H & Sano-M, The *l*-Forbidden Magnetic Dipole Transitions 567
Asahi-T (see Hori-J) 523

Baliga-B. B (see Ghoshal-S. N) 556
Bhatia-A. B & Shah-S.M, Saturation of Nuclear Forces.....Neutron Proton Scattering 561

Das-A, Effects of the Central Spin on Planetary Motion 373
Donovan-B, Kanazawa-H & March-N. H, Effect of Long-Range Correlations on Free Electron Diamagnetism .....813(L)

Fujii-Y (see Hara-O) . .313,819(L), 820(L), 822(L)
Fujino-H & Osada-J, Investigation of Chew-Low Equation by Iteration 751
Fujita-J & Miyazawa-H, Pion Theory of Three-Body Forces 360
Fujita-J & Miyazawa-H, Spin-Orbit Coupling in Heavy Nuclei 366
Fukuda-H (see Amai-S) 241
Fukutome-H, Low's Scattering Equation and S-Matrix 383
Furuichi-S, Kodama-T, Ogawa-S, Sugahara-Y, Wakasa-A & Yonezawa-M, On the Spins and Decay Interactions of $K_{\mu 3}$ and K_{e3} 89

Galasiewicz-Z, Interaction with the Sound Quanta in a System consisting of Two Kinds of Charged Particles .....613(L)
Ghosh-N. N, On a Solution of Field Equations in Einstein's Unified Field Theory, II 131
Ghoshal-S. N & Baliga-B. B, On the Elastic Scattering of Charged Particles by Nuclei in the Intermediate Energy Region..... 556
Goto-S, On the Universality of the Weak Interactions 107
Goto-S, A Remark on the Lepton Processes . . ..517(L)
Goto-T (see Araki-H) 419
Goto-T (see Nozawa-M) .....504(L)

Hanawa-S, Mathematical Formulation of the Gell-Mann-Nishijima Scheme for New Particles 592
Hara-O & Fujii-Y, An Attempt of Generalizing the Hypothesis of Charge Independence, I . . 313
Hara-O, Fujii-Y & Ohnuki-Y, An Attempt of Generalizing the Invariance of the Theory under the Charge Conjugation .....819(L)
Hara-O, Fujii-Y & Ohnuki-Y, Pion-Nucleon Interaction according to the Previous Theory . . .. 820(L)
Hara-O, Fujii-Y & Ohnuki-Y, The Annihilation of Antinucleon according to the Previous Theory .....822(L)
Hasegawa-K (see Nakabayasi-K) 581
Hasegawa-S, On the Air Shower Core . .603(L)
Hayakawa-S & Marumori-T, Relation between the Nuclear Collective Model and the Individual Particle Model with Configurational Mixing . . 43
Hayashi-C, Nakano-T, Nishida-M, Suekane-S & Yamaguchi-Y, The Catalysis of Nuclear Fusion Reactions by μ-Mesons .....615(L)
Hayashi-C, Giant Stars with Shell Source of C-N and p-p Reactions 737
Hiida-K, Some Identities for Renormalization Constants in S(S) and PS(PS) Meson Theories ..... 520(L)
Hiida-K, A Kinematical Test for the Relation between the Coupling Constants in Meson Theory ..... 605(L)
Hori-J & Asahi-T, On the Vibration of Disordered Linear Lattice 523
Horie-H (see Arima-A) 567
Huzinaga-S, One-Center Expansion of Molecular Wave Function, II, III 162, 169
Huzinaga-S, He-He Repulsive Potential . .512(L)

Ida-M, On the Systematization of the Strongly Interacting Particles .....733(L)
Igarasi-S & Takano-T, Elastic Scattering of 24.6 Mev Alpha Particles by Au.....124(L)
Iso-C (see Amai-S) 241
Itabashi-K, Electromagnetic Properties of Deuteron, II..... 80
Iwamoto-F & Yamada-M, Cluster Development

Method in the Quantum Mechanics of Many Particle System, I 543

Kakudo-Y & Kobayakawa-K, The Production Ratio of K^+ to K^- Mesons ..... 823(L)
Kammuri-T, Nakasima-R & Takagi-S, On the Exchange Effect in the Stripping Reactions ..... 306(L)
Kanamori-J, Theory of the Magnetic Properties of Ferrous and Cobaltous Oxides, I, II..177, 197
Kanazawa-H, Coulomb Interactions and the Diamagnetism of Free Electrons, II 1
Kanazawa-H, Note on Bloch-Nordsieck Transformation and Electron-Lattice Interaction..304(L)
Kanazawa-H (see Donovan-B) ..... 813(L)
Kanki-T, Murata-K & Sunakawa-S, The Functional Integrals in Quantum Theory.... 7
Katayama-Y, Quantum Numbers for Lepton Processes ..... 510(L)
Katsumori-H, Mass Difference of Σ^+, Σ^- and Σ^0 Hyperons 803
Kawaguchi-M (see Araki-H) 419
Kikuchi-K, The Penetration of the Diffuse Nuclear Potential Barrier 643
Kikuchi-K (see Nakasima-R) ..... 816(L)
Kita-H & Wakano-M, On the Bethe-Salpeter Equation and the Excited States of the Hydrogen Atom 63
Kitao-K (see Terashima-Y) ..... 814(L)
Koba-Z, Note on Models of Multiple Meson Production at Extremely High Energy 288
Kobayakawa-K (see Kakudo-Y) ..... 823(L)
Kodama-T (see Furuichi-S) 89

March-N. H (see Donovan-B) ..... 813(L)
Marumori-T (see Hayakawa-S) 43
Matsubara-T (see Matsuda-H) 19
Matsuda-H & Matsubara-T, A Lattice Model of Liquid Helium, II 19
Minami-S, K-π Interactions and K^+-N Scattering ..... 508(L)
Minami-S & Yamaguchi-Y, The Photon-Nucleon Scattering 651
Miyachi-S, The Coupling Constants of Strong Interactions ..... 515(L)
Miyazawa-H (see Fujita-J) 360, 366
Miyazawa-H (see Ando-F) ..... 607(L)
Morita-M (see Ohmura-T) 326
Munakata-Y (see Araki-H) 419
Murota-K (see Kanki-T) 7
Murota-T (see Amai-S) 807

Nakai-S, Note on the Isotopic Spin of the Anti-particle 139
Nakabayasi-K & Hasegawa-K, Pion-Nucleon Scattering and Structure of Fourth-Order Matrix Elemehts in Ps(Ps) Meson Theory 581
Nakanishi-N, General Integral Formula of Perturbation Term in the Quantized Field Theory 401
Nakano-H, A Method of Calculation of Electrical Conductivity 145
Nakano-T (see Hayashi-C) ..... 615(L)
Nakasima-R (see Kammuri-T) ..... 306(L)
Nakasima-R (see Nozawa-M) ..... 504(L)
Nakasima-R & Kikuchi-K, Interpretation on the Ratio of the Emission Probabilities for Proton and Neutrons ..... 816(L)
Nanda-V. S (see Trikha-S. K) ..... 126(L), 679
Nishida-M (see Hayashi-C) ..... 615(L)
Nsshida-M (see Amai-S) 807
Nishida-Y, Remarks on the Elastic Scattering of Deuterons by Heavy Nuclei ..... 506(L)
Nishijima-K, On the Asymptotic Conditions in Quantum Field Theory 765
Nishiyama-T, On Collective and Internal Motions in a Bose System 711
Nozawa-M, Goto-T, Yajima-N & Nakasima-R, Antiproton-Nucleus and Antiproton-Proton Interactions ..... 504(L)

Ogawa-K (see Terashima-Y) ..... 814(L)
Ogawa-S (see Furuichi-S) 89
Ogimoto-T & Yamaguchi-Y, Some Remarks on the Charged Mesic Decay of $^3H_\Lambda$ ....817(L)
Oguchi-T, Note on the Ferromagnetic Resonance Absorption 659
Ohmura-T, Morita-M & Yamada-M, Effect of Hard Cores on the Binding Energies of H^3 and He^3, II 326
Ohnuki-Y (see Hara-O)..819(L), 820(L), 822(L)
Okabayashi-T & Sato-S, A Method of Renormalization for Unstable Particles 30
Okai-S, A Model for the Inelastic Scattering of Nucleons ..... 308(L)
Osada-J (see Fujino-H) 751
Ouchi-T, Selection Rule Imposed by Mass Reversal 743

Rustgi-O. P (see Trikha-S. K) ..... 303(L)

Sano-M (see Arima-A) 567
Sato-M (see Amai-S) 241
Sato-S (see Okabayashi-T) 30
Senba-K, On a Classification of Light Particles ..... 519(L)

Shah-S. M (see Bhatia-A. B) 561
Suekane-S & Yamaguchi-Y, The Magnetic Octupole Moments of Nuclei 443
Suekane-S & Yamaguchi-Y, The Hexadecapole Moments of Atomic Nuclei 449
Suekane-S (see Hayashi-C) ..... 615(L)
Sugahara-Y (see Furuichi-S) 89
Sunakawa-S (see Kanki-T) 7

Tachiki-M & Yosida-K, Antiferromagnetism of Zn-Ferrite 223
Tachiki-M (see Yosida-K) 331
Takagi-S (see Kammuri-T) ..... 306(L)
Takano-T (see Igarasi-S) ..... 124(L)
Taniuti-T, On the Wave Propagation in the Non-Linear Fields 461
Terashima-Y, Kitao-K & Ogawa-K, Acceleration of Cosmic Rays by Hydromagnetic Wave ..... 814(L)
Tomita-K, Equivalence between the Line Width and the Rate of Energy Relaxation in Inversion Spectrum of Ammonia ..... 513(L)
Trikha-S. K & Nanda-V. S, On the Effective Normal Fluid Density of Solution of He^3 in He^4 ..... 126(L)
Trikha-S. K & Rustgi-O. P, The Surface Tension of Liquid He^3 ..... 303(L)
Trikha-S. K, Viscosity and Heat Conductivity of Liquid He^3 ..... 609(L)
Trikha-S. K & Nanda-V. S, On Solutions of He^3 and He^4 679

Wakano-M (see Kita-H) 63
Wakasa-A (see Furuichi-S) 89
Yajima-N (see Nozawa-M) ..... 504(L)
Yamada-M (see Ohmura-T) 326
Yamada-M (see Iwamoto-F) 543
Yamaguchi-Y (see Suekane-S) 443, 449
Yamaguchi-Y, On the Annihilation Cross Section of the Antinucleon and the Nucleon ..... 612(L)
Yamaguchi-Y (see Hayashi-C) ..... 615(L)
Yamaguchi-Y (see Minami-S) 651
Yamaguchi-Y (see Ogimoto-T) ..... 817(L)
Yomosa-S, Theory of the Susceptibility and Specific Heat of He^3 487
Yonezawa-M (see Furuichi-S) 89
Yosida-K & Tachiki-M, On the Origin of the Magnetic Anisotropy Energy of Ferrites 331
Yosida-K (see Tachiki-M) 223

Errata 129, 522, 618, 734, 825

Contents of Journ. of Phys. Soc. of Japan
Vol. 11, No. 12 130
Vol. 12, No. 1 & 2 ..... 311, 312
Vol. 12, No. 3 & 4 ..... 735, 736
Vol. 12, No. 5 826

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Coulomb Interactions and the Diamagnetism of Free Electrons, II

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The effect of the term $\Delta \boldsymbol{j}(\boldsymbol{x})$, which should be added to the current density operator $\boldsymbol{j}(\boldsymbol{x})$, on the Landau diamagnetism is investigated. It is shown that the magnitude of the diamagnetic susceptibility slightly increases due to the long-range part of the Coulomb interactions between electrons.

§ 1. Introduction

In the preceding paper (to be referred to as I) we have developed the collective description of the organized behavior due to the Coulomb interactions in an electron gas in a magnetic field, and have investigated the effect of the long-range part of the Coulomb interactions on the Landau diamagnetism. It was shown that the magnetic susceptibility of alkali metals slightly increased.

The Hamiltonian of our system is given by

$$H=\sum_i \boldsymbol{\pi}_i^2/2m+2\pi e^2\hbar^2 \sum_{\substack{i\neq j \\ k}} \exp\{i/\hbar\cdot\boldsymbol{k}(\boldsymbol{x}_i-\boldsymbol{x}_j)\}/k^2, \tag{1·1}$$

where $\boldsymbol{\pi}_i=\boldsymbol{p}_i+e/c\cdot\boldsymbol{A}(\boldsymbol{x}_i)$, $\boldsymbol{A}(\boldsymbol{x}_i)$ being the vector potential of the magnetic field. The Hamiltonian (1·1) is equivalent to the following Hamiltonian with a set of subsidiary conditions

$$\begin{aligned} H=&\sum_i [\boldsymbol{\pi}_i+(4\pi e^2)^{1/2}\sum_{k<k_c} q_k \boldsymbol{\varepsilon}_k \exp\{i/\hbar\cdot\boldsymbol{k}\boldsymbol{x}_i\}]^2/2m \\ &-1/2\cdot\sum_{k<k_c} p_k p_{-k}+2\pi e^2\hbar^2\sum_{\substack{i\neq j \\ k>k_c}} \exp\{i/\hbar\cdot\boldsymbol{k}(\boldsymbol{x}_i-\boldsymbol{x}_j)\}/k^2 \\ &-2\pi n e^2\hbar^2\sum_{k<k_c} 1/k^2, \end{aligned} \tag{1·2}$$

where $\varepsilon_k=\boldsymbol{k}/|\boldsymbol{k}|$. The subsidiary conditions are given by

$$\xi_k \Phi=0 \qquad (k<k_c), \tag{1·3}$$

where

$$\xi_k=p_{-k}-i(4\pi e^2\hbar^2/k^2)^{1/2}\sum_i \exp(-i/\hbar\cdot\boldsymbol{k}\boldsymbol{x}_i). \tag{1·4}$$

The current density operator in the new representation, in which the Hamiltonian is given by (1·2) is $\boldsymbol{j}(\boldsymbol{x})+\Delta\boldsymbol{j}(\boldsymbol{x})$, where

$$\boldsymbol{j}(\boldsymbol{x}) = -e/2m \cdot \sum_{i}' [\boldsymbol{\pi}_i \delta(\boldsymbol{x}-\boldsymbol{x}_i) + \delta(\boldsymbol{x}-\boldsymbol{x}_i)\boldsymbol{\pi}_i], \tag{1·5}$$

$$\Delta\boldsymbol{j}(\boldsymbol{x}) = -(4\pi e^4/m^2)^{1/2} \sum_{i,\, k<k_c}' q_k \boldsymbol{\varepsilon}_k \exp(i/\hbar \cdot \boldsymbol{k}\boldsymbol{x}_i)\, \delta(\boldsymbol{x}-\boldsymbol{x}_i). \tag{1·6}$$

If we apply the unitary transformation $\Psi = S\psi$, where

$$S = \exp\Big[-1/\hbar \cdot \sum_{i,\, k<k_c}' (4\pi e^2\hbar^2/k^2)^{1/2} q_k \exp(i/\hbar \cdot \boldsymbol{k}\boldsymbol{x}_i)\Big], \tag{1·7}$$

we get

$$\boldsymbol{j}(\boldsymbol{x}) + \Delta\boldsymbol{j}(\boldsymbol{x}) \to S^{-1}[\boldsymbol{j}(\boldsymbol{x}) + \Delta\boldsymbol{j}(\boldsymbol{x})]S = \boldsymbol{j}(\boldsymbol{x}), \tag{1·8}$$

which is the current density operator in the old representation in which the Hamiltonian is given by (1·1). The appearance of $\Delta\boldsymbol{j}(\boldsymbol{x})$ in the new representation is due to the fact that the velocity of the electron in the new representation is given by

$$\begin{aligned} \boldsymbol{v}_i = \dot{\boldsymbol{x}}_i &= i/\hbar \cdot [H, \boldsymbol{x}_i] \\ &= [\boldsymbol{\pi}_i + (4\pi e^2)^{1/2} \sum_{k<k_c}' q_k \boldsymbol{\varepsilon}_k \exp(i/\hbar \cdot \boldsymbol{k}\boldsymbol{x}_i)]/m. \end{aligned} \tag{1·9}$$

In the next section we shall investigate the effect of $\Delta\boldsymbol{j}(\boldsymbol{x})$ on the Landau diamagnetism.

§ 2. Contribution from $\Delta\boldsymbol{j}(\boldsymbol{x})$

(1·6) may be written as

$$\Delta\boldsymbol{j}(\boldsymbol{x}) = -e^2/m \cdot \sum_{i,\, k<k_c}' (2\pi\hbar/\omega)^{1/2} \boldsymbol{\varepsilon}_k \exp(i/\hbar \cdot \boldsymbol{k}\boldsymbol{x}_i)\, (a_k - a_{-k}^*)\, \delta(\boldsymbol{x}-\boldsymbol{x}_i), \tag{2·1}$$

where a_k^* and a_k are the creation and destruction operators of plasma quanta respectively.[1] In I we made a canonical transformation on the Hamiltonian (1·2). The generating function of the canonical transformation is given by

$$\begin{aligned} S = -(ei/m) \sum_{i,\, k<k_c}' (2\pi\hbar/\omega)^{1/2} \Big[& \frac{\boldsymbol{\varepsilon}_k \cdot (\boldsymbol{\Pi}_i - \boldsymbol{k}/2)}{\hbar\omega - \boldsymbol{k}\cdot\boldsymbol{\Pi}_i/m + k^2/2m} A_k \exp(i/\hbar \cdot \boldsymbol{k}\boldsymbol{X}_i) \\ & - \exp(-i/\hbar \cdot \boldsymbol{k}\boldsymbol{X}_i) A_k^* \frac{\boldsymbol{\varepsilon}_k \cdot (\boldsymbol{\Pi}_i - \boldsymbol{k}/2)}{\hbar\omega - \boldsymbol{k}\cdot\boldsymbol{\Pi}_i/m + k^2/2m} \Big]. \end{aligned} \tag{2·2}$$

Let us make the transformation on $\Delta\boldsymbol{j}(\boldsymbol{x})$:

$$\begin{aligned} \Delta\boldsymbol{j}(\boldsymbol{x}) &= e^{-iS} \Delta\boldsymbol{J}(\boldsymbol{x}) e^{iS} \\ &= \Delta\boldsymbol{J}(\boldsymbol{x}) + \Delta\boldsymbol{J}^{(1)}(\boldsymbol{x}) + \cdots, \end{aligned} \tag{2·3}$$

where

$$\Delta\boldsymbol{J}(\boldsymbol{x}) = -e^2/m \cdot \sum_{i,\, k<k_c}' (2\pi\hbar/\omega)^{1/2} \boldsymbol{\varepsilon}_k \exp(i/\hbar \cdot \boldsymbol{k}\boldsymbol{X}_i)\, (A_k - A_{-k}^*)\, \delta(\boldsymbol{x}-\boldsymbol{X}_i), \tag{2·4a}$$

$$\Delta\boldsymbol{J}^{(1)}(\boldsymbol{x}) = i[\Delta\boldsymbol{J}(\boldsymbol{x}), S]$$

$$=e^3/m^2\cdot\sum_{i,\,k<k_c}(2\pi\hbar/\omega)\Big[\varepsilon_k\exp\{i/\hbar\cdot\boldsymbol{k}(\boldsymbol{X}_i-\boldsymbol{X}_j)\}\,\delta(\boldsymbol{x}-\boldsymbol{X}_i)$$

$$\cdot\frac{\varepsilon_k\cdot(\boldsymbol{\Pi}_j-\boldsymbol{k}/2)}{\hbar\omega-\boldsymbol{k}\cdot\boldsymbol{\Pi}_j/m+k^2/2m}+\varepsilon_k\exp\{-i/\hbar\cdot\boldsymbol{k}(\boldsymbol{X}_i-\boldsymbol{X}_j)\}\,\delta(\boldsymbol{x}-\boldsymbol{X}_i)$$

$$\cdot\frac{\varepsilon_k\cdot(\boldsymbol{\Pi}_j+\boldsymbol{k}/2)}{\hbar\omega-\boldsymbol{k}\cdot\boldsymbol{\Pi}_j/m-k^2/2m}+A_kA_k^*\Big\{\frac{\varepsilon_k\cdot(\boldsymbol{\Pi}_i-\boldsymbol{k}/2)}{\hbar\omega-\boldsymbol{k}\cdot\boldsymbol{\Pi}_i/m+k^2/2m}$$

$$\cdot\varepsilon_k\delta(\boldsymbol{x}-\boldsymbol{X}_i)-\varepsilon_k\delta(\boldsymbol{x}-\boldsymbol{X}_i)\frac{\varepsilon_k\cdot(\boldsymbol{\Pi}_i+\boldsymbol{k}/2)}{\hbar\omega-\boldsymbol{k}\cdot\boldsymbol{\Pi}_i/m-k^2/2m}\Big\}$$

$$+\cdots\Big]. \qquad (2\cdot4b)$$

In (2·4b) we have neglected a number of terms containing a factor $A_l^*A_l$, $A_kA_l^*$ $(\boldsymbol{k}\neq\boldsymbol{l})$ $A_k^*A_l^*$ or A_kA_l. These terms make no contribution to the magnetic moment.[1]

We use the formalism of second quantization. Then we obtain

$$\Delta\boldsymbol{J}(\boldsymbol{x})=-e^2/m\cdot\sum_{\substack{p\sigma p'\sigma\\ k<k_c}}(2\pi\hbar/\omega)^{1/2}\varepsilon_k\exp\{i/\hbar\cdot(\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k})\boldsymbol{x}\}\,(A_k-A_{-k}^*)\,b_{p'\sigma}^*b_{p\sigma}, \qquad (2\cdot5)$$

$$\Delta\boldsymbol{J}_0^{(1)}(\boldsymbol{x})\approx e^3/m^2\cdot\sum_{\substack{p\sigma p'\sigma\\ k<k_c}}(2\pi\hbar/\omega)\,(1/\hbar\omega)\,\varepsilon_k(\varepsilon_k\cdot(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k}))\exp\{i/\hbar\cdot(\boldsymbol{p}-\boldsymbol{p}')\boldsymbol{x}\}\,b_{p'\sigma}^*b_{p\sigma}, \qquad (2\cdot6)$$

$$\Delta\boldsymbol{J}_1^{(1)}(\boldsymbol{x})\approx 2e^4/m^2c\cdot\sum_{\substack{p\sigma p'\sigma\\ k<k_c}}(2\pi\hbar/\omega)\,(1/\hbar\omega)\,\varepsilon_k(\varepsilon_k\cdot\boldsymbol{A}(\boldsymbol{x}))\exp\{i/\hbar\cdot(\boldsymbol{p}-\boldsymbol{p}')\boldsymbol{x}\}\,b_{p'\sigma}^*b_{p\sigma}. \qquad (2\cdot7)$$

In deriving (2·6) and (2·7),* we have made an approximation of replacing $1/(\hbar\omega-\boldsymbol{k}\cdot\boldsymbol{\Pi}_i/m+k^2/2m)$ by $1/\hbar\omega$ and we have replaced $A_kA_k^*$ by 1, because $\langle A_kA_k^*\rangle_{\mathrm{vac}}=1$. $b_{p\sigma}^*$ and $b_{p\sigma}$ are the creation and destruction operators for the electrons and σ refers to the electron spin.

The mean current density at temperature T due to the additional current density operator $\Delta\boldsymbol{j}(\boldsymbol{x})$ is given by[1]

$$\Delta\boldsymbol{i}(\boldsymbol{x})=\sum_{\nu\nu'}{\sum_{\substack{k<k_c\\ k_z>0}}}'\langle\boldsymbol{\Pi}\,\delta(\xi_k^+)\,\delta(\xi_k^-)\rangle_{\nu\nu'}\exp\beta(\zeta n+\Omega_0-E_\nu)$$

$$\cdot\Big[\langle\Delta\boldsymbol{J}_1^{(1)}(\boldsymbol{x})\rangle_{\nu'\nu}+\sum_\lambda\frac{1-\exp\beta(E_\nu-E_\lambda)}{E_\nu-E_\lambda}\{\langle\Delta\boldsymbol{J}_0^{(1)}(\boldsymbol{x})\rangle_{\nu'\lambda}\langle H'\rangle_{\lambda\nu}$$

$$+\langle\Delta\boldsymbol{J}(\boldsymbol{x})\rangle_{\nu'\lambda}\langle H_{\mathrm{e.p.}}\rangle_{\lambda\nu}\}\Big], \qquad (2\cdot8)$$

where

$$H'=e/2m^*c\cdot\sum_{p\sigma p'\sigma}(\boldsymbol{p}+\boldsymbol{p}')\cdot\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{p}')\,b_{p'\sigma}^*b_{p\sigma}, \qquad (2\cdot9)$$

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* Terms which depend on two electron coordinates in (2·4b) cancel to the first order in the magnetic field.

$$H_{e\cdot p.}=e^2/2m^2c\cdot\sum_{\substack{pp'\sigma\\ k<k_c}}(2\pi\hbar/\omega)^{1/2}(1/\hbar\omega)\{-(\boldsymbol{p}+\boldsymbol{p}'+\boldsymbol{k})\cdot[\boldsymbol{\varepsilon}_k\times[(\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k})$$
$$\times\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k})]]b^*_{p'\sigma}b_{p\sigma}A_k+(\boldsymbol{p}+\boldsymbol{p}'+\boldsymbol{k})\cdot[\boldsymbol{\varepsilon}_k\times[(\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k})$$
$$\times\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k})]]b^*_{p'\sigma}b_{p\sigma}A^*_k\}. \tag{2·10}$$

The mass m^* is slightly larger than m: $m^*=m\times 3n/(3n-n')$. Ω_0 is given by[1]

$$\exp(-\beta\Omega_0)\approx(1/2\pi)^{n'}\prod_{\substack{k<k_c\\ k_z>0}}\frac{4\pi}{\sum_q g^2(\boldsymbol{k},\boldsymbol{q})\{f_q(1-f_{q+k})+f_{q+k}(1-f_q)\}}$$
$$\cdot\prod_p[1+\exp\{-\beta(\varepsilon(\boldsymbol{p})-\zeta)\}]^2. \tag{2·11}$$

After some calculations quite analogous to those in I, we obtain

$$\Delta i_\mu(\boldsymbol{q})\approx\sum_\nu K_{\mu\nu}(\boldsymbol{q})A_\nu(\boldsymbol{q}), \tag{2·12}$$

where

$$K_{\mu\nu}(\boldsymbol{q})=-(e^4/m^*m^2c)\sum_p(2\pi n'/3\omega^2)[q_\mu q_\nu\beta f_p(1-f_p)$$
$$-q^2(\beta^2/m^*)p_\mu p_\nu\{f_p(1-f_p)-2f_p^2(1-f_p)\}$$
$$-(\beta^2/m^*)\sum_\lambda(q_\nu q_\lambda p_\mu p_\lambda+q_\mu q_\lambda p_\nu p_\lambda)\{f_p(1-f_p)-2f_p^2(1-f_p)\}$$
$$-\sum_{\lambda,\rho}(2\beta^3/3m^{*2})q_\lambda q_\rho p_\mu p_\nu p_\lambda p_\rho\{6f_p^2(1-f_p)^2-f_p(1-f_p)\}]$$
$$+(2\pi e^4/m^3c)\sum_{\substack{p\\ k<k_c}}(1/\hbar^2\omega^4)(\boldsymbol{\varepsilon}_k)_\mu[\hbar\omega\{\sum_\lambda(\boldsymbol{\varepsilon}_k)_\lambda q_\nu q_\lambda-(\boldsymbol{\varepsilon}_k)_\nu q^2\}$$
$$\cdot f_p(1-f_{p-k})+2'\sum_{\lambda,\rho}\{(\boldsymbol{\varepsilon}_k)_\nu p_\lambda q_\lambda-(\boldsymbol{\varepsilon}_k)_\lambda p_\nu q_\lambda\}\{(\boldsymbol{p}-\boldsymbol{k})_\rho q_\rho$$
$$\cdot f_p(1-f_{p-k})/m^*-(\boldsymbol{p}-\boldsymbol{k})_\rho q_\rho(\beta\hbar\omega/m^*)f_pf_{p-k}(1-f_{p-k})\}]$$
$$+0(q^4). \tag{2·13}$$

Performing the summations (see Appendix I of I), we get

$$K_{\mu\nu}(q)=(\delta_{\mu\nu}q^2-q_\mu q_\nu)(2\pi e^2/3mc)(2m^*\zeta)^{1/2}/\hbar^3\cdot\delta, \tag{2·14}$$

where

$$\delta=(4\pi e^2n'/3m\omega^2)+(16\pi^2e^2m^*\zeta k_c^3/3m^2\hbar\omega^3\hbar^3)[-2/3$$
$$+(\zeta/\hbar\omega)(3\kappa/8-\kappa^3/24+3\kappa^5/1280)]$$
$$\approx(n'/n)[1/3+(\zeta/\hbar\omega)\{-2/3+(\zeta/\hbar\omega)(3\kappa/8-\kappa^3/24$$
$$+3\kappa^5/1280)\}]. \qquad(\kappa\equiv k_c/p_0) \tag{2·15}$$

Hence the contribution to the susceptibility is given by

$$\Delta\chi = -\delta \cdot \chi_0^{*}. \tag{2·16}$$

Thus $(\Delta\chi)_{\text{l.r.}}$, which represents the effect of the long-range part of the Coulomb interactions on the diamagnetic susceptibility, is given by[1]

$$(\Delta\chi)_{\text{l.r.}} = (\alpha + 8m^{*}\zeta\gamma - \varepsilon - \delta)\,\chi_0, \tag{2·17}$$

where

$$\alpha \approx (n'/n)\,(\zeta/\hbar\omega)\,(\kappa^2/5),$$

$$8m^{*}\zeta\gamma \approx (n'/n)\,(2\zeta/3\hbar\omega),$$

$$-\varepsilon \approx (n'/n)\,(\zeta/\hbar\omega)^2\,[3\kappa/4 + \kappa^3/12 - \kappa^5/640 - (\zeta/\hbar\omega)\cdot 8\kappa^2/25].^{**}$$

Table 1

	Li	Na	K	Rb	Cs
m_{eff}/m[2]	1.45	0.98	0.93	0.89	0.83
κ[2]	0.634	0.703	0.780	0.800	0.834
α	0.006	0.011	0.017	0.020	0.023
$8m^{*}\zeta\gamma$	0.050	0.074	0.093	0.103	0.113
$-\varepsilon$	−0.018	−0.032	−0.042	−0.048	−0.053
$-\delta$	−0.003	−0.002	−0.008	−0.009	−0.012
$(\Delta\chi)_{\text{l.r.}}/\chi_0$	0.035	0.051	0.060	0.066	0.071

The values of $(\Delta\chi)_{\text{l.r.}}/\chi_0$ for alkali metals are given in Table 1. The values in Table 1 have been calculated by replacing the electron mass by m_{eff}. The errors in our calculation come mainly from the approximation of replacing $1/(\hbar\omega - \cdot\boldsymbol{k}\boldsymbol{\Pi}_i/m + k^2/2m)$ by $1/\hbar\omega_p$. Errors in $(\Delta\chi)_{\text{l.r.}}/\chi_0$ might be considered to be 20 percent at most.

References

1) H. Kanazawa, Prog. Theor. Phys. **15** (1956), 273.
2) F. Seitz and D. Turnbull, *Solid State Physics* Vol. 1 (Academic Press, New York, 1955), p. 368.

* χ_0 is the diamagnetic susceptibility for the perfect electron gas.
** There was a mistake in (3·44) of I.

Errata

Coulomb Interactions and the Diamagnetism of Free Electrons

H. KANAZAWA

Prog. Theor. Phys. **15** (1956), 273

		originally	correctly
p. 280	eq. (3·19)	$1-(1/\hbar\omega)^2(\boldsymbol{k}\cdot\boldsymbol{p}/m+k^2/2m)$	$1-(1/\hbar\omega)^2(\boldsymbol{k}\cdot\boldsymbol{p}/m+k^2/2m)^2$
p. 285	eq. (3·44)	$[-16/45+\cdots+\kappa^7/45\}]$	$[1/3\kappa+\kappa/27-\kappa^3/1440-32/225\cdot(\zeta/\hbar\omega)]$
p. 289	l·11	$+\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{k})/m$	$-\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{k})/m$

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The Functional Integrals in Quantum Theory

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Matthews and Salam expressed the propagator of the interacting fields as a functional integral by using the Feynman principle of the "sum over histories". However, when we are concerned with the boundary condition of the propagator and the definition of the functional integrals, their treatment seems to us somewhat unsatisfactory. In this paper, we discuss the reason why they got the correct results in spite of the existing defects.

Introduction

Recently, the propagator of the interacting fields has been investigated in the form of the functional integral. Particularly, Edwards and Peierls[1] expressed the modified propagator as a functional integral, starting with the Schwinger theory of Green-functions[2] and ignoring the contributions from the closed loops. On the other hand, Matthews and Salam[3] succeeded in giving an exact expression to the propagator by directly applying the Feynman principle[4] of the "sum over histories" in particle mechanics. Their treatment, however, seems to be somewhat unsatisfactory as will be seen below.

The Feynman principle in the field theory states that

$$\langle \varphi(\boldsymbol{x}', t') = \varphi', t' | \varphi(\boldsymbol{x}'', t'') = \varphi'', t'' \rangle = N \int\int_{\varphi''}^{\varphi'} e^{iI^*[\varphi]} \delta\varphi, \qquad (0\cdot 1)$$

and

$$\langle \varphi(\boldsymbol{x}', t') = \varphi', t' | T(\varphi(1)\varphi(2)) | \varphi(\boldsymbol{x}'', t'') = \varphi'', t'' \rangle = N \int\int_{\varphi''}^{\varphi'} \varphi(1)\varphi(2) e^{iI^*[\varphi]} \delta\varphi, \qquad (0\cdot 2)$$

where $I^*[\varphi]$ is the action of the system and the asterisk means addition of the infinitesimal negative imaginary constant $-i\epsilon$ to the mass in the Lagrangian in order to make the functional integral over the action converge. Matthews and Salam defined the propagator by

$$\langle 0, t' | T(\varphi(1)\varphi(2)) | 0, t'' \rangle / \langle 0, t' | 0, t'' \rangle = \int\int \varphi(1)\varphi(2) e^{iI^*[\varphi]} \delta\varphi / \int\int e^{iI^*[\varphi]} \delta\varphi, \qquad (0\cdot 3)$$

where $|0, t\rangle$ means the true vacuum state of the system. Carrying out the functional integration in the right-hand side of $(0\cdot 3)$, they derived the well-known expressions of Δ_F, S_F and S'_F. It seems, however, very obscure how the boundary condition (vacuum states) of the left-hand side is reflected in the functional integrals of the right-hand side.

In order to clarify the problem, let us review the definition of the propagator of the interacting fields. Gell-Mann and Low[5] defined it as

$$\Psi_0^* T(\varphi(1)\varphi(2))\Psi_0 / \Psi_0^* \Psi_0, \tag{0·4}$$

where Ψ_0 is the true vacuum state and $\varphi(x)$ is a field operator in the Heisenberg representation. Denoting by $|A(t)=A',\ t\rangle$ a state where a Heisenberg operator $A(t)$ has the eigenvalue A' at a time t, we can easily see that

$$|A(t)=A',\ t\rangle = e^{iHt}|A(0)=A',\ 0\rangle,$$

where H is the total Hamiltonian of the system. From this relation we have

$$|H=E_0,\ t\rangle = e^{iE_0 t}\Psi_0, \tag{0·5}$$

where E_0 is the eigenvalue of the true vacuum state. In view of (0·4) and (0·5), we have an alternative expression for the propagator :

$$\langle H=E_0,\ t'|T(\varphi(1)\varphi(2))|H=E_0,\ t''\rangle / \langle H=E_0,\ t'|H=E_0,\ t''\rangle, \tag{0·6}$$

which is Schwinger's definition.[2] Inserting a complete set of functionals into (0·6), we obtain

$$= \frac{\int\int d\varphi' d\varphi'' \langle H=E_0,\ t'|\varphi(t')=\varphi',\ t'\rangle \langle \varphi(t')=\varphi',\ t'|T(\varphi(1)\varphi(2))|\varphi(t'')=\varphi'',\ t''\rangle}{\int\int d\varphi' d\varphi'' \langle H=E_0,\ t'|\varphi(t')=\varphi',\ t'\rangle \langle \varphi(t')=\varphi',\ t'|\varphi(t'')=\varphi'',\ t''\rangle} \times \frac{\langle \varphi(t'')=\varphi'',\ t''|H=E_0,\ t''\rangle}{\langle \varphi(t'')=\varphi'',\ t''|H=E_0,\ t''\rangle}.$$

Then the Feynman principles, (0·1) and (0·2), lead to

$$\frac{\langle H=E_0,\ t'|T(\varphi(1)\varphi(2))|H=E_0,\ t''\rangle}{\langle H=E_0,\ t'|H=E_0,\ t''\rangle} = \frac{\int\int d\varphi' d\varphi'' \langle E_0|\varphi'\rangle \int\int_{\varphi'}^{\varphi''} \varphi(1)\varphi(2) e^{iI} \partial\varphi \langle \varphi''|E_0\rangle}{\int\int d\varphi' d\varphi'' \langle E_0|\varphi'\rangle \int\int_{\varphi'}^{\varphi''} e^{iI} \partial\varphi \langle \varphi''|E_0\rangle}, \tag{0·7}$$

where we have used the relation

$$\langle H=E_0,\ t'|\varphi(t')=\varphi',\ t'\rangle = \langle H=E_0,\ 0|\varphi(0)=\varphi',\ 0\rangle = \langle E_0|\varphi'\rangle.$$

In our following arguments, we shall start with the definition (0·7), because (0·7) seems more rigorous than (0·3).

Besides the above mentioned problem concerning the boundary value, there is another doubtful point in Matthews and Salam's paper. Namely, the validity of the change of integration variables in the functional integrals used by Matthews and Salam is not assured in the computation of the Green-functions.

The purpose of this paper is to obtain a condition under which Matthews and Salam's method can give a correct result. For the sake of simplicity, we consider only the case of the one-dimensional coupled oscillators, but we can readily extend our method to the case of the field theory. In addition, we shall treat only the case of the interacting boson fields, because there are some obscure points in treating fermion fields.

§ 1. The free harmonic oscillator

In the first place, let us compare the result derived from our definition (0·7) with

that of Matthews and Salam, and investigate under what condition the both results become identical. As a simple example, let us consider the propagator of the one-dimensional free harmonic oscillator with the Lagrangian

$$L=\tfrac{1}{2}(\dot{q}^2-\omega_0^2 q^2). \tag{1·1}$$

By using (0·7), the propagator $\Delta(t_1, t_2)$ of this system is given by

$$\Delta(t_1, t_2)=\frac{\int dq'dq''\langle E_0|q'\rangle\int\int_{q'}^{q''}q(1)q(2)e^{iI^*[q]}\delta q\langle q''|E_0\rangle}{\int dq'dq''\langle E_0|q'\rangle\int\int_{q'}^{q''}e^{iI^*[q]}\delta q\langle q''|E_0\rangle}, \tag{1·2}$$

where E_0 is the lowest eigenvalue of the Hamiltonian, and I^* is

$$I^*[q]=\int_{t''}^{t'}L^*(q,\dot{q})\,dt,\quad L^*=\tfrac{1}{2}(\dot{q}-\omega^2q^2)\quad\text{and}\quad\omega=\omega_0-i\epsilon. \tag{1·3}$$

Divide $q(t)$ into two parts in the following way:

$$q(t)=q_c(t)+\chi(t), \tag{1·4}$$

where $q_c(t)$ is the solution of the classical equation of motion $\ddot{q}_c+\omega^2q_c=0$ which satisfies the initial and final conditions, i.e. $q_c(t')=q'$ and $q_c(t'')=q''$, and $\chi(t)$ is an arbitrary function with the boundary values $\chi(t')=\chi(t'')=0$. Substituting (1·4) into (1·3), we get

$$\begin{aligned}I^*[q]&=\tfrac{1}{2}q_c\dot{q}_c\Big|_{t''}^{t'}+\tfrac{1}{2}\int_{t''}^{t'}(\dot{\chi}^2-\omega^2\chi^2)\,dt\\&=F(q', q''; t', t'')+I^*[\chi].\end{aligned} \tag{1·5}$$

Then, (1·2) becomes

$$\Delta(t_1, t_2)=\frac{\int\int dq'dq''\langle E_0|q'\rangle q_c(1)q_c(2)e^{iF}\langle q''|E_0\rangle}{\int\int dq'dq''\langle E_0|q'\rangle e^{iF}\langle q''|E_0\rangle}+\frac{\int\int_0^0\chi(1)\chi(2)e^{iI^*[\chi]}\delta\chi}{\int\int_0^0 e^{iI^*[\chi]}\delta\chi}, \tag{1·6}$$

where we have used the fact that such a term as $\int\int_0^0\chi e^{iI^*[\chi]}\delta\chi$ vanishes in virtue of the antisymmetric property of the integrand. We must note that in the first term of the right-hand side of (1·6) the functional integrals in the numerator and the denominator have cancelled out each other and this term depends on the boundary condition. On the other hand, the second term does not depend on the boundary condition, and can be easily computed in the same way as in Matthews and Salam's paper. Namely, we can derive the correct propagator by changing the integration variables from χ to a set of integration variables $\{a_n\}$ introduced by the definition

$$\chi=\sum a_n\chi_n,$$

where $\{\chi_n\}$ is some complete set of functions. (See Appendix.) The result gives the expression (0·3) which is just the same as was given by Matthews and Salam. Therefore, the first term is expected to vanish by virtue of the condition that both the initial and final states are the vacuum state.

In order to test this conjecture, let us integrate the first term of (1·6). The solution of the classical equation of motion is

$$q_c(t\,;t',t'')=\frac{1}{\sin\omega\tau}\{q'\sin\omega(t-t'')+q''\sin\omega(t'-t)\}, \tag{1·7}$$

where $\tau=t'-t''$. From (1·5) and (1·7), we obtain

$$F(q',q''\,;t',t'')=\tfrac{1}{2}q_cq_c\Big|_{t''}^{t'}=\frac{\omega}{2\sin\omega\tau}[(q'^2+q''^2)\cos\omega\tau-2q'q'']. \tag{1·8}$$

The eigenfunction $\langle E_0|q'\rangle$ is given by

$$\langle E_0|q'\rangle=(\omega_0/\pi)^{1/4}e^{-\omega_0 q'^2/2}. \tag{1·9}$$

Inserting (1·7), (1·8) and (1·9) into the first term of (1·6), we have the following result:

$$\begin{aligned}[\text{the first term of }(1\cdot6)]=&\frac{1}{2\omega}\cdot\frac{1}{a^2e^{2i\omega\tau}-1}\\ &\times\Big[\Big\{\frac{e^{i\omega(t_1+t_2)}}{e^{2i\omega t'}-e^{2i\omega t''}}+\frac{e^{-i\omega(t_1+t_2)}}{e^{-2i\omega t''}-e^{-2i\omega t'}}\Big\}\{(e^{2i\omega\tau}+1)a-2\}\\ &+\Big\{\frac{e^{i\omega(t_1-t_2)}}{e^{2i\omega\tau}-1}+\frac{e^{-i\omega(t_1-t_2)}}{e^{2i\omega\tau}-1}\Big\}\{e^{2i\omega\tau}(1-2a)+1\}\Big],\end{aligned} \tag{1·10}$$

where

$$a=1+(i\epsilon/2\omega)\,(1-e^{-2i\omega\tau}).$$

As will be easily seen, (1·10) does not vanish against our expectation as long as t' and t'' have the finite values. Thanks to the fact $\omega=\omega_0-i\epsilon$, however, we can see that (1·10) vanishes in the limit, $t'=-t''\to+\infty$. In this limit, however, (1·7) and (1·8) lead to

$$\lim_{t'=-t''\to+\infty}q_c(t\,;t',t'')=0 \quad\text{and}\quad \lim_{t'=-t''\to\infty}F(q',q''\,;t',t'')=\frac{i}{2}\omega(q'^2+q''^2)$$

for the finite value of t. Therefore, the first term of (1·6) vanishes irrespectively of the boundary condition, and we have the following curious result;

$$\lim_{t'=-t''\to\infty}\frac{\langle E_n,t'|T(q(1)q(2))|E_n,t''\rangle}{\langle E_n,t'|E_n,t''\rangle}=\lim_{t'=-t''\to\infty}\frac{\langle E_0,t'|T(q(1)q(2))|E_0,t''\rangle}{\langle E_0,t'|E_0,t''\rangle}$$

for an arbitrary excited state $|E_n,t\rangle$. Besides this, it must be noted that the propagators defined by (0·4) and (0·5) are independent of the initial and final times t' and t'' respectively, whereas the final expressions, (1·6) and (1·10), depend on them in a complicated way (see Appendix).

These difficulties are due to the inadequate method of the integration over all paths. Originally, Feynman defined the functional integrals in the following way.[4] He divides the time interval into a set of small time segments and defines the path integral as a

multiple integral with respect to q at each dividing point. With this definition, he proved the equivalence between his formalism and the ordinary quantum mechanics. But the method of integration used by Matthews and Salam is very different from Feynman's original one (see Appendix) and is justified by B. Davison[6] only in the case of computing the transformation function $\langle q', t'|q'', t''\rangle$. When we consider such a matrix element as $\langle q', t'|q(1)q(2)|q'', t''\rangle$, the validity of the change of integration variables used by Matthews and Salam seems to be quite doubtful.

In fact, Feynman's original definitions of the transformation function and of the matrix element give (1·2) the following expression:

$$(1\cdot 2)=\frac{\int dq'dq''\langle E_0|q'\rangle\left[\int dq_1dq_2\left(\int\int_{q_1}^{q'}e^{i/\hbar[t',t_1]}\mathfrak{D}q\right)q_1\left(\int\int_{q_2}^{q_1}e^{i/\hbar[t_1,t_2]}\mathfrak{D}q\right)q_2\left(\int\int_{q''}^{q_2}e^{i/\hbar[t_2,t'']}\mathfrak{D}q\right)\right]\langle q''|E_0\rangle}{\int dq'dq''\langle E_0|q'\rangle\left[\int dq_1dq_2\left(\int\int_{q_1}^{q'}e^{i/\hbar[t',t_1]}\mathfrak{D}q\right)\left(\int\int_{q_2}^{q_1}e^{i/\hbar[t_1,t_2]}\mathfrak{D}q\right)\left(\int\int_{q''}^{q_2}e^{i/\hbar[t_2,t'']}\mathfrak{D}q\right)\right]\langle q''|E_0\rangle}.$$

After dividing the path $q(t)$ into the classical one $q_c(t)$ and the remainder, we can see that the functional integrals in the numerator and the denominator just cancel out each other and that the remaining ordinary integrals give the correct result without any contradiction for the case of the free harmonic oscillator. Namely, Feynman's result has the correct dependence on the boundary condition and is independent of t' and t''.

Let us now compare our results (1·6) and (1·10) with Feynman's one. In this case, the both results also agree with each other under the same condition $t'=-t''\to+\infty$ as before for the case of the vacuum state. Therefore, we are led to the following conclusion (I):

When the limiting procedure $t'=-t''\to\infty$ is carried out, the Matthews and Salam method of integration gives the correct result, at least, for the case of the free propagator whose boundary conditions are given by the vacuum state.

As was already seen, in computing the free propagator with the Feynman method, the main problem turned out to seek for the eigenfunction of the total Hamitonian instead of computing the functional integral. This new problem of finding the eigenfunction is in general quite difficult to solve. On the other hand, the Matthews and Salam method permit us to write down the propagators in the functional integral forms in spite of the above stated difficulties and their method will give us a quite powerful means in treating the practical problems. Therefore, it is desirable to apply their method to the case of the interacting fields. Our next problem is to get another condition in addition to our conclusion (I) in order to derive a correct result from the Matthews and Salam method for this general case, or, at least, to make their result agree with that derived from the perturbation method. In the following sections, we shall investigate this problem in the case of the propagators of the interacting oscillators.

§ 2. The forced harmonic oscillator

As was stated in §§ 0 and 1, Matthews and Salam succeeded in giving a correct expression for the propagator in spite of their dubious treatment of the boundary condition and of the functional integrals for the case of the interacting fields.

Here and in the next section, we shall start with the correct definition (0·7) of the propagator and find a condition which leads us to the correct result though we shall use the conventional method of integration of Matthews and Salam in the course of calculation.

In this section, we shall consider the propagator of an oscillator interacting with a given external field on the basis of the Matthews and Salam method and of the limiting process discussed in § 1. The Lagrangian of the system is

$$L=(1/2)\,(\dot{q}^2-\omega_0^2q^2-u(t)q^2), \tag{2·1}$$

where $u(t)$ is an external field which satisfies the following condition:

$$u(t)=\begin{cases}\text{finite for } -T<t<T<\infty,\\ 0 \quad \text{otherwise.}\end{cases} \tag{2·2}$$

The propagator of this system is given by

$$S(t_1, t_2)=$$

$$\frac{\int\int dq'dq''\langle H_0(0)=E_0|q(0)=q'\rangle\langle q_I(\infty)=q',\ \infty|T(Sq_I(1)q_I(2))\,|q_I(-\infty)=q'',\ -\infty\rangle}{\int dq'dq''\langle H_0(0)=E_0|q(0)=q'\rangle\langle q_I(\infty)=q',\ \infty|S|q_I(-\infty)=q'',\ -\infty\rangle}$$

$$\times\frac{\langle q(0)=q''|H_0(0)=E_0\rangle}{\langle q(0)=q''|H_0(0)=E_0\rangle}, \tag{2·3}$$

where q_I is an operator in the interaction representation and $H_0(0)=\frac{1}{2}\,(p^2(0)+\omega_0^2q^2(0))$[†]. Introducing the Feynman principle into (2·3), we have

$$S(1\cdot 2)=\frac{\int\int dq'dq''\langle E_0|q'\rangle\int\int_{q'}^{q''}q(1)q(2)e^{iI^*}\partial q\langle q''|E_0\rangle}{\int\int dq'dq''\langle E_0|q'\rangle\int\int_{q'}^{q''}e^{iI^*}\partial q\langle q''|E_0\rangle}, \tag{2·4}$$

where

$$\langle E_0|q'\rangle\equiv\langle H_0(0)=E_0|q(0)=q'\rangle,$$

$$I^*=\int_{-\infty}^{\infty}L^*(q,\dot{q},u)\,dt$$

and

$$L^*=\tfrac{1}{2}\,(\dot{q}^2-\omega^2q^2-u(t)q^2). \tag{2·5}$$

In the same way as in § 1, let us put $q(t)$ as

$$q(t)=q_c(t)+\chi(t), \tag{2·6}$$

where $q_c(t)$ is the solution of the equation $\ddot{q}_c+(\omega^2+u(t))\,q_c=0$ which has the boundary values $q_c(+\infty)=q'$ and $q_c(-\infty)=q''$, and $\chi(t)$ is an arbitrary function which is $\chi(\pm\infty)=0$. From (2·5) and (2·6) we get

$$I^*=F(q', q'')+I^*[\chi, u], \tag{2·7}$$

[†] In (2·3) we have used the fact that

$$\langle H_{0I}(\pm\infty)=E_0,\ \pm\infty|q_I(\pm\infty)=q',\ \pm\infty\rangle=\langle H_0(0)=E_0|q(0)=q'\rangle.$$

where

$$F(q', q'') = \tfrac{1}{2} q_c \dot{q}_c \Big|_{-\infty}^{\infty}$$

and

$$I^*[\chi, q] = \tfrac{1}{2} \int_{-\infty}^{\infty} (\dot{\chi}^2 - (\omega^2 + u(t))\chi^2)\, dt.$$

Inserting (2·6) and (2·7) into (2·4), we obtain

$$S(1, 2) = \frac{\int\int dq' dq'' \langle E_0|q'\rangle q_c(1) q_c(2) e^{iF(q', q'')} \langle q''|E_0\rangle}{\int\int dq' dq'' \langle E_0|q'\rangle e^{iF(q', q'')} \langle q''|E_0\rangle} + \frac{\int\int_0^0 \chi(1)\chi(2) e^{iI^*[\chi, u]} \mathcal{D}\chi}{\int\int_0^0 e^{iI^*[\chi, u]} \mathcal{D}\chi} \quad . \quad (2\cdot 8)$$

In order to investigate the first term of (2·8), let us calculate the classical path $q_c(t)$. It is unfortunately impossible for us to obtain an exact solution of the classical equation of motion, so we shall solve it by means of the iteration method, assuming the convergence of the series derived by iteration. For the sake of convenience, we shall replace the boundary conditions $q_c(+\infty) = q'$ and $q_c(-\infty) = q''$ with those for the instants t' and t'' respectively;

$$q_c(t') = q' \quad \text{and} \quad q_c(t'') = q'',$$

where $t' = -t'' \quad T$. Integrating the classical equation of motion, we have the following integral eqation;

$$q_c(t\,;\, t', t'') = q_F(t, ;\, t', t'') + \int_{t''}^{t'} K(t, s\,;\, t', t'')\, u(s)\, q_c(s\,;\, t', t'')\, ds,$$

where

$$q_F(t\,;\, t', t'') = \frac{1}{\sin \omega\tau} \{q' \sin \omega(t - t'') + q'' \sin \omega(t' - t)\}, \quad (2\cdot 9)$$

$$K(t, s\,;\, t', t'') = \frac{1}{\omega \sin \omega\tau} [\sin \omega(t - t'') \sin \omega(t' - s)\, \theta(s - t) + \sin \omega(t' - t) \sin \omega(s - t'')\, \theta(t - s)] \quad (2\cdot 10)$$

and

$$\theta(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

In view of the conditions (2·2) and $t' = -t'' \gg T$, it runs

$$q_c(t\,;\, t', t'') = q_F(t\,;\, t', t'') + \int_{-T}^{T} K(t, s\,;\, t', t'')\, u(s)\, q_c(s\,;\, t', t'')\, ds. \quad (2\cdot 11)$$

The iteration method gives the expression

$$\int_{-T}^{T} \cdots \int_{-T}^{T} K(t, 1)\, u(1) K(1, 2)\, u(2) \cdots K(n-1, n)\, u(n)\, q_F(n\,;\, t', t'')\, d1 d2 \cdots dn \quad (2\cdot 12)$$

to the n-th order term of the power series concerning the external field $u(t)$.
Insert (2·9) and (2·10) into (2·12) and take the limit $t' = -t'' \to +\infty$ in (2·12)

for the finite value of t. Then, thanks to $i\epsilon$ in ω, we can easily see that

$$(2\cdot12)\quad \propto \lim_{t'=-t''\to+\infty} \frac{1}{(\sin\omega\tau)^{n+1}} e^{i\omega_0(t'-t'')+\epsilon(t'-t'')} e^{i\omega_0 t''+\epsilon t''}\cdot\int_{-T}^{T}\cdots\int_{-T}^{T} G(t, s_1, s_2, \ldots s_n; q', q'')\, ds_1 \ldots ds_n$$

$$\propto \lim_{t'=-t''\to+\infty} e^{i\omega_0 t''+\epsilon t''}\cdot\int_{-T}^{T}\cdots\int_{-T}^{T} G(t, s_1, s_2, \ldots s_n; q', q'')\, ds_1 \ldots ds_n$$

$$= \lim_{t'=-t''\to+\infty} O(e^{+\epsilon t''}) = 0. \qquad (2\cdot13)$$

From $(2\cdot13)$ we arrive at the conclusion that

$$\lim_{t'=-t''\to\infty} q_c(t; t', t'') = \lim_{-t''\to\infty} O(e^{\epsilon t''}) = 0 \qquad (2\cdot14)$$

for a finite value of t. Therefore, the first term of $(2\cdot11)$ vanishes in our limit and we have the result

$$S(1, 2) = \frac{\int\int_{\sigma}^{\sigma}\chi(1)\chi(2)\, e^{i I[\chi, u]}\hat{\partial}\chi}{\int\int_{\sigma}^{\sigma} e^{i I[\chi, u]}\hat{\partial}\chi}, \qquad (2\cdot15)$$

which is just Matthews and Salam's expression.

In the above argument it must be remarked that, besides the condition already mentioned in (I), another condition

$$\begin{gathered} u(t') = u(t'') = 0 \\ \text{for} \quad t' > T \quad \text{and} \quad t'' < -T \end{gathered} \qquad \text{(II)}$$

plays quite an important role in making the first term of $(2\cdot8)$ vanish.

§ 3. The coupled oscillators

We now consider the case of two oscillators interacting with each other. The Lagrangian is given by

$$L = L_q + L_u + L_{in},$$

where $L_q = \frac{1}{2}(\dot{q}^2 - \omega_0^2 q^2)$, $L_u = \frac{1}{2}(\dot{u}^2 - \nu_0^2 u^2)$ and $L_{in} = -\frac{1}{2} g(t)\, u\cdot q^2$. $g(t)$ is a given positive definite function of t describing the variable coupling strength and vanishes for $|t| > T$ and is assumed to have small non-vanishing value for $-T < t < T$ in order to guarantee the applicability of the perturbation method. (Condition (II'))

One of the propagators of this system is given by

$$S'(1, 2) = \frac{\int dq'dq''du'du'' \left\langle H_0(0) = E_0 \middle|_{u(0)=u'}^{q(0)=q'} \right\rangle}{\int dq'dq''du'du'' \left\langle H_0(0) = E_0 \middle|_{u(0)=u'}^{q(0)=q'} \right\rangle}$$

$$\times \frac{\left\langle {}_{u_I(\infty)=u'}^{q_I(\infty)=q'},\ \infty \middle| T(S q_I(1) q_I(2)) \middle|_{q_I(-\infty)=u''}^{u_I(-\infty)=q''},\ -\infty \right\rangle \left\langle {}_{u(0)=u''}^{q(0)=q''} \middle| H_0(0) = E_0 \right\rangle}{\left\langle {}_{u_I(\infty)=u'}^{q_I(\infty)=q'},\ \infty \middle| S \middle|_{u_I(-\infty)=u''}^{q_I(-\infty)=q''},\ -\infty \right\rangle \left\langle {}_{u(0)=u''}^{q(0)=q''} \middle| H_0(0) = E_0 \right\rangle},$$

where $H_0(0)$ is the free Hamiltonian defined by

$$H_0(0)=\tfrac{1}{2}(p^2(0)+\omega_0^2q^2(0))+\tfrac{1}{2}(\pi^2(0)+\nu_0^2u^2(0)),$$

and q_I and u_I are the operators in the interaction representation which is reduced to the Heisenberg and Schrödinger representation at $t=0$. By virtue of the Feynman principle, we may write

$$S'(1,2)=\frac{\int dq'dq''du'du''\langle E_0|q',u'\rangle\int\int_{q'''u'''}^{q'u'}q(1)q(2)e^{iI^*[q,u]+iI^*[u]}\partial q\partial u\cdot\langle q'',u''|E_0\rangle}{\int dq'dq''du'du''\langle E_0|q',u'\rangle\int\int_{q'''u'''}^{q'u'}e^{iI^*[q,u]+iI^*[u]}\partial q\partial u\cdot\langle q'',u''|E_0\rangle}, \tag{3·1}$$

where

$$\langle E_0|q',u'\rangle \quad \left\langle H_0(0)=E_0\middle|\begin{matrix}q(0)=q'\\u(0)=u'\end{matrix}\right\rangle,$$

$$I^*[q,u]=\int_{-\infty}^{\infty}(L_q^*+L_{in})\,dt \quad \text{and} \quad I^*[u]=\int_{-\infty}^{\infty}L_u^*dt. \tag{3·2}$$

We devide $q(t)$ and $u(t)$ as

$$q(t)=q_c(t)+\chi(t) \tag{3·3}$$

and $$u(t)=u_c(t)+\eta(t),$$

where

$$\begin{aligned}&\ddot{q}_c+(\omega^2+g(t)u(t))q_c=0,\ q_c(+\infty)=q',\ q_c(-\infty)=q'',\\&\ddot{u}_c+\nu^2u_c=0,\ u_c(+\infty)=u',\ u_c(-\infty)=u'',\\&\chi(\pm\infty)=0 \quad\text{and}\quad \eta(\pm\infty)=0.\end{aligned} \tag{3·4}$$

From (3·2), (3·3) and (3·4) we have

$$I^*[q,u]+I^*[u]=I^*[\chi,u]+I^*[\eta]+F, \tag{3·5}$$

where

$$I^*[\chi,u]=\tfrac{1}{2}\int_{-\infty}^{\infty}dt(\dot{\chi}^2-(\omega^2+g(t)u(t))\chi^2),$$

$$I^*[\eta]=\tfrac{1}{2}\int_{-8}^{\infty}dt(\dot{\eta}^2-\nu^2\eta^2)$$

and

$$F=\tfrac{1}{2}[q_cq_c+u_c\dot{u}_c]\Big|_{-\infty}^{\infty}.$$

Introducing (3·3) and (3·5) into (3·1) and using the expression (2·15), we get

$$\begin{aligned}S'(1,2)=&\frac{\int dq'dq''du'du''\langle E_0|q',u'\rangle\int\int_0^0q_c(1)q_c(2)N[u_c+\eta]e^{iI^*[\eta]}e^{iF}\partial\eta\cdot\langle q'',u''|E_0\rangle}{\int dq'dq''du'du''\langle E_0|q',u'\rangle\int\int_0^0N[u_c+\eta]e^{iI^*[\eta]}e^{iF}\partial\eta\cdot\langle q'',u''|E_0\rangle}\\&+\frac{\int dq'dq''du'du''\langle E_0|q',u'\rangle\int\int_0^0S(1,2;u_c+\eta)N[u_c+\eta]e^{iI^*[\eta]}e^{iF}\partial\eta\cdot\langle q'',u''|E_0\rangle}{\int dq'dq''du'du''\langle E_0|q',u'\rangle\int\int_0^0N[u_c+\eta]e^{iI^*[\eta]}e^{iF}\partial\eta\cdot\langle q'',u''|E_0\rangle},\end{aligned} \tag{3·6}$$

where

$$N[u_c+\eta]=\int\int_0^0e^{iI^*[\chi,u_c+\eta]}\partial\chi$$

and

$$S(1,\,2\,;\,u_c+\eta)=\iint_0^0\chi(1)\,\chi(2)\,e^{iI^*[\chi,\,u_c+\eta]}\partial\chi\Big/\iint_0^0 e^{iI^*[\chi,\,u_c+\eta]}\partial\chi.$$

We can, however, prove that F is independent of η and that $N[u_c+\eta]$ and $S(1,\,2,\,u_c+\eta)$ are independent of u_c, as will be seen afterwards. Therefore, we can write

$$S'(1,\,2)=\frac{\int dq'dq''du'du''[\langle E_0|q',\,u'\rangle e^{iF}\langle q'',\,u''|E_0\rangle\int\int^0 q_c(1)\,q_c(2)\,N[\eta]e^{iI^*[\eta]}\partial\eta]}{[\int dq'dq''du'du''\langle E_0|q',\,u'\rangle e^{iF}\langle q'',\,u''|E_0\rangle]\cdot[\int\int_0^0 N[\eta]e^{iI^*[\eta]}\partial\eta]}$$

$$+\frac{\int\int_0^0 S(1,\,2\,;\,\eta)\,N[\eta]e^{iI^*[\eta]}\partial\eta}{\int\int_0^0 N[\eta]e^{iI^*[\eta]}\partial\eta}\,. \qquad (3\cdot7)$$

In the same way as in the preceding section, we can see that $\lim_{t'=-t''\to\infty} q_c(t\,;\,t',\,t'')=0$, so the first term of the right-hand side of $(3\cdot7)$ vanishes. Finally, we obtain the Matthews and Salam expression:

$$S'(1,\,2)=\iint_0^0 S(1,\,2\,;\,\eta)\,N[\eta]e^{iI^*[\eta]}\partial\eta\Big/\iint_0^0 N[\eta]e^{iI^*[\eta]}\partial\eta.$$

Let us now prove that F is independent of η. Inserting the solution of $(3\cdot4)$ into $(3\cdot5)$, we obtain

$$F(t'\to+\infty,\,t''\to-\infty)=(i/2)\,\omega\,(q'^2+q''^2)+(i/2)\,(u'^2+u''^2)$$

$$-\lim_{t'=-t''\to\infty}\frac{1}{4i\sin\omega\tau}\int_{-T}^{T}\{q'\,(e^{i\omega(s-t'')}-e^{-i\omega(s-t'')})$$

$$+q''\,(e^{+i\omega(t'-s)}-e^{-i\omega(t'-s)})\}\,g(s)\,u(s)\,q_c(s\,;\,t',t'')\,ds. \qquad (3\cdot8)$$

Because of the finiteness of the integral in $(3\cdot8)$ and of the fact that $q_c(s\,;\,t',\,t'')=0(e^{t''})$, the last term vanishes and thus F is independent of η.

In the next place, we shall show that $N[u_c+\eta]$ and $S(1,\,2\,;\,u_c+\eta)$ are independent of u_c. Our task is only to prove that $I^*[\chi,\,u_c+\eta]$ is independent of u_c. From the fact that

$$I^*[\chi,\,u_c+\eta]=\tfrac{1}{2}\int_{-\infty}^{\infty}(\dot\chi^2-(\omega^2+g(t)\,\eta(t))\,\chi^2)\,dt-\tfrac{1}{2}\int_{-\infty}^{\infty}u_c(t)\,g(t)\,\chi^2(t)\,dt,$$

we consider only the last term of the right-hand side. Since $g(t)=0$ for $|t|>T$, we can write it as

$$\lim_{t'=-t''\to\infty}(-\tfrac{1}{2})\int_{-T}^{T}u_c(t\,;\,t',\,t'')\,g(t)\,\chi^2dt=0,$$

where we have used the following property

$$\lim_{t'=-t''\to\infty}u_c(t\,;\,t',\,t'')=\lim_{t'=-t''\to\infty}\frac{1}{\sin\nu\tau}\{u'\sin\nu\,(t'-t'')+u''\sin\nu\,(t'-t)\}=0$$

for a finite t.

In the above argument, we see that the same conditions as those used in § 2 play an essential role in deriving Matthews and Salam's result.

In conclusion, we can get the correct expressions for the propagators for the vacuum state by starting with the correct definition (0·7) and by using the Matthews and Salam method of integration, if we use the condition (I) and (II′).

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Appendix

We shall calculate the second term of (1·2) :

$$\Delta(t_1, t_2) = \iint_0^0 \chi(1)\chi(2)\, e^{iI[\chi]}\delta\chi \Big/ \iint_0^0 e^{iI[\chi]}\delta\chi \tag{A·1}$$

along the line of reasoning used in Matthews and Salam's paper. Let us introduce a complete set of the functions $\chi_n(t)$ which are normalized in the following way;

$$\tfrac{1}{2}\int_{t''}^{t'} \chi_m(t)\left(\frac{d^2}{dt^2}+\omega^2\right)\chi_n(t)\,dt = \delta_{nm}, \tag{A·2}$$

where $\chi_n(t') = \chi_m(t'') = 0$. Expanding $\chi(t)$ in terms of them as

$$\chi(t) = \sum_{n=1}^{\infty} a_n \chi_n(t), \tag{A·3}$$

we have

$$\Delta(t_1, t_2) = \frac{\sum_m \chi_m(t_1)\chi_m(t_2)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} a_m^2 e^{-i\sum_n a_n^2} H_s da_s}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i\sum_n a_n^2} H_s da_s} \tag{A·4}$$

$$= -\frac{i}{2}\sum_{n=1}^{\infty}\chi_n(t_1)\chi_n(t_2).$$

It must be noted that (A·4) depends implicitly on t' and t''. As will be easily seen, one of such complete sets that satisfy the condition (A·2) is given by

$$\chi_n(t) = \frac{2}{\sqrt{\tau}}\left[\frac{1}{\omega^2-(n\pi/\tau)^2}\right]^{1/2}\sin\frac{n\pi}{\tau}(t-t''). \tag{A·5}$$

From (A·4) and (A·5) we have

$$\Delta(t_1, t_2) = \frac{-i}{\tau}\sum_{n=1}^{\infty}\left[\frac{1}{\omega^2-(n\pi/\tau)^2}\right]\cdot\left[2\sin\frac{n\pi}{\tau}(t_1-t'')\cdot\sin\frac{n\pi}{\tau}(t_2-t'')\right]$$

$$= \frac{-i}{\pi}(\pi/\tau)\sum_{n=1}^{\infty}\left[\frac{1}{\omega^2-(n\pi/\tau)^2}\right]\cdot\left[\cos\frac{n\pi}{\tau}(t_1-t_2)-\cos\frac{n\pi}{\tau}(t_1+t_2-2t'')\right].$$

When τ is very large, we can replace the summation with the integration as

$$= \frac{-i}{\pi}\int_0^\infty \frac{dE}{\omega^2-E^2}[\cos E(t_1-t_2)-\cos E(t_1+t_2-2t'')]$$

$$= \frac{-i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iE(t_1-t_2)}}{\omega^2-E^2}\,dE+\left(\frac{-i}{2\pi}\right)\int_{-\infty}^{\infty} \frac{e^{iE(t_1+t_2-2t'')}}{\omega^2-E^2}\,dE,$$

where $n\frac{\pi}{\tau}\to E$. Using the fact that $\omega=\omega_0-i\epsilon$ and $t_1+t_2-2t''>0$, we obtain

$$= \frac{1}{2\omega}e^{-i\omega_0|t_1-t_2|-\epsilon|t_1-t_2|}+\frac{1}{2\omega}e^{-i\omega_0(t_1+t_2-2t'')-\epsilon(t_1+t_2-2t'')}.$$

In the limit $t''\to-\infty$, the second term vanishes and the first term gives the correct result for the propagator.

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A Lattice Model of Liquid Helium, II

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The partition function of the lattice model of liquid helium introduced in the preceding paper is calculated by making use of the Kikuchi's approximation. Setting two parameters, the effective mass $m=1/7m_0$ (m_0 is the mass of He atom), and the lattice spacing $d=3.1$Å, an excellent agreement of the density dependence of λ-temperature with experiment is obtained. The general trend of other thermodynamical quantities calculated in this paper is in good accordance with the observation.

1. Introduction

In the preceding paper,[1] hereafter referred to as I, we have introduced Hamiltonian of the lattice model of liquid helium and proved its equivalence to that of a spin system with anisotropic exchange coupling. The λ-transition was shown there to correspond to the spin system, and making use of the molecular field approximation, we could obtain right dependence of the λ-temperature on density. Phonons in liquid helium were proved to correspond to spin waves in the spin system.

All the properties of our lattice model, however, were derived by translating the results obtained for the spin system, under certain assumptions which were not necessarily without question open to criticism. For example, in the molecular field approximation we assumed the existence of the long range order of magnetization in the xy plane, but there is some difficulty in defining the long range order because of the fact that the magnetization in the xy plane is not constant of motion owing to the anisotropic exchange coupling. Furthermore, the crude approximation adopted in I fails to afford not only quantitative conclusions on specific heat and pressure, etc., but also qualitative explanation for the last of the three questions proposed at the begining of I. Namely as to the question why liquid helium has negative thermal expansion coefficient just below the λ-point, the previous treatment seems unsatisfactory for obtaining a correct solution.

To remedy these points, we shall employ in this paper a rather direct method of approximation without referring to the spin system for calculating the partition function of the lattice model, and try to understand qualitatively the thermodynamical behavior of

the liquid helium near the λ-point.

§ 2. Reduction of the partition function

The Hamiltonian of the lattice model has been taken in I as

$$H=(\hbar^2/2md^2)\sum_{\langle ij\rangle}(a_i^*-a_j^*)(a_i-a_j)-v_0\sum_{\langle ij\rangle}a_i^*a_ia_j^*a_j, \tag{1}$$

where m is the effective mass of an atom, d is the lattice constant and $\langle ij\rangle$ means to take the summation over all the nearest neighbor pair points. The operators a_i and a_i^* satisfy the commutation relations:

$$[a_i, a_j]_-=[a_i^*, a_j^*]_-=[a_i, a_j^*]_-=0 \text{ for } i\neq j, \tag{2}$$

$$[a_i, a_i]_+=[a_i^*, a_i^*]_+=0, \quad [a_i, a_i^*]_+=1 \text{ for } i=j. \tag{3}$$

Here, for the sake of simplicity, we put $v_0=0$, because the weak attractive potential will not play any essential role in the peculiar behaviors of this substance. In evaluating the partition function, we regard the total number of atoms $N=\sum_i a_i^*a_i$ as a given constant. Then, the Hamiltonian may be written apart from the additive constant as

$$H=-B\sum_{\langle ij\rangle}a_i^*a_j, \tag{4}$$

where

$$B=\hbar^2/2md^2.$$

The problem is to calculate the partition function

$$Z=\mathrm{Trace}\,[\exp(-\beta H)].$$

In the representation where $a_i^*a_i$'s are diagonal, $a_i^*a_j$ is such an operator that transfer an atom from the j-th to the i-th lattice point if the j-th position is occupied by an atom and the i-th by a hole. Otherwise it gives a zero. Therefore, when we expand Z as

$$Z=\mathrm{Trace}\,[\sum_n(-\beta H)^n/n!] \tag{5}$$

and substitute (4) into (5), we see that there appear in the right-hand side of (5) infinitely many terms, to each of which certain transfers of atoms correspond. If we represent each transfer of an atom by a line connecting the initial and final position of the atom, each term of (5) can be described by a set of closed polygons which we shall call a graph (Fig. 1). That is, Z can be written as

$$Z=\sum_n G_{2n}(\beta B)^{2n}/(2n)! \tag{5'}$$

and the problem is reduced to counting the number of graphs G_{2n} produced by $2n$ transfers. In a graph there may be double or triple, etc., lines corresponding to twice or three times, etc., repeated transfers of an atom along the same path and this situation introduces a formidable complication which prevents us from counting G_{2n} in a simple

manner. In what follows, we shall give up the rigorous calculation of G_{2n} and make a simplification so as to attain an approximate expression of Z on the basis of the theory of probability.

First, we shall define for each graph its skeleton which is a figure obtained from a graph by replacing all the multiple lines with single lines (Fig. 2). When we transfer atoms only along the lines of skeleton, a set of closed polygons is again formed corresponding to a graph with the smallest number of transfers in a given skeleton (Fig. 3). We call the graph thus obtained a fundamental figure (abbreviated as F. G.) corresponding to the original graph. We can always reduce each graph to its F. G. by the above mentioned procedures, and therefore we may classify the graphs by the corresponding F. G.'s.

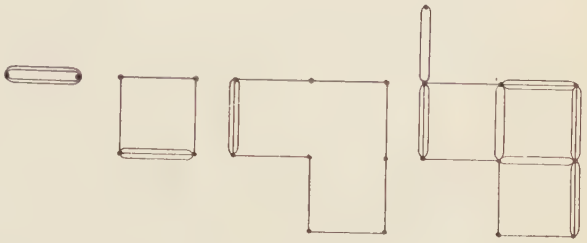

Fig. 1. Examples of allowed graphs.

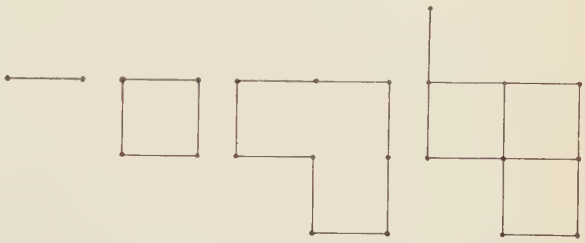

Fig. 2. The skeleton corresponding to Fig. 1.

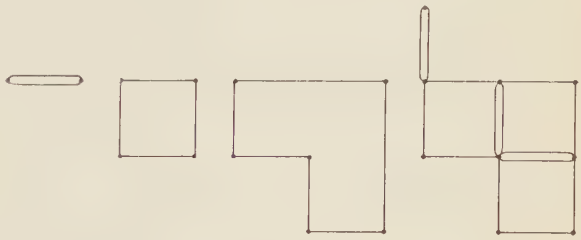

Fig. 3. The fundamental figures corresponding to Fig. 1.

Now let us fix the reference configuration of atoms and holes at any one of $[M!/(M-N)!N!]$ configurations where M denotes the total number of lattice points. If we are concerned with the term of order $2n$ in (5'), the central problem we have to solve is: In how many ways can we transfer atoms on a F. G. and return to the initial configuration after $2n$ transfers?

Before we enter into this problem, we shall define several notations. By the term "unit line" we mean a line connecting nearest neighbor lattice points, and let the total number of unit lines in a skeleton be $\bar{m}$ (the number of bonded pairs), and the total number of lattice points forming the skeleton be s (the number of bonded apices). Further, let the number of unit lines included in the F. G. be $2m$ which we call the side length of the F. G.* $\rho_0=(M-N)/M$ represents the probability of finding a hole at

* This is necessarily an even number.

a lattice point and $\rho_1 = M/N$ the probability of finding an atom at a lattice point.

In order to settle the above mentioned problem, we shall base our calculation on the following assumptions :

Assumption I

When we want to transfer an atom along a unit line, this unit line should have one end occupied by an atom and the other end by a hole. The probability that a unit line is " transferable " (i. e. along which one can transfer an atom) is, therefore, equal to $2_f\rho_0\rho_1$. Among m unit lines in a skeleton, there are $2m_f\rho_0\rho_1$ transferable unit lines. If we regard for a moment $2n$ transfers of atoms on a graph as independent events to each other, there are $[2\bar{m}_f\rho_0\rho_1]^{2n}$ ways of arranging these $2n$ transfers.

Assumption II

We have to correct the error produced by assuming $2n$ *independent* transfers. As a nature of trace, atoms should return to their initial configuration after $2n$ transfers. If we assume that each atom almost forgets its initial configuration during each travel, the probability for returning to the initial configuration will be approximated by the reciprocal of the number of ways that $s_f\rho_1$ atoms and $s_f\rho_0$ holes can be distributed among s lattice points of the skeleton, or by $[s!/(s_f\rho_0)!(s_f\rho_1)!]^{-1}$.

Assumption III

In order that a graph drawn by arranging $2n$ unit lines according to assumptions I and II belongs to the original F. G. with side length $2m$, it is necessary that each of $2m$ unit lines of F. G. is used up, at least once, as the path of transfer. This restriction will be approximately taken into account by a factor $f(m, n)$, which is the probability that $2n$ indistinguishable particles can be taken out of a set composed of equal number of a great many particles of $2m$ species, with a restriction that each species should appear at least once in $2n$ particles. Evidently for $n < m$

$$f(m, n) = 0, \tag{6}$$

and for $m < n \ll 2m$

$$f(m, n) \simeq [(2n)!/2^{2(n-m)}]\binom{2m}{2(n-m)}(2m)^{-2n}.$$

Multiplying these three factors together with the temperature factor $(\beta B)^{2n}/(2n)!$ and the total number of configurations $(M!/(M-N)!\,N!)$, and summing up with respect to both n and all possible F. G., we have

$$Z = \frac{M!}{(M-N)!\,N!}\,\mathbf{S}\sum_{n=0}^{\infty} f(m,n)[2\bar{m}_f\rho_0\rho_1]^{2n}\frac{(\beta B)^{2n}}{(2n)!}\left[\frac{s!}{(s_f\rho_0)!(s_f\rho_1)!}\right]^{-1}, \tag{7}$$

where $\mathbf{S}$ means the summation over all possible F. G.'s.

Each F. G. will be classified by a set of parameters $\{x_i\}$ and $\{y_i\}$ which are to be introduced in the next section. The values of m, m and s then become functions of $\{x_i\}$ and $\{y_i\}$. Denoting by $\eta(\{x_i\}, \{y_i\})$ the number of F. G.'s belonging to the set of parameters $\{x_i\}$ $\{y_i\}$, (7) can be rewritten as

$$Z=\frac{M!}{(M-N)!\,N!}\sum_{\{x\}\{y\}}\sum_{n=0}^{\infty} g(\{x\},\{y\})f(m,n)[2\bar{m}\rho_0\rho_1]^{2n}\frac{(\beta B)^{2n}}{2n!}\left[\frac{s!}{(s\rho_0)!(s\rho_1)!}\right]^{-1}. \tag{7'}$$

We further approximate Z by the maximum term of the right-hand side of (7'), the logarithm of which is denoted by $P(\{x_i\},\{y_i\})$. Then from $\partial P(\{x_i\},\{y_i\})/\partial n=0$ we have

$$n=m\left[1+\frac{1}{1+4m/\gamma}\right] \tag{8}$$

for $m<n\leqslant 2m$, where $\gamma=2\bar{m}\rho_0\rho_1\beta B$. We shall see in the next section that at the λ-point $\gamma\leq m/2$, so that $n\leq m(1+1/9)$. So far as we are concerned with the temperature range near the λ-point, we may therefore put $n\simeq m$.

§ 3. Introduction of Kikuchi's approximation

In evaluating $g(\{x_i\},\{y_i\})$, we shall use so-called Kikuchi's approximation. In his paper[2] on the λ-transition of liquid helium, Kikuchi undertook a calculation of the total number of ways $g(L)$ that polygons can be drawn by connecting nearest neighbor points of a simple cubic lattice so that the total number of sides of all the polygons altogether is L. Since he made use of the Feynman's formulation,[3] the rule of drawing polygons in his case is somewhat different from that in our F. G. For instance, such a polygon as (d) in Fig. 3 is not allowed in Kikuchi's case while it cannot be excluded according to our definition of F. G. But in the temperature region near the λ-point where, as we shall see presently, the polygons of one-sided type like (b) and (c) in Fig. 3 have just become important, it will not lead to a serious error even if we neglect such complicated polygons as (d) in Fig. 3. Then we may put the analogous restrictions as Kikuchi's case concerning polygons. They are:

(i) between two adjacent lattice points only one side is drawn except for two sided polygons, and

(ii) polygons must not come into contact.

We need not distinguish two directions of cyclic transfer because its effect is already taken into account by a factor 2 in the probability factor $[2\bar{m}\rho_0\rho_1]^{2n}$.

Under these restrictions concerning polygons, we have now table I for the possible configurations of a lattice point and table II for that of a bond.

The x's and y's are the probabilities of finding one of the respective configurations and the values of α's and β's are the numbers of different configurations having the same probability on the basis of symmetry requirement. With the aid of tables I and II, we can express m, $\bar{m}$ and s in terms of x's and y's. When $m<n\leqslant 2m$

$$\begin{aligned} 2n\simeq 2m&=3M(2y_4+25y_5),\\ \bar{m}&=3M(y_4+25y_8),\\ s&=M(6x_2+15x_3). \end{aligned} \tag{9}$$

Table I Configuration of a point

Configuration	Probability	α_i
	x_1	1
	x_2	6
	x_3	15

Table II Configuration of a bond

Configuration	Probability	β_i
	y_1	1
	y_2	10
	y_3	25
	y_4	1
	y_5	20
	y_6	100
	y_7	100
	y_8	25

According to Kikuchi's approximation for the case of simple cubic lattice, $g(\{x_i\},\{y_i\})$ is approximated by

$$g(\{x_i\},\{y_i\})=X^5\,Y^{-3}(M!)^2, \tag{10}$$

where

$$X=\prod_{i=1}^{3}(x_i M)!^{3_i}, \qquad Y=\prod_{i=1}^{8}(y_i M)!^{8_i}.$$

Putting (10) into (7′), we finally obtain for the logarithm of the maximum term of the partition function per lattice point

$$P=\operatorname*{Max}_{M}^{P(\{x_i\}\ \{y_i\})}=\operatorname{Max}\Big[5\sum_{i=1}^{3}\alpha_i x_i\ln x_i-3\sum_{i=1}^{8}\beta_i y_i\ln y_i+3(2y_4+25y_8)I'$$
$$+(6x_2+15x_3)A-A\Big], \tag{11}$$

where we put

$$I'\equiv\ln\gamma_0=\ln(\rho_0\rho_1\beta B),$$
$$A\equiv\rho_0\ln\rho_0+\rho_1\ln\rho_1. \tag{12}$$

In deriving (11) we have made a simplification that $m\simeq m$ which will be justified later on.

§ 4. Region just below the λ-point

From the tables I and II, the following geometrical relations among x's and y's are obtained :

$$\begin{aligned} 1 &= x_1 + 6x_2 + 15x_3, \\ x_1 &= y_1 + 5y_2 + 10y_5, \\ x_2 &= y_2 + 5y_3 + 10y_6 = y_4, \\ x_3 &= y_5 + 5y_6 + 10y_7 = 5y_8. \end{aligned} \tag{13}$$

If we choose y_3, y_4, y_6, y_7 and y_8 as independent variables, the other parameters can be expressed as linear combinations of them, as shown in table III.

Table III. Relations between the dependent and independent variables.
The meaning of this table is, for example, $x_1 = 1 - 6y_4 - 75y_8$.

	1	y_3	y_4	y_6	y_7	y_8
x_1	1		−6			−75
x_2			1			
x_3						5
y_1	1	25	−11	100	100	−125
y_2		−5	1	−10		
y_5				−5	−10	5

Maximizing $P(\{x_i\}, \{y_i\})$ given by (11) with respect to these independent variables, one gets

$$y_1 y_3 = y_2^2, \tag{14a}$$

$$y_1 y_6 = y_2 y_5, \tag{14b}$$

$$y_1 y_7 = y_5^2, \tag{14c}$$

$$\exp\left[-2(A+\Gamma)\right](y_2^{10} y_4 / y_1^{11}) = (x_2/x_1)^{10}, \tag{14d}$$

$$\exp\left[-(A+\Gamma)\right](y_5^4 y_8 / y_1^5) = (x_3/x_1)^5. \tag{14e}$$

These equations together with the relations in table III can be easily solved by substitutions

$$y_2/y_1 = v, \qquad y_5/y_1 = w, \tag{15}$$

yielding

$$v = 1/5 \exp\left[A+\Gamma\right] = 1/5 \exp(A) \rho_0 \rho_1 \beta B, \tag{16}$$

$$w = 1/10\,(4 \exp\left[A+\Gamma\right] - 1) = 2v - 1/10. \tag{17}$$

The other variables can be written as functions of v and w.

Since both y_5 and y_1 are positive, w should be also positive, and accordingly eq. (17)

shows that the solution obtained above is valid only when the temperature is lower than the critical temperature T_λ defined by

$$4\exp\lfloor A+I'\rfloor-1=0$$

or

$$T_\lambda=(4\rho_0\rho_1/k)\exp(A)\,B. \tag{18}$$

At $T=T_\lambda$, it holds that

$$\gamma=2\bar{m}\rho_0\rho_1\beta B\simeq\tfrac{1}{4}m\exp(-A)\leq m/2. \tag{19}$$

This justifies the approximation $n\simeq m$ in a self-consistent way.

Just below the critical temperature where our approximation is valid, the logarithm of the partition function per lattice point is given by

$$\bar{P}=3\ln(1+5v+10w)-2\ln(1+6v+15w)-A. \tag{20}$$

The energy E, heat capacity at constant volume C_{II}, and pressure p of the system can be readily calculated as

$$E=-\frac{3}{5}Me^{A}\rho_0\rho_1 B\left[\frac{24}{1+6v+15w}-\frac{25}{1+5v+10w}\right], \tag{21}$$

$$C_{II}=3M(\tfrac{1}{5}\rho_0\rho_1 Be_A)^2\frac{1}{kT^2}\left[\frac{864}{(1+6v+15w)^2}-\frac{625}{(1+5v+10w)^2}\right], \tag{22}$$

$$p=\frac{kT}{d^3}\frac{\partial P}{\partial M}=\frac{1}{d^3}\left[kT\left(\bar{P}+\rho_1\ln\frac{\rho_1}{\rho_0}\right)+\frac{3}{5}e^{A}\rho_0\rho_1 B\right.$$
$$\left.\times\left\{\frac{75}{1+5v+10w}-\frac{72}{1+6v+15w}\right\}\left\{\frac{\rho_1}{\rho_0}-1-\rho_1\ln\frac{\rho_1}{\rho_0}\right\}\right]. \tag{23}$$

According to (22), the magnitude of specific heat at the critical point amounts to

$$C_{II}/Mk=33/43. \tag{24}$$

§ 5. The region above the λ-point

When w is zero, x_3 and y_3 to y_8 also vanish as is seen from (13) and (14). Returning to the meanings of these parameters, we see that this corresponds to a state in which only two-sided polygons exist. Then, of the five equations (14a) to (14e), all but (14a) and (14d) become trivial identities. We can still use v as defined in (16) for this case, but the results are different, namely

$$v=(r-1)/10 \tag{25}$$

with

$$r^2=1+20\exp\lfloor 2(A+I')\rfloor.$$

In this region, the partition function per lattice point, the total energy, heat capacity at

constant volume, and pressure of the system are respectively given by

$$\bar{P}=3\ln(1+5v)-2\ln(1+6v)-A, \tag{26}$$

$$E=2M\left(\frac{12}{1+6v}-\frac{15}{1+5v}\right)(e^{A}\rho_0\rho_1 B)^2\frac{1}{rkT} \tag{27}$$

$$C_1=\frac{6Mk}{r^2}(e^{A}\rho_0\rho_1\beta B)^2\left[\left\{r-\frac{20}{r}(e^{A}\rho_0\rho_1\beta B)^2\right\}\left(\frac{5}{1+5v}-\frac{4}{1+6v}\right)\right.$$
$$\left.-2(e^{A}\rho_0\rho_1\beta B)^2\left\{\frac{25}{(1+5v)^2}-\frac{24}{(1+6v)^2}\right\}\right], \tag{28}$$

$$P=\frac{MkT}{d^3}\left[P+\rho_1\ln\frac{\rho_1}{\rho_0}+\left(\frac{15}{1+5v}-\frac{12}{1+6v}\right)\right.$$
$$\left.\times\frac{2}{r}e^{2A}(\rho_0\rho_1\beta B)^2\left(\frac{\rho_1}{\rho_0}-1-\rho_1\ln\frac{\rho_1}{\rho_0}\right)\right]. \tag{29}$$

At the critical temperature, (28) gives

$$C_{1\lambda}/Mk=11/169. \tag{30}$$

Comparing this with C_{1l} derived in § 4 and noting that E and p are continuous at T_λ, we can conclude that this temperature T_λ corresponds to the critical point of second-kind transformation and may be identified with the λ temperature of liquid helium.

§ 6. Comparison with experiments and discussions

In order to compare our results with experiments, we must first fix two parameters, the effective mass m and the lattice spacing d. If we choose

$$d=3.1\text{Å}$$
$$m=0.14m_0\simeq m_0/7 \tag{31}$$

where m_0 is the mass of He^4 atom. Eq. (18) gives an excellent agreement with experiment of the density dependence of the λ temperature as is shown in Fig. 4. Along the λ-line, the value of ρ_1 changes from 0.65 under the pressure of 1 atm. to 0.78 under the pressure of 25 atm.

We have obtained rather small value for m. One of the reasons for this will be due to too severe restriction of F. G. or neglect of the complicated polygons such as (d) in Fig. 3. The restriction, indeed, will impede possible particle transfers when the temperature is lowered. In other words, it will make one underestimate the quantum effect and consequently will tend to lower the λ-temperature, resulting in smaller value of m in order to have the agreement with experiments.

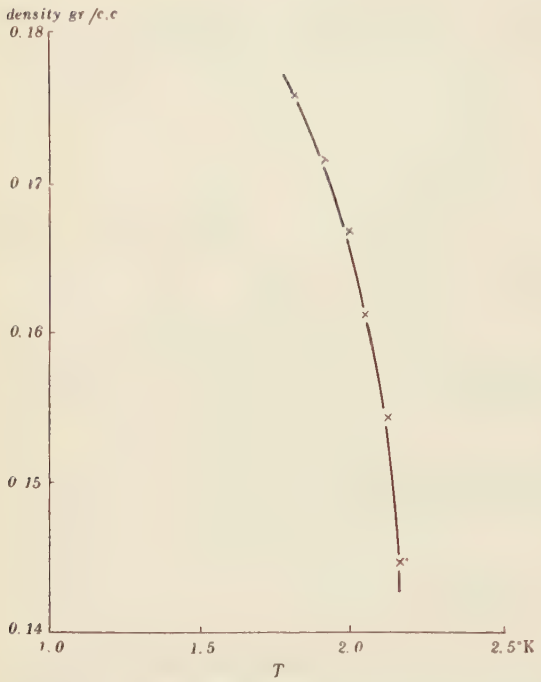

Fig. 4. The λ-line calculated from eq. (18). $\times$correspond to the experimental values due to Keesom & Miss Keesom.[4]

Figs. 5 and 6 show respectively the curves of heat capacity and isopycnals calculated basing on the present approximation. The general trend of these curves agrees fairly well with observations in a qualitative sense. Although Fig. 6 does not show the existence of negative thermal expansion coefficient in the density range $0.6 \leq \rho_1 \leq 0.8$, to which the density of actual liquid helium is supposed to correspond, the gradient of each curve has at the λ-line a discontinuity which becomes larger with increasing density as is observed in liquid helium.

The negative thermal expansion coefficient will be expected to come out from the fact that the lower the temperature is, the more important becomes the contribution to the partition function from a graph with large polygons which is sensitive to the density variation. For $\rho_1 = 0.9$ the pressure increases considerably with decreasing temperature just below the λ-point, showing the possibility of the existence of negative thermal expansion coefficient.

As we have omitted the attractive potential from the beginning, numerical values of pressure can not be compared with experiments. Neglect of certain types of F. G. makes one underestimate the value of $\bar{P}$ less and less as the temperature is lowered below the λ-point. This will be one of the main reasons why the values of specific heats in Fig. 5 are smaller than those of experiments.

Finally we note that the formulation in this paper has a certain similarity with Feynman-Kikuchi's paper.[2),3)] The interpretation of the transition is just the same, and actually we owe our calculation in § 3 to § 5 much to Kikuchi's paper. The main difference lies in the reduction of the partition function. Feynman introduced the effective mass m' which is a complicated function of density and temperature, and he separated the part of the partition function that is not essential for the transition. Our calculation will correspond to the estimation of density and temperature dependence of the part left out of consideration in his theory, thus enabling us to see qualitatively the phenomena depending on density.

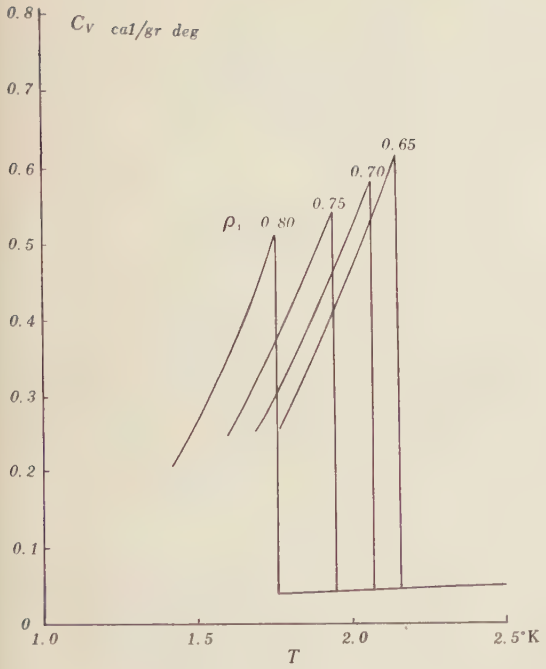

Fig. 5. Specific heats at constant volume derived from (22) and (28).

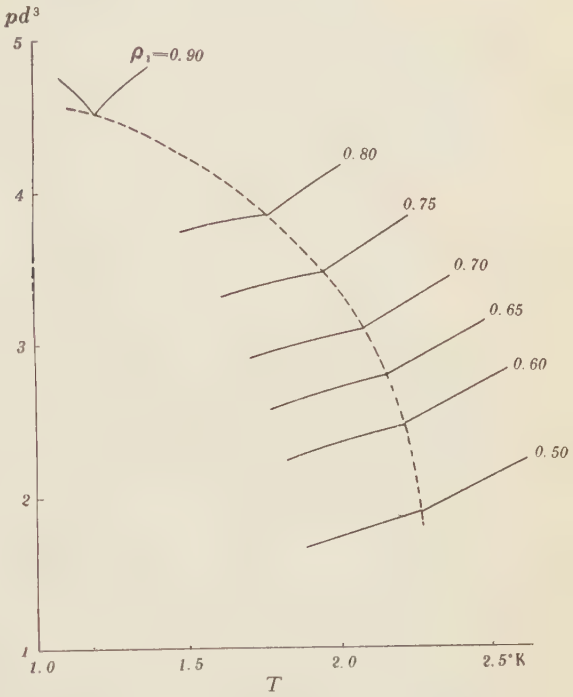

Fig. 6. Isopycnals of liquid helium calculated from (23) and (29).

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A Method of Renormalization for Unstable Particles

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We investigate the Lee's model which is modified in such a way as the V state is in the unstable mass relation. A renormalization recipe is proposed, which reduces to the conventional one in the stable case but is essentially different from the latter in some respects. The result obtained by this method represents certainly the unstability of the V state. However, its decay character is more complicated than the ordinarily accepted exponential time-dependence. Some new features introduced by our method are discussed.

§ 1. Introduction

Quantum electrodynamics has made so great progress that it is not only nearly complete in its form but also it gives satisfactory results in comparison with experiments. In its course of development, it has been the guiding principle that the theory must be formulated in a relativistically invariant form and that it is sufficient to treat only observable quantities. Further, quantum electrodynamics plays an important role as model in analysis of the phenomena in which so-called elementary particles take parts. The present field theory thus constructed, however, still involves many unsolved problems, and some of them seem to compel us to introduce some new concept or to reconstruct the field theory from somewhat different standpoint. It has so closed a form that any partial modification destroy the theorys at all. Therefore, it might be necessary to reformulate the theory from the beginning by introducing a new idea.

Let us now take out the following two questions from those which we bear in mind.

i) Although the quantitative agreement between theoretical calculations and experiments is very surprising in quantum electrodynamics, meson theory is still nothing but a phenomenology in spite of its qualitative success. Why is this the case?

ii) The field theory has succeeded in quantum electrodynamics to a certain extent, but at the present stage it owes the finite results to the temporal method of renormalization. How should we treat this problem in future step?

Here we may remember the fact that the present field theory (especially the second quantization and renormalization) has been constructed on the ground of the stable particle image. If we could reconstruct field theory in such a way that it would contain the conventional field theory as the limit where all concerned particles are stable, the interpretation of the first problem would become clear. That is, nucleons and mesons are more or less unstable in contrast to electrons and photons, and, in addition, meson

theory must be very closely related to unstable heavy particles and isobaric states (if they are present). Of course, as is ordinarily thought, we may start with the smallness of the coupling constant in quantum electrodynamics, the peculiar character of photon, and the analysis of high energy phenomena. At this point, we may emphasize that if we consider this problem as that of approximation method it means the inadequacy of the model for nucleon or meson, and we must interprete why $e^2/\hbar c$ is such a small quantity as 1/137.

On the other hand, very interesting consideration with regards the second problem is the investigations by Stückelberg and his collaborators.[1] The divergences occuring in the present field theory are due to the requirement of microscopic causality even to the two world points at the same time. To the points in such a relation, however, causality has no correspondence-theoretical meaning. If the theory could be reconstructed upon the basis of unstable particle picture, its mass might have a width due to uncertainty principle. Therefore, the time of creation and annihilation might be extended, and it would result in extending the vertex effectively. Non-local field theory and theories of non-local interaction have been proposed to smooth out interactions and studied by many authors. However, apart from the reasonableness of these models, it is a critical defect of them that no self-consistent calculation method has been constructed. If a non-point interaction, in any sense, is introduced, it must not be done merely formally, but automatically upon physical background. Then, their physical background would lead to a self-consistent method of calculation.

From the points discussed above, it would be very interesting to reinvestigate the problem of unstable particle, although our present field theory is practical enough to estimate the lifetime of unstable particles.

Now, it is not clear whether the renormalization method can be used successfully for unstable particles, because it was constructed on the basis of stable particle image. If it is possible, the modification which becomes necessary under the special situation of the unstable case might give some clue to the problems mentioned above.

Peierls[2] has investigated properties of a particle with metastable isobaric states, assuming that its Green's function should have poles in the lower half of the complex energy plane. This idea is based on the Weisskopf-Wigner's perturbation method, and it is clear that such a characteristic form of Green's function can be obtained only by an appropriate approximation. Taking account of this situation, we will develop the theory in a quite different form.

In this paper we investigate the Lee's[3] model in the case in which the mass of V-state is in an unstable mass relation, and propose a procedure to renormalize divergent quantities so that unstable particles* decay certainly. Then we take care particularly of the physical meaning of the procedure and notice to discuss a general features without use of the special character of the model.

* Here the word "unstable particle" means a particle which has larger mass than the decay products and interaction describing the decay.

§ 2. Preliminary calculations

In this paper, from the view-point that the decay of an unstable particle is essentially a non-stationary phenomenon, we solve the time-dependent Schrödinger equation directly, using the Arnous-Zienau's[1] formalism which is a very skillful field-theoretical rearrangement of the Weisskopf-Wigner's method.

According to this formalism the solution of the time-dependent Schrödinger equation

$$i\cdot\partial\Omega(t)/\partial t=(H_0+H)\,\Omega(t) \tag{1}$$

is given by following expressions when the given initial state is described by φ_0;

$$\Omega(t)=R(t)\varphi_0\,,$$

$$R(t)=\frac{i}{2\pi}\int_{-\infty}^{\infty}dE\,e^{-iEt}\left(1+\frac{1}{E-H_0+i\epsilon}U(E)\right)\lambda(E),$$

$$\lambda(E)^{-1}=E-H_0+i/2\cdot\Gamma(E)+i\epsilon,$$

$$\Gamma(E)=2i\{H+H(E-H_0+i\epsilon)^{-1}U(E)\}_{d},$$

$$U(E)=\{(1-H(E-H_0+i\epsilon)^{-1})\,(H+X)\}_{n.d.},$$

$$X(E)=-[\{1-H(E-H_0+i\epsilon)^{-1}\}_{d}]^{-1}[\{1-H(E-H_0+i\epsilon)^{-1}\}H]_{d},$$

where $\{\ \}_{d}$ and $\{\ \}_{n.d}$ mean diagonal and non-diagonal part of the concerning operators, with respect to the complete system of eigenfunctions of H_0. We may call attention here to the situation that $\{R(t)\}_{d}$'s are the Green's functions of corresponding states and $U(E)$ is a non-diagonal operator, so each element of $i\lambda(E)$ is a Fourier transform of the Green's function.

In the Lee's model, Hamiltonian is given by

$$\begin{aligned}H_0&=m_V a_V^{\dagger}a_V+m_N a_N^{\dagger}a_N+\sum_{k}{}'\,\omega_k a_k^{\dagger}a_k\,,\\ H&=g\sum_{k}{}'(2\omega_k\Omega)^{-1/2}(a_V^{\dagger}a_N a_k+a_N^{\dagger}a_V a_k^{\dagger})+\delta m a_V^{\dagger}a_V,\end{aligned} \tag{2}$$

where $a_V^{\dagger}$, a_V ; $a_N^{\dagger}$, a_N ; $a_k^{\dagger}$, a_k are creation and annihilation operators of V, N, θ with momentum $\boldsymbol{k}$ (energy $\omega_k=\sqrt{\boldsymbol{k}^2+\mu^2}$), respectively, and m_V, m_N, μ are observed masses of these particles, and δm is a counterterm due to self-energy of V. The unstability of V state is described by the relation $m_V>m_N+\mu$.

Taking $|V>$ (bare V state) and $a_k^{\dagger}|N>$ (the state where an N and a θ with momentum $\boldsymbol{k}$ exist) as initial states, the necessary matrix-elements of U and Γ are easily calculated by elementary matrix calculation. Here we write down only the results (The suffix $l|V$, for example, indicates that it is the matrix element between $|V>$ and $a_l^{\dagger}|N>$).

$$U_{l|V}=H_{l|V}=g(2\omega_l\Omega)^{-1/2},$$

$$U_{V|l}=g(2\omega_l\Omega)^{-1/2}M_l^{-1},$$

where

$$M_l \equiv N + g^2 (2\omega_l \Omega)^{-1} (E - m_V + i\epsilon)^{-1} (E - E_l + i\epsilon)^{-1},$$

$$N \equiv 1 - (\delta m + g^2 \sum (2\omega_k \Omega)^{-1} (E - E_k + i\epsilon)^{-1}) (E - m_V + i\epsilon)^{-1},$$

$$E_l = m_N + \omega_l ,$$

$$U_{l|V} = g^2 (2\omega_l \Omega)^{-1/2} (2\omega_{l'} \Omega)^{-1/2} M_{l'}^{-1} (E - m_V + i\epsilon)^{-1},$$

$$= 2i (\delta m + g^2 \sum (2\omega_k \Omega)^{-1} (E - E_k + i\epsilon)^{-1}),$$

that is

$$Im\, \Gamma'_{V|V} = 2\left(\delta m + g^2 \sum \frac{1}{2\omega_k \Omega} \frac{\mathcal{P}}{E - E_k}\right),$$

$$Re\, \Gamma'_{V|V} = 2\pi g^2 \sum (2\omega_k \Omega)^{-1} \delta (E - E_k),$$

$$\to g^2/2\pi \cdot \sqrt{(E - m_N)^2 - \mu^2}\, \theta (E - m_N - \mu).$$

Here $\to$ means the limit when Ω tends to infinity, and

$$\theta (x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Moreover, if we use straight cut-off with cut-off momentum K, we obtain

$$Re\, \Gamma'_{V|V} \to g^2/2\pi \cdot \sqrt{(E - m_N)^2 - \mu^2}\, \{\theta (E - m_N - \mu) - \theta (E - m_N - \omega_K)\}.$$

Finally

$$\Gamma'_{l|l} = 2ig^2 (2\omega_l \Omega)^{-1} M_l^{-1} (E - m_l + i\epsilon)^{-1}.$$

There is a simple relation between $\lambda_{K|K}$ and $\lambda_{V|V}$

$$\lambda_{K|K} (E)^{-1} = M_k (E)^{-1} (E - E_k + i\epsilon) (E - m_V + i\epsilon)^{-1} \lambda_{V|V} (E)^{-1}, \tag{3}$$

which follows directly from

$$N = 1 + i/2 \cdot \Gamma'_{V|V} (E) (E - m_l + i\epsilon)^{-1}.$$

Now, if we expand $\Omega (t)$ with respect to eigenstates of H_0 as

$$\Omega (t) = \sum_n b_{n|0} (t) e^{-iE_n t} \varphi_n ,$$

b's are given by

$$b_{0|0} (t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE\, e^{i(E_0 - E)t} \lambda_{0|0} (E),$$

$$b_{A|0} (t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dE\, e^{i(E_A - E)t} \frac{1}{E - E_A + i\epsilon} U_{A|0} (E) \lambda_{0|0} (E), \tag{4}$$

where $A \neq 0$.

§ 3. Mass renormalization

For a stable particle, the physical requirement to determine δm is expressed as follows in our formalism: "*When the bare particle state is taken as the initial state,* $\Omega(t)$ *must have time dependence of the form* e^{-imt} *as* $t\to\infty$". This is equivalent to require that the Fourier transform of the Green's function must have a pole at $E=m$.

If we impose a similar requirement for an unstable particle,

$$\lambda_{V|V}(E)=[\{E-m_V-\tfrac{1}{2}Im\,\Gamma'_{V|V}(E)\}+i/2\cdot Re\,\Gamma'_{V|V}(E)]^{-1} \tag{5}$$

must have a complex pole at $E=m_V-i/2\cdot\gamma$ $(\gamma>0)$. However, if we put the denominator of eq. (5) equal to zero and then put $E=m_V-i/2\cdot\gamma$, we obtain the following equation to determine γ;

$$\gamma\{1+g^2\sum(2\omega_k\Omega)^{-1}[(m_V-E_k)^2+1/4\cdot\gamma^2]^{-1}\}=0.$$

This equation, however, has no solution except $\gamma=0$. This result is inconsistent with the above requirement. However, as will be explained later, V decays in spite of the absence of complex pole, and the decay character is very much complicated than $\exp(-1/2\cdot\gamma t)$. This method is to be considered to determine m_V as zero point of $Re\,\lambda^{-1}_{V|V}$ after all.

As the usually accepted method to determine δm failed, we must impose another requirement. Now, we require that *the energy spectrum of* θ *particles, which emerge in V-decay, must have a maximum point at* $E=m_V$.

The energy spectrum is given by

$$I(\omega_p)=|b_{p|V}(\infty)|^2\rho_p,$$

where ρ_p is the density of the final states. Using the relation

$$-(1/2\pi i)\lim_{t\to\infty}e^{-ixt}\,1/(x+i\epsilon)=\delta(x)$$

and

$$\rho_p=(2\pi)^{-3}\Omega\cdot 4\pi p\omega_p,$$

we get

$$I(\omega_p)=\frac{g^2}{4\pi^2}\frac{p}{\sqrt{\{E_p-m_V-\tfrac{1}{2}Im\,\Gamma'_{V|V}(E_p)\}^2+\{\tfrac{1}{2}Re\,\Gamma'_{V|V}(E_p)\}^2}}.$$

Our requirement is expressed as

$$dI(\omega_p)/d\omega_p|_{\omega_p=m_V-m_N}=0.$$

After some calculation, we obtain

$$\delta m=\delta m_L-(p_0^2/\omega_0)(1/Z_L)(1-\sqrt{1+(g^4/16\pi^2)Z_L^2(\omega^2/p_0^2)}) \tag{6}$$

where

$$\omega_0\equiv m_V-m_N,\qquad p_0\equiv\sqrt{\omega_0^2-\mu^2},$$

$$\delta m_L \equiv -g^2 \sum \frac{1}{2\omega_k \Omega} \frac{\mathcal{P}}{m_V - E_k},$$

$$Z_L^{-1} \equiv 1 + g^2 \sum \frac{1}{2\omega_k \Omega} \frac{\mathcal{P}}{(m_V - E_k)^2}.$$

Here $\mathcal{P}\sum \frac{1}{2\omega_k\Omega} \frac{1}{(m_V - E_k)^2}$ is defined by $-\frac{d}{dE}\mathcal{P}\sum \frac{1}{2\omega_k\Omega}\frac{1}{E-E_k}\Big|_{E=m_V}$. In this expression, the term $g^4/16\pi^2 \cdot Z_L^2 \omega_0^2/p_0^2$ under root symbol originates in $Re\, \Gamma'_{V|V}(E_p)$. Therefore, if $m_V < \mu + m_N$, it vanishes and δm reduces to δm_L, the value in the case of stable V state, as it must be so.

Of course, in order to determine δm, it is sufficient to impose only one physical requirement, and the requirement used above is merely a possible one, and not unique. For example, if we require*) that

$$Re\, \lambda_{V|V}(E)^{-1} = 0, \text{ at } E = m_V,$$

we obtain $\delta m = \delta m_L$, and the arguments of subsequent sections hold as they are. However, our requirement seems to be more physical than others.

§ 4. Charge renormalization

If we use the expression of δm obtained in the last section, $\lambda_{V|V}(E)$ becomes definite. However, it still contains a divergent integral. To remove this divergence, we must renormalize the coupling constant.

In the stable case, charge renormalization is performed by defining Z as probability amplitude of the bare V state in the physical V state. But, in the unstable case, physical V state changes every moment. So, we must seek another method to determine Z.

Now, we consider an eigenstate of total Hamiltonian, in which the configuration of θ field corresponds to the standing wave condition. We write this state as $|E_r>$. Then we define Z as probability amplitude of the bare V state in this state,**) that is

$$(H_0 + H)|E_r> = E_r|E_r>,$$

$$|E_r> = Z^{1/2}\left\{|V> + g\sum \frac{1}{\sqrt{2\omega_k\Omega}} \frac{\mathcal{P}}{E_r - E_k} \alpha_k^\dagger |N>\right\}. \tag{7}$$

Accordingly, E_r is determined by

$$E_r = m_V + \delta m + g^2 \sum \frac{1}{2\omega_k \Omega} \frac{\mathcal{P}}{E_r - E_k}, \tag{8}$$

---

* This alternative can be considered as such an approximation in which ω_p-dependence in $U_{pV}(E_p)$ and ρ_p are all neglected. The difference between two methods is the order of half-width, numerically.

** Z^{-1} given under eq. (8) has a definite value and a definite meaning, if we stay in finite volume. But in the limit of $\Omega \to \infty$, we must use a technic shown there, and the physical meaning becomes ambiguous. We have not yet any definite method to treat virtual level when $\Omega \to \infty$. So, we must be content to use such a temporal technic.

and Z is defined by

$$Z^{-1}=1+g^2\sum\frac{1}{2\omega_k\Omega}\frac{\mathcal{P}}{(E_r-E_k)^2},$$

where $\mathcal{P}\sum(2\omega_k\Omega)^{-1}(E_r-E_k)^{-2}$ has the same meaning as before.

Now, we introduce renormalized coupling constant g_c by

$$g_c^2=Zg^2. \tag{9}$$

If we express Z in terms of g_c, it becomes

$$Z=1-g_c^2\sum\frac{1}{2\omega_k\Omega}\frac{\mathcal{P}}{(E_r-E_k)^2}, \tag{10}$$

and here we face to a well-known difficulty[5]. Therefore, to perform the above process consistently, we must introduce a cut-off so as to satisfy $0\leq Z\leq 1$.

Using these quantities, we obtain

$$\begin{aligned}\lambda_{V|V}(E)^{-1}&=E-m_V+i/2\cdot\Gamma_{V|V}(E)\\&=Z^{-1}\Big\{E-m_V-(E-m_V)g_c^2\mathcal{P}\sum\frac{1}{2\omega_k\Omega}\frac{Em_V-E_r^2+E_k(2E_r-E-m_1)}{(E_r-E_k)^2(m_V-E_k)(E-E_k)}\\&\qquad+i\pi g_c^2\sum\frac{1}{2\omega_k\Omega}\delta(E-E_k)+\Delta\Big\},\end{aligned}$$

where

$$\Delta\equiv\alpha\frac{p_0^2}{\omega_0}\left(1-\sqrt{1+\frac{g_c^4}{16\pi^2}\frac{1}{\alpha^2}\frac{\omega_0^2}{p_0^2}}\right),$$

$$\alpha\equiv\frac{Z}{Z_L}=1+g_c^2\sum\frac{1}{2\omega_k\Omega}\mathcal{P}\frac{E_r^2-m_V^2+2E_k(m_V-E_r)}{(E_r-E_k)^2(m_V-E_k)^2}.$$

Further, using eq. (8) and eliminating Δ, we obtain

$$\lambda_{V|V}(E)^{-1}=Z^{-1}\,\lambda_{V|V}^{(c)}(E)^{-1}, \tag{11}$$

$$\begin{aligned}\lambda_{V|V}^{(c)}(E)^{-1}=(E-E_r)\Big\{1-(E-E_r)g_c^2\mathcal{P}\sum\frac{1}{2\omega_k\Omega}\frac{1}{(E_r-E_k)^2(E-E_k)}\\+i\pi g_c^2\sum\frac{1}{2\omega_k\Omega}(E-E_k)\Big\}.\end{aligned} \tag{12}$$

Here $\lambda_1^{(c)}{}_1$ is the renormalized propagator and contains no divergent integral even if momentum cut-off is removed.

After the renormalization process is performed, b's become as follows:

$$b_{V|V}(t)=Z\frac{i}{2\pi}\int_{-\infty}^{\infty}dE\,e^{i(m_V-K)t}\lambda_{1|1}^{(c)}(E), \tag{13}$$

$$b_{k|V}(t)=Z^{1/2}\frac{g_c}{\sqrt{2\omega_k\Omega}}\frac{i}{2\pi}\int_{-\infty}^{\infty}dE\,e^{i(E_k-E)t}\frac{1}{E-E_k+i\epsilon}\lambda^{(c)}_{V|V}(E), \tag{14}$$

$$b_{V|k}(t)=e^{-i(E_k-m_V)t}b_{k|V}(t), \tag{15}$$

$$b_{\substack{k'|k\\k'\neq k}}(t)=\frac{g_c^2}{2\sqrt{\omega_k\omega_{k'}}\,\Omega}\frac{i}{2\pi}\int_{-\infty}^{\infty}dE\,e^{i(E_{k'}-E)t}\frac{1}{(E-E_k+i\epsilon)(E-E_{k'}+i\epsilon)}\lambda^{(c)}_{V|V}(E), \tag{16}$$

$$b_{k|k}(t)=1+\lim_{k'\to k}b_{k'|k}(t). \tag{17}$$

Similarly to the case of δm, the definition of Z is not unique, but we have chosen the above definition because of mathematical and physical simplicity, that is, in $\lambda^{(c)}_{V|V}$, J is eliminated and E_r appears instead of m_V. E_r will turn out to be a resonance energy of θ-N scattering (this fact will be shown later), and this is consistent with the configuration of θ-field in $|E_r>$. Further, Z reduces to Z_L when the V state is stable.

§ 5. Properties of $b_{f|i}$

According to the results in preceding sections, $b_{k'|k}$ and $b_{k|k}$ contain no divergent quantity, but $b_{V|V}$, $b_{k|V}$ and $b_{V|k}$ have a Z factor, respectively. At first sight this fact seems to show that our recipe is not sufficient, but we can see that these factors are necessary for $b_{f|i}$ to satisfy the initial condition and the conservation of probability. This section will be devoted to show these matters.

For the sake of simplicity we calculate in the case of $\mu=0$, and use a straight cut-off*, but the general features are not affected by these simplifications at all. In this case, we have

$$\begin{aligned}
&\sum\frac{1}{2\omega_k\Omega}\frac{\mathcal{P}}{E-E_k}\to-\frac{1}{4\pi^2}\left\{(E-m_V)\log\left|\frac{E-m_N-K}{E-m_N}\right|+K\right\},\\
&\sum\frac{1}{2\omega_k\Omega}\frac{\mathcal{P}}{(E-E_k)^2}\to-\frac{1}{4\pi^2}\left\{1+\frac{E-m_N}{m_N+K-E}+\log\left|\frac{E-m_N}{E-m_N-K}\right|\right\}.
\end{aligned} \tag{18}$$

Using these relations, $\lambda^{(c)}_{V|V}(E)^{-1}$ becomes

$$\begin{aligned}
\lambda^{(c)}_{V|V}(E)^{-1}=&(E-E_r)\left\{1+\frac{g_c^2}{4\pi^2}\left(1+\frac{E_r-m_N}{m_N+K-E_r}+\log\left|\frac{E_r-m_N}{E_r-m_N-K}\right|\right)\right\}\\
&+(E-m_N)\frac{g_c^2}{4\pi^2}\left\{\log\left|\frac{E-m_N-K}{E-m_N}\right|+i\pi\left(\theta(E-m_N)-\theta(E-m_N-K)\right)\right\}\\
&-(E_r-m_N)\frac{g_c^2}{4\pi^2}\log\left|\frac{E_r-m_N-K}{E_r-m_N}\right|.
\end{aligned} \tag{19}$$

* Strictly speaking, the straight cut-off causes a difficulty due to the discontinuity at the cut-off momentum, that is, $\lambda_{V|V}{}^{(c)}(E)$ becomes to have a pole in the range $E>m_N+K$. If we use a sufficiently smooth cut-off function, this pole disappears. Therefore, we may use the straight cut-off and neglect this abnormal pole.

Now, consider a function of complex variable

$$F(E) \equiv (E-E_r)\left\{1+\frac{g_c^2}{4\pi^2}\left(1+\frac{E_r-m_N}{m_N+K-E_r}+\log\frac{E_r-m_N}{m_N+K-E_r}\right)\right\}$$

$$-(E-m_N)\frac{g_c^2}{4\pi^2}\log\frac{m_N-E}{m_N+K-E}+(E_r-m_N)\frac{g_c^2}{4\pi^2}\log\frac{E_r-m_N}{m_N+K-E_r}, \tag{20}$$

with branch-points at $E=m_N$ and $E=m_N+K$, and a cut* indicated in Fig. 1. It is easily seen that $F(E)$ is equal to $\lambda^{(c)}_{V|V}(E)^{-1}$ on the real axis, so that the former is the analytic continuation of the latter. When the multiplicative factor of $\lambda^{(c)}_{V|V}(E)$ in the integrand of $b_{f|i}$ has no pole on the real axis (for example, eq. (13)), the above cut can be taken along the real axis ($E>m_N$). Then, as explained before, $F(E)^{-1}$ has no pole. Taking a contour indicated by the dotted line in Fig. 1, we obtain the following results:**

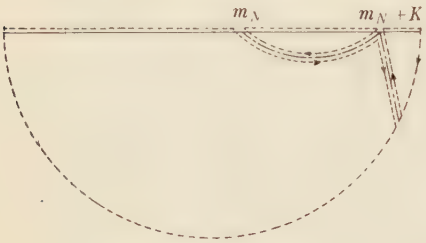

Fig. 1.

$$e^{-im_Vt}b_{V|V}(t)=Z\frac{i}{2\pi}\int_c dE\,e^{-iEt}D(E), \tag{21}$$

$$e^{-iE_kt}b_{k|V}(t)=e^{-im_Vt}b_{V|k}(t)$$

$$=Z^{1/2}\frac{g_c}{\sqrt{2\omega_k\Omega}}\left\{\frac{e^{-iE_kt}}{F(E_k)}+\frac{i}{2\pi}\int_c dE\,e^{-iEt}\,\frac{D(E)}{E-E_k}\right\}, \tag{22}$$

$$b_{k'|k}(t)=\frac{g_c^2}{2\sqrt{\omega_k\omega_{k'}}\,\Omega}\left\{\frac{1}{E_{k'}-E_k}\,\frac{1}{F(E_{k'})}\,\frac{e^{i(E_{k'}-E_k)t}}{(E_k-E_{k'})F(E_k)}\right.$$

$$\left.+\frac{i}{2\pi}\int_c dE\,e^{i(E_{k'}-E)t}\,\frac{1}{(E-E_{k'})(E-E_k)}D(E)\right\}, \tag{23}$$

$$b_{k|k}(t)=1+\lim_{k'\to k}b_{k'|k}(t). \tag{24}$$

Here, $D(E)$ is the jump of $F(E)^{-1}$ at the cut, and $\int_c dE$ means integration along the cut.

We now examine the initial condition. As $b_{k|V}(t)$, $b_{V|k}(t)$ and $b_{k'|k}(t)$ have a

* To introduce such a cut is not proper to our special model or simplification. The combination of $\log\left|\frac{E-m_N-K}{E-m_N}\right|$ and $i\pi\{\theta(E-m_N)-\theta(E-m_N-K)\}$ comes from $\delta_+(E-E_k)=i/2\pi\cdot(E-E_k+i\epsilon)^{-1}$. This quantity can be expressed by $\frac{i}{2\pi}\frac{d}{dE}\log(E-E_k)$, with understanding that we have a cut in the lower half plane from the point $E=E_k$. It is a general character, and essential points of our discussion are due to this fact.

** In eqs. (22), (23), and (24), cut must be shifted from the real axis so that $F(E)^{-1}$ may have some poles in the lower half plane. In these equations, contributions of those poles are not included, but even if they are taken into account, final conclusions are not changed.

factor of the order E^{-1} besides $F(E)^{-1}$, respectively, $b_{V|K}(0)$, $b_{K|V}(0)$ and $b_{K'|K}(0)$ can be evaluated by elementary contour integration. Owing to the facts that the usual infinite semicircle in the upper half plane makes no contribution, and that $F(E)$ has no pole in the upper half plane, we obtain immediately

$$b_{K|V}(0)=b_{V|K}(0)=b_{K'|K}(0)=0.$$

For $b_{V|V}(t)$ and $b_{K|K}(t)$, using eqs. (21) and (29), we obtain

$$b_{V|V}(0)=Z\frac{i}{2\pi}\int_c dE\,D(E),$$

$$b_{K|K}(0)=1\,.$$

We shall show in Appendix that the relation

$$\frac{i}{2\pi}\int_c dE\,D(E)=Z^{-1} \tag{25}$$

holds. Therefore the initial condition is completely satisfied. From the above calculation, it turns out that the factor Z in eq. (21) is necessary for the initial condition to be satisfied.

We show that the factor $Z^{1/2}$ in eq. (22) is necessary to the conservation of probability. From eqs. (21) and (22) we get

$$|b_{V|V}(\infty)|^2+\sum|b_{K|V}(\infty)|^2=Zg_c^2\sum\frac{1}{2\omega_k\Omega}\frac{1}{|F(E_k)|^2},$$

and the right-hand side of this equation is just $Z\dfrac{i}{2\pi}\displaystyle\int_c dE\,D(E)$. Therefore, owing to eq. (25), we obtain

$$|b_{V|V}(\infty)|^2+\sum|b_{K|V}(\infty)|^2=1.$$

Next, to explain the physical meaning of E_r, we calculated the cross-section of θ-N scattering. The result is

$$d\sigma/d\Omega=(4\pi/k^2)\,|\sin\delta|^2,$$

$$\tan\delta=(g_c^2/4\pi)\,k/Re\,F(k).$$

From these equations and eq. (8) it turns out that E_r corresponds to the resonance energy of θ-N scattering as mentioned before.

Finally, we make some remarks concerning the stable case. In this case, the function which corresponds to $F(E)$ in the untable case is

$$\overline{F}(E)=(E-m_V)\left[1+\frac{g_L^2}{4\pi^2}\left(1+\frac{m_V-m_N}{m_V-m_N-K}+\log\left|\frac{m_N-m_V}{m_N+K-m_V}\right|\right)\right]$$

$$-(E-m_N)\frac{g_L^2}{4\pi^2}\log\frac{m_N-E}{m_N+K-E}$$

$$+(m_V-m_N)\frac{g_L^{\circ}}{4\pi^2}\log\left|\frac{m_N-m_V}{m_N+K-m_V}\right|,$$

and it has a zero point at $E=m_V$. So we get

$$e^{-im_Vt}b_{V|V}(t)=Z_L\Big(e^{-im_Vt}+\frac{i}{2\pi}\int_c dE\,e^{-iEt}\bar{D}(E)\Big),$$

$$e^{-iE_kt}b_{k|V}(t)=Z_L^{1/2}\frac{g_c}{\sqrt{2\omega_k\Omega}}\frac{e^{-im_Vt}}{E-E_k}$$

$$+(\textit{right-hand side of eq. (24), where } Z \textit{ is replaced by } Z_L).$$

Therefore the part of $R(t)|V>$ which contains a factor e^{-im_Vt} is just the physical V state. Here, $\bar{D}(E)$ is the jump of $\bar{F}(E)^{-1}$, and satisfies, in stead of eq. (25),

$$Z_L^{-1}=1+\frac{i}{2\pi}\int_c dE\,\bar{D}(E).$$

Comparing these two (stable and unstable) cases, we see that Z factors appear in $b_{k|V}$ and $b_{V|V}$ in the same manner. These factors correspond to the renormalization of initial and final lines in the Feynman graph.

The unstability of the V state is expressed by the fact

$$\lim_{t\to\infty} e^{-im_Vt}\,b_{V|V}(t)=0$$

in contrast to the stable case, where

$$b_{V|V}(\infty)=Z_L\neq 0.$$

§ 6. Concluding remarks

It is now clear that we can obtain finite and physically meaningful results for such problems as involving unstable particles. Our recipe to remove divergent quantities was to renormalize mass by considering spectrum of decay products, and then to renormalize charge by refering to the probability of the bare V state in the virtual level state concerning resonance scattering. This method reduces to the conventional recipe for the stable case, but it is not unique as mentioned before. Charge renormalization is not unique in the conventional renormalization method, but there mass renormalization is unique. In our case, even mass renormarization is not unique according to the treatment of imaginary part of the Green's function. This situation is concerned with the uncertainty property of energy for decaying particles.

A remarkable feature in our results is the fact that the time-dependence, which characterizes the unstability, is not expressed in the form $\exp(-\gamma/2\cdot t)$, and has a more complicated character. This situation is not changed in the alternative treatment mentioned in section 3. This is not surprising, because the time dependence $\exp(-\gamma/2\cdot t)$ was got only in an asymptotic expression in the case of the excited state of atom, where renormalization was not performed.[6]

Here it may be a question, at first sight, that we gave a bare state with mass m_1 at the sharp time $t=0$. But, we can see that this assignment corresponds to take as the initial state virtual section of the Feynman's diagram where the clothed particle becomes temporarily bare, as will be understood by analysing the method of Arnous-Zienau. It is nonsense to take as the initial state such an eigen state as $|E_r>$, because it is stationary. One may take as the initial state some wave packet instead of the pure state. But the width thus introduced is essentially different from the one treated in this paper, and we can not expect for decay character such as $\exp(-\gamma/2\cdot t)$ to be recovered. If it might be so, it would be equivalent to give the width from the beginning. This is a kind of renormalization of half-width.

A further point to be mentioned is the situation that we used $b_{p|V}(\infty)$ to determine δm instead of $b_{V|V}$. It makes the procedure very complicated, and we could carry out the program owing to the simplicity of the Lee's model. We have no good idea at present how we could perform the approximate calculation in practical field theory. But we might find a more elegant method by noticing that $\lambda_{V|V}$ plays an essential role in $b_{p|V}(t)$ and $\lambda_{V|V}$ is the Green's function.

Finally, it is noticable that we have one resonance level for each unstable particle and the level is shifted from the mass of that unstable particle. This shift is of the order of half-width. If such a situation is established experimentally, our method will become more significant not only in the academic point of view but also in the practial use.

The most interesting point is what we can learn from the fact that a renormalization method in such a mode does exist. This problem will remain for future discussion.

Appendix

Integrate $F(E)^{-1}$ along the contour indicated in Fig. 2. As $F(E)^{-1}$ has no pole inside the contour, we obtain

$$0=\frac{1}{2\pi i}\left\{\int_{c_0}\frac{1}{F(E)}dE+\int_{c_1}\frac{1}{F(E)}dE+\int_{c_2}\frac{1}{F(E)}dE\right\}$$

$$=\frac{1}{2\pi i}\left\{\int_{c_0}\frac{1}{F(E)}dE+\int_{m_N}^{R}dE\,D(E)\right\}.$$

So we get

$$\frac{1}{2\pi i}\int_{m_N}^{R}dE\,D(E)=-\frac{i}{2\pi}\int_{c_0}\frac{1}{F(E)}dE.$$

As $F(E)$ behaves like ZE as $E\to\infty$, when we make R tend to infinity, we obtain

$$\frac{1}{2\pi i}\int_{c}dE\,D(E)=Z^{-1}.$$

This is eq. (25).

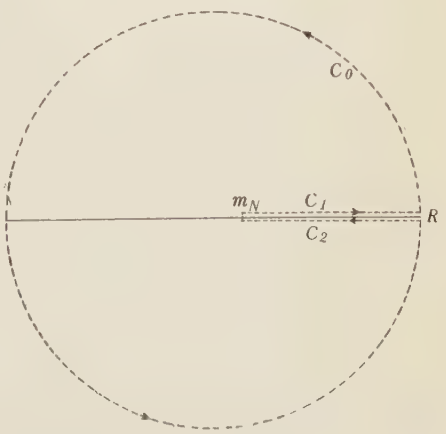

Fig. 2.

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Relation between the Nuclear Collective Model and the Individual Particle Model with Configurational Mixing

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By using the method of the quantum mechanical description of collective motion, the relation between the collective model and the individual particle model with configurational mixing is discussed on the basis of the following postulates: (I) The wave function of the nucleus, as a system composed of actual nucleons, implies strong correlations, so that the Hartree approximation makes little sense. Consequently, the method of configurational mixing currently employed as well as the shell model may not be usable without altering the representation: (II) The strong correlation gives rise to collective motion, chiefly the surface oscillation at low energies. The individual motion of nucleons which remains after subtraction of the collective motion is only weakly correlated, so that the shell model is a good approximation to the individual motion.

Leaving aside the essential problem of proving this statement, we reach the following conclusions concerning the relationship between the collective and the shell models of nuclei. i) The individual particle model with configurational mixing does in fact hold in the case of small nuclear deformation, provided that the individual particle motion in the "collective represention" can be treated by means of the adiabatic approximation. ii) The effective inter-particle force responsible for the configurational mixing consists of the following two parts, one arising from the exchange of surfons between an extra-particle and a particle in the core and the other arising from a direct inter-particle interaction. iii) As far as the quadrupole moment is concerned, the collective model and the method of configurational mixing do not show any essential difference. However, there exists an essential difference in these two methods for the interpretation of the magnetic moments.

§ 1. Introduction

There have been proposed a number of nuclear models that can, at least in part, explain various nuclear properties. Among them the success of the shell model is well known.[1),2)] The shell model is, however, to be modified in the following two ways in order to interpret the detailed behaviour of nuclear moments.[3)] One is the method of configurational mixing,[1),5)] in which single particle states in the shell structure of a core are mixed through the perturbation caused by extra-particles.*) The other is the collective model, in which the particles forming the core undergo a collective motion which may couple with the motion of extra-particles. The purpose of this paper is to make clear the relation between these two points of view.

*) Of course, the mixing of states of extra-particles may be caused. In this paper, however, we are rather interested in the relation between the method of configurational mixing and that of the collective model so we shall focus the discussion on the behaviour of the core in these two methods

Table I. Calculated and observed values of quadrupole moments in 10^{-24} cm^2

Nucleus	Proton (neutron) configuration	Single particle model	Hydro-dynamical model	Configu-rational mixing	Collec-tive model	Observed value
$^{27}_{13}Al_{14}$	$(1d_{5/2})^{-1}$	0.062	0.24	0.16	0.37	0.16
$^{37}_{17}Cl_{20}$	$1d_{3/2}$	−0.051	−0.25	−0.08	−0.087	−0.062
$^{59}_{27}Co_{32}$	$(1f_{7/2})^{-1}$	0.12	0.90	0.19	0.30	0.5
$^{69}_{31}Ga_{38}$	$(2p_{3/2})^{-1}$	0.08	0.52	0.15	0.20	0.23
$^{71}_{31}Ga_{40}$	$(2p_{3/2})^{-1}$	0.08	0.52	0.15	0.14	0.14
$^{141}_{59}Pr_{82}$	$2d_{5/2}$	−0.18	−1.7	−0.30	−0.35	−0.05
$^{209}_{83}Bi_{126}$	$1h_{9/2}$	−0.30	−5.2	−0.53	−0.55	−0.4
$^{17}_{8}O_{9}$	$1d_{5/2}$	−0.0013*)	−0.14	−0.04	−0.031	−0.004
$^{33}_{16}S_{17}$	$1d_{3/2}$	0	−0.20	−0.09	−0.054	−0.064
$^{35}_{16}S_{19}$	$(1d_{3/2})^{-1}$	0	0.20	0.09	0.036	0.045
$^{73}_{32}Ge_{41}$	$1g_{9/2}$	0	−1.1	−0.43	−0.11	−0.2

*) The motion of the center of mass is taken into account.

The method of configurational mixing worked out by Arima and Horie[5] and the collective model calculated by Marumori et al.[6] are found to have a common origin in accounting for nuclear quadrupole moments in the sense that the nuclear core is perturbed through interactions with extra-nucleons; the interaction in the former method is ascribed to a nuclear force between a nucleon in the core and an extra-nucleon, while that in the latter is due to the deformation of the core induced by an extra-nucleon. The fluctuation in the values of quadrupole moments from nucleus to nucleus essentially depends upon how strongly the excited states of the core particles are admixed to their ground state. As this situation holds for both methods, one can expect similar values of quadrupole moments to result. Indeed, as shown in Table I, these values are quite similar as far as comparison can be made, and reproduce observed quadrupole moments fairly well in most cases, except those in which the deformation and hence the quadrupole moment is large.

Now we will ask why these two methods predict nearly the same result, although their starting points seem to be different. First of all, we notice that in the method of configurational mixing an inter-nucleon force is introduced which is assumed to be nothing but the one giving rise to the pairing energy.[5] It does not seem to be justified that such a force is equal to the nuclear force acting between two free nucleons, but should be rather regarded as an "effective" inter-nucleon force acting in a nucleus.

The origin of the effective inter-nucleon force seems to have an important bearing on the validity of the shell model. The nuclear force observed in the two nucleon system is so strong that a single particle orbit can hardly persist for a sufficiently long time, if the same force exists between nucleons in a nucleus. In this connection, one might emphasize the importance of the Pauli principle which could effectively diminish the inter-nucleon force.[7] This is certainly correct qualitatively, but quantitatively the free nucleon force is a little too strong to account correctly for the absorption of a nucleon in nuclear

matter.[8),9),10)] However, there is some indication that the effective force is actually weaker than the free force. Vogt[11)] has shown that the imaginary part of the optical potential may be reduced when the nucleus undergoes deformation in correlation with the motion of an impinging nucleon, otherwise the imaginary part is found to be too large[12)] to account for the shape elastic scattering of neutrons.[13)]

Such an analysis together with the partial validity of the shell and collective models leads us to the following view of nuclear structure. Although the nuclear force is so strong as to give rise to a strong correlation between nucleons, the strongly correlated part contributes mainly to the collective motion and the remainig part is indeed weak enough for the shell structure to hold. We have not been able to prove this statement, but in the present paper regard this as a postulate. On the basis of this postulate our supposition may be described as follows.

(I) The wave function of the nucleus as a system composed of actual nucleons implies the strong correlation, so that the Hartree approximation makes little sense. Consequently, the method of configurational mixing as well as the shell model may not be usable without altering the representation.

(II) The strong correlation gives rise to collective motion, chiefly the surface oscillation at low energies, the individual motion of the nucleons which remains after subtraction of the collective motion being only weakly correlated. Thus the shell model is a good approximation to the individual motion.

The picture of (II) is described in a representation which results by means of a unitary transformation from the representatation used in (I).[6)] Hence the " nucleon " described in (II) is different from the actual nucleon described in (I) and is subject to a kind of constraint conditions. However, owing to the transformation function being symmetric with respect to the nucleon coordinates, an essential feature of the nucleon, namely the Fermi statistics, is maintained. The symmetric character results in the irrotational flow of the collective motion. Thus the " nucleons " can be regarded as very similar to actual nucleons and, as a first approximation, may form a shell structure.

As discussed in our previous paper,[6)] cited as M-S-Y, the separation of the collective motion by means of the adiabatic approximation is carried out for the system of " nucleons " forming a core. The " nucleons " which form the core are then described by a product wave function in the Hartree approximation. An extra-" nucleon " perturbs the shell structure of the core through collective deformation. This can be described in such a way that a collective quantum, a surfon, is exchanged virtually between the extra-" nucleon " and one of the individual " nucleons " in the core. This results in a force between them that may be responsible for the configurational mixing of the shell states In addition to this force there may exist a weak direct force between " nucleons ".

Thus it becomes evident why nearly identical values of quadrupole moments are obtained by the configurational mixing method and the collective model. The inter-nucleon force assumed in the former is nothing but the force caused by the exchange of a surfon and the additional direct force. Moreover, as will be shown in a separate paper, the pairing energy, which is assumed due to the inter-nucleon force, can be accounted for in

terms of the exchange of a surfon between two extra-nucleons and also of the direct force between them.

As far as the quadrupole moment is concerned, it is not easy to distinguish between the surfon exchange force and the direct force, because the quadrupole moment arises from the second order deformation caused directly by the collective motion. The situation is different in the case of the magnetic dipole moment. Here, the direct force seems more effective than the surfon exchange force because only the former contributes directly to the magnetic moment, while the contribution of the latter appears only in higher orders. In analogous ways, one can estimate the respective contributions of these two kinds of forces when various nuclear properties are to be considered. In the present paper however, we shall be concerned only with the nuclear moments mentioned below and shall leave other cases for later consideration.

§ 2. Method of configurational mixing and its underlying assumptions

In this section we shall first give a review of the method of configurational mixing[1),5)] and try to clarify our supposition (I).

According to the method of configurational mixing, the wave function of a nucleus, Φ_{mixed}, is expressed by a superposition of those shell states which have a common total angular momentum and the same z-component of angular momentum. Thus,

$$\Phi_{mixed}=\Phi_0+\sum_n \beta_n \Phi_n \qquad (2\cdot1)$$

where Φ_0 corresponds to the ground state configuration and Φ_n to the excited states. The amplitude for a mixed state, β_n, is given by means of the perturbation theory as

$$\beta_n=(\Phi_n, \sum_{i<k} V_{ik}\Phi_0)/(E_0-E_n), \qquad (2\cdot2)$$

where E_0 and E_n indicate the energy levels of the ground and excited shell states respectively. V_{ik} is the interaction potential between the i^{th} and k^{th} nucleons.

The expectation value of any physical operator, O, is given by

$$(\Phi_{mixed}, O \Phi_{mixed})=(\Phi_0, O \Phi_0) + \{\sum_n \beta_n^*(\Phi_n, O \Phi_0)+\text{comp. conj.}\}, \qquad (2\cdot3)$$

in which the second term on the right hand side is added, as an effect of configurational mixing, to the first one which is the expectation value in the perfect shell model. It has been suggested on an intuitive basis that the correction term corresponds to the effect of the core deformation in the collective model.[11)] Leaving this problem to later sections, we shall recall the assumptions employed in the method of configurational mixing.

We start from the Schrödinger equation for a nucleus composed of A nucleons, whose positions are $\boldsymbol{x}_1\cdots$, $\boldsymbol{x}_A$, respectively,

$$(E-\mathcal{H}_A)\Phi(\boldsymbol{x}_1,\cdots\boldsymbol{x}_A)=0, \qquad (2\cdot4)$$

with

$$\mathcal{H}_A=\sum_{i=1}^{A} T_i+V_A(\boldsymbol{x}_1\cdots\boldsymbol{x}_A). \tag{2·5}$$

In this representation the kinetic energy of the system is expressed as the sum of that of an individual nucleon, T_i, and the potential energy, V_A, is a function of the positions of A nucleons. By introducing a Hartree potential $U(\boldsymbol{x}_i)$, (2·4) may be rewritten as

$$(\mathcal{H}_0+\mathcal{H}_{int})\,\Phi=E\Phi, \tag{2·4′}$$

where

$$\mathcal{H}_0=\sum_{i=1}^{A}(T_i+U(\boldsymbol{x}_i)),\quad \mathcal{H}_{int}=V_A(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)-\sum_{i=1}^{A}U(\boldsymbol{x}_i). \tag{2·6}$$

$U(\boldsymbol{x}_i)$ is chosen in such a way that $\mathcal{H}_{int}$ is ineffective in the ground state:

$$(\Phi_0,\ \mathcal{H}_{int}\ \Phi_0)=0. \tag{2·7}$$

The validity of the configurational mixing lies in the fact that *not only the diagonal elements but also the non-diagonal elements of* $\mathcal{H}_{int}$ *are small enough to justify the expansion*, (2·1). Actually, in view of the strong nuclear forces, the non-diagonal elements are expected to be considerable.

This difficulty may be overcome, if we follow the argument given by Eden and Francis.[15] According to them, the shell model is valid in a representation, which is different from Φ that obeys (2·4). The new representation is attained by a model operator M which leaves quantum numbers unchanged:

$$\Psi=M\Phi. \tag{2·8}$$

In this representation the nucleon no longer maintains its original nature, but is transformed into something which may be called the "particle". Interactions between the "particles" are now weak enough to ensure the validity of the shell structure of the "particle" system.

However, the physical content of this way of thinking is not specified unless a particular form of the model operator be chosen. We should prefer that model operator which results in a representation in which the expectation values of as many as possible physical operators defined in the original representation are as close as possible to the experimental values. As will be shown in what follows, if the choice of the model operator be inadequate, the expectation values of some of the operators defined in the original representation may differ from the experimental values.

Suppose a nuclear state of configurational mixing, like the one in (2·1), be given by a model operator defined as in (2·8). Then the expectation value of an operator in this representation is

$$(\Psi,\ M\,O\,M^{-1}\,\Psi)=(\Psi,\,O\,\Psi)+(\Psi,[M,O]M^{-1}\,\Psi). \tag{2·9}$$

On the other hand, the expectation value in the configurational mixing method is usually calculated by $(\Psi,\ O\ \Psi)$, similar to (2·3). These two expressions are not in general identical; if (2·3) can give agreement with experiments, (2·9) may fail. Keeping this

fact in mind, we shall in a later section introduce the model operator that gives the collective representation.

§ 3. Collective model

Our supposition, (II), will be discussed along the line developed in our previous work, M-S-Y.[6] Firstly we consider the case where A nucleons form a core. The Schrödinger equation, (2·4), is transformed by a model operator,

$$M=\exp[i(m/\hbar)\sum_{i=1}^{A}\phi(\boldsymbol{x}_i)]=\exp[i(m/\hbar)\int\rho(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}], \tag{3·1}$$

which is U given in (2·3) of M-S-Y. Here m is the nucleon mass and $\rho(\boldsymbol{x})$ the density operator; $\rho(\boldsymbol{x})=\sum_{i=1}^{A}\delta(\boldsymbol{x}-\boldsymbol{x}_i)$. $\phi(\boldsymbol{x})$ corresponds to the velocity potential of an irrotational and incompressible carrier fluid and is related to the collective momenta, π_{lm}, through

$$\phi(\boldsymbol{x})=\sum_{lm}\pi_{lm}^{*}F_{lm}(\boldsymbol{x}). \tag{3·2}$$

The model transformation, M, yields the "collective" state,

$$\Psi(\boldsymbol{x}_1,\cdots\boldsymbol{x}_A,\alpha)=M\Phi(\boldsymbol{x}_1\cdots\boldsymbol{x}_A). \tag{3·3}$$

The extra-degrees of freedom, α_{lm}*), thus introduced require the subsidiary conditions,

$$(\alpha_{lm}-\sum_{i=1}^{A}F_{lm}(\boldsymbol{x}_i))\Psi(\boldsymbol{x}_1,\cdots\boldsymbol{x}_A,\alpha)=0. \tag{3·4}$$

Since the transformation operator M does not involve the momenta of the nucleons, only the kinetic energy $\sum_i T_i$ in (2·5) is transformed, yielding the kinetic energy of the collective motion, $m/2\cdot\int\rho(\boldsymbol{x})(grad\,\phi(\boldsymbol{x}))^2d\boldsymbol{x}$, and the interaction between the individual particles' and collective velocities, $m\int\boldsymbol{v}(\boldsymbol{x})\cdot grad\,\phi(\boldsymbol{x})d\boldsymbol{x}$, as additional terms. On the other hand, the potential energy in (2·5), which is for the time being assumed to be a function of position alone, remains unchanged. In the collective representation however, owing to the subsidiary condition (3·4), there arises a dependence of V on α. As a rule, the α-dependence of V is given uniquely by the method developed by Tomonaga,[16] but we shall not touch on this point in this paper. Thus we are able to write down the transformed Hamiltonian as

$$H=H_0+H_{int}, \tag{3·5}$$

$$H_0=\sum_{i=1}^{A}(T_i+U(\boldsymbol{x}_i,\alpha))+\frac{m}{2}\int\rho(\boldsymbol{x})(grad\,\phi(\boldsymbol{x}))^2d\boldsymbol{x}$$

*) α_{lm} are the expansion parameters of the nuclear surface defined by

$$R(\theta,\varphi)=R_0[1+\sum_{lm}\alpha_{lm}Y_{lm}(\theta,\varphi)]$$

where R_0 is the equilibrium radius.

$$+m\int \boldsymbol{v}(\boldsymbol{x})\, grad\, \phi(\boldsymbol{x})\, d\boldsymbol{x}, \tag{3·5a}$$

$$H_{int}=V(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)-\sum_{i=1}^{A}U(\boldsymbol{x}_i,\ \alpha). \tag{3·5b}$$ *)

With the above expression for the Hamiltonian, the Schrödinger equation in the collective representation is given by

$$(H-E)\Psi=0. \tag{3·6}$$

In order to solve this equation we use the adiabatic approximation**) as in M-S-Y:

$$\Psi(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)=\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A;\ \alpha)\chi(\alpha). \tag{3·7}$$

ψ^{core} describes the individual motion of the core for a given collective deformation α, and is the solution of

$$[\sum_{i=1}^{A}(T_i+U(\boldsymbol{x}_i,\ \alpha))+H_{int}-E^{core}(\alpha)]\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A;\ \alpha)=0. \tag{3·8}$$

Moreover, because of the condition (3·4), ψ^{core} must satisfy the condition:

$$(\psi^{core},\ \sum_{i=1}^{A}F_{lm}(\boldsymbol{x}_i)\psi^{core})=\alpha_{lm}. \tag{3·9}$$

With the aid of (3·8) and (3·9), the Schrödinger equation, (3·6), with (3·4) is reduced to the equation of motion of collective modes:

$$[m/2\int(\psi^{core},\ \rho(\boldsymbol{x})\psi^{core})\,(grad\,\phi(\boldsymbol{x}))^2 d\boldsymbol{x}+E^{core}(\alpha)]\chi(\alpha)=E\chi(\alpha). \tag{3·10}$$

Since we are interested in the case in which the deformation of the core is small, we expand $\sum_{i=1}^{A}U(\boldsymbol{x}_i,\ \alpha)$ and $H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)$ in (3·8) around $\alpha_{lm}=0$:

$$\begin{aligned}\sum_{i=1}^{A}U(\boldsymbol{x}_i,\ \alpha)&=\sum_{i=1}^{A}(U^{(0)}(\boldsymbol{x}_i)+\sum_{lm}U_{lm}^{(1)}(\boldsymbol{x}_i)\alpha_{lm}+\sum_{lm}U_{lm}^{(2)}(\boldsymbol{x}_i)|\alpha_{lm}|^2)\\ H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)&=H_{int}^{(0)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)+\sum_{lm}H_{int;lm}^{(1)}\cdot\alpha_{lm}+\sum_{lm}H_{int;lm}^{(2)}|\alpha_{lm}|^2.\end{aligned} \tag{3·11}$$

Regarding $\sum_i\sum_{lm}U_{lm}^{(1)}\alpha_{lm}$, $\sum_i\sum_{lm}U^{(2)}|\alpha_{lm}|^2$ and $H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)$ as small perturbation terms, we can solve eq. (3·8). We obtain

$$\begin{aligned}\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A;\ \alpha)=\psi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)+\sum_n\frac{1}{E_0^{(0)}-E_n^{(0)}}(\psi_n,[H_{int}^{(0)}+\sum_i\sum_{lm}U_{lm}^{(1)}\alpha_{lm}\\ +\sum_{lm}H_{int;lm}^{(1)}\alpha_{lm}]\psi_0)\times\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A),\end{aligned} \tag{3·12}$$

$$E^{core}(\alpha)=E^{(0)}+\sum_{lm}E_{lm}^{(1)}\alpha_{lm}+\sum_{lm}E_{lm}^{(2)}|\alpha_{lm}|^2. \tag{3·13}$$

*) As is shown in M-S-Y, A. Bohr's model corresponds to neglecting H_{int}.

**) As will be discussed later, if the suppositions (I) and (II) in § 1 be correct, the tenability of this approximation may be significant for the shell model.

Here $\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)$ is the solution of the equation

$$\sum_{i=1}^{A}(T_i+U^{(0)}(\boldsymbol{x}_i))\,\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)=E_n^{(0)}\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A), \tag{3·14}$$

and

$$E^{(0)}=E_0^{(0)}+\sum_n|(\psi_n,\ H_{int}^{(0)}\ \psi_0)|^2/(E_0^{(0)}-E_n^{(0)}), \tag{3·15a}$$

$$\sum_{lm}E_{lm}^{(1)}\ \alpha_{lm}=\sum_{lm}(\psi_0,\ \sum_i U_{lm}^{(1)}(\boldsymbol{x}_i)\,\alpha_{lm}\ \psi_0)+2\sum_n\frac{1}{E_0^{(0)}-E_n^{(0)}}(\psi_n,\ H_{int}^{(0)}\psi_0)\times$$
$$\times(\psi_0,\ \sum_{lm}[\sum_i U_{lm}^{(1)}(\boldsymbol{x}_i)+H_{int;lm}^{(1)}]\ \alpha_{lm}\ \psi_n), \tag{3·15b}$$

$$\sum_{lm}E_{lm}^{(2)}|\alpha_{lm}|^2=\sum_{lm}(\psi_0,\ \sum_i U_{lm}^{(2)}(\boldsymbol{x}_i)\,|\alpha_{lm}|^2\psi_0)+\sum_n\frac{1}{E_0^{(0)}-E_n^{(0)}}\times$$
$$\times|(\psi_n,\ \sum_{lm}[\sum_i U_{lm}^{(1)}(\boldsymbol{x}_i)+H_{int;lm}^{(1)}]\cdot\alpha_{lm}\ \psi_0)|^2, \tag{3·15c}$$

provided that the Hartree potential $\sum_{i=1}^{A}U(\boldsymbol{x}_i,\alpha)$ is chosen so as to satisfy the condition

$$(\psi_0,\ H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)\ \psi_0)=0.$$

It should be noted that ψ_0 has a vanishing angular momentum because we consider the core to be a closed shell, and that the quantities α_{lm} lose their meaning if l becomes smaller than 2. Therefore, if we neglect the second term in (3·15b), which is a higher order correction to the first term, it is readily seen that the term linear in α_{lm} does not appear in $E^{core}(\alpha)$.

Thus we have

$$E^{core}(\alpha)=E^{(0)}+\sum_{lm}E_{lm}^{(2)}|\alpha_{lm}|^2. \tag{3·16}$$

By using (3·16) and (3·12), (3·10) becomes

$$\sum_{lm}\left[\frac{1}{2B_{lm}}|\pi_{lm}|^2+\frac{1}{2}\ C_{lm}|\alpha_{lm}|^2\right]\chi(\alpha)=(E-E^{(0)})\,\chi(\alpha), \tag{3·17}$$

where $1/2\cdot C_{lm}\equiv E_{lm}^{(2)}$ and

$$1/B_{lm}=m\int(\psi_0,\ \rho(\boldsymbol{x})\psi_0)\,|grad\,F_{lm}(\boldsymbol{x})|^2d\boldsymbol{x}.$$

B is constant in the adiabatic approximation, whereas the non-adiabatic contributions give rise to the dependendce of B on the core structure, as discussed by Inglis[18] and by Bohr and Mottelson.[19] In the present paper we shall not consider the non-adiabatic contributions.

The collective surface motion, which plays a role in the deformation of the closed shell nuclei, is described by the Schrödinger equation, (3·17). This surface motion gives rise to energy $\sum n\hbar\omega_l$, where $\omega_l=(C_{lm}/B_{lm})^{1/2}$ and n is the number of surfons excited. Thus the energy eigenvalue in the ground state of eq. (3·6) is equal to $E^{(0)}$, provided

that the zero-point energy of the collective motion is omitted.

It should be noted that $E^{(0)}$ is the energy eigenvalue of the equation

$$\left[\sum_i (T_i + U^{(0)}(\boldsymbol{x}_i)) + H_{int'}^{(0)}(\boldsymbol{x}_1 \cdots \boldsymbol{x}_A)\right] \Psi^s(\boldsymbol{x}_1 \cdots \boldsymbol{x}_A) = E^{(0)} \Psi^s(\boldsymbol{x}_1 \cdots \boldsymbol{x}_A), \qquad (3\cdot 18)$$

where

$$\Psi^s(\boldsymbol{x}_1 \cdots \boldsymbol{x}_A) = \psi_0(\boldsymbol{x}_1 \cdots \boldsymbol{x}_A) + \sum_n \frac{(\psi_n, H_{int}^{(0)} \psi_0)}{E_0^{(0)} - E_n^{(0)}} \psi_n(\boldsymbol{x}_1 \cdots \boldsymbol{x}_A).$$

In the above discussion we have shown that the shell model should be a good approximation, provided the collective motion be separated so as to diminish the inter-nucleon forces, although we have not been able to prove the ineffectiveness of $H_{int'}$. This way of approach should be distinguished from that in Section 2, where no possibility exists for making $\mathcal{H}_{int}$ ineffective. On the other hand, we have found a device for making $H_{int'}$ small, so that the shell model is indeed a good approximation. Thus we are allowed to start from the Schrödinger equation for the shell model, (3·18), and to treat the non-diagonal parts as perturbation. This will be the basis for the following discussion.

§ 4. Interactions between the core and an extra-particle*)

We shall now discuss the behaviour of the nucleus consisting of the core and one extra-particle, the latter being the cause of the core deformation. The coordinates of nucleons forming the core are designated by the suffices $1, \cdots N$ and that of the extra-particle is by the suffix 0. Since we restrict ourselves to the case of only one extra-particle, the approximations we adopt, such as the perturbation or weak coupling treatment, may be valid and unnecessary complication can be avoided.**)

In this case our Schrödinger equation in the collective representation is given by

$$\left[H_{core} + H_{int} + H_{0.c} + T_0 + U_0(\boldsymbol{x}_0, \alpha)\right] \Psi(\boldsymbol{x}_1 \cdots \boldsymbol{x}_N, \boldsymbol{x}_0, \alpha) = E\Psi(\boldsymbol{x}_1 \cdots \boldsymbol{x}_N, \boldsymbol{x}_0, \alpha). \qquad (4\cdot 1)$$

with the subsidiary conditions

$$\left[\alpha_{lm} - \sum_{i=1}^{N} F_{lm}(\boldsymbol{x}_i)\right] \Psi(\boldsymbol{x}_1 \cdots \boldsymbol{x}_N, \boldsymbol{x}_0, \alpha) = 0, \qquad (4\cdot 2)$$

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*) For the sake of simplicity, we do not take into account the Pauli principle explicitly.

**) The method of configurational mixing in the shell model is based on taking into account the effect of the core particles as a small perturbation. Therefore, it is clear intuitively that the weak coupling scheme in the collective model, in which the effect of the core deformation is treated as a perturbation, corresponds to the method of configurational mixing. This is one of the reasons why we consider the "core+one extra-particle" type of nucleus. In the case of the "core+several extra-particles" type of nucleus, the situation may be more complicated; we must take into account not only the collective motion of the core but also that of the extra-particles and the subsidiary condition (4·2) may have to be modified suitably. Furthermore, the weak coupling scheme may be unsatisfactory in many cases.

For nuclei for which the configurational mixing method is successful however, the situation may, as a rule, be the same as that discussed in this section.

where $H_{0,c}$ represents the direct interaction between the extra-particle and the core particles and is expressed by

$$H_{0,c}=V_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0,\ \alpha)-U_0(\boldsymbol{x}_0,\ \alpha). \tag{4·3}$$

In the original representation, $V_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)$ corresponds to the interaction potential between the extra-nucleon and the core nucleons, and the α-dependence of $V_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0,\ \alpha)$ in the collective representation can also, as a rule, be given by Tomonaga's procedure. H_{core} and H_{int} are the same as those given by (3·5a) and (3·5b), except for the change of the number of particles from A to N.

Since our interest in this section will be directed to the interaction between the core and the extra-particle, we shall hereafter neglect the effect of H_{int}, having discussed it in the preceding section.

As discussed in M-S-Y, the correspondence to A. Bohr's model is made clear, if we neglect $H_{0,c}$ and expand $U_0(\boldsymbol{x}_0,\ \alpha)$ around $\alpha_{lm}=0$;

$$U_0(\boldsymbol{x}_0,\ \alpha)=U_0^{(0)}(\boldsymbol{x}_0)+\sum_{lm}U_{0;lm}^{(1)}(\boldsymbol{x}_0)\alpha_{lm}+\sum_{lm}{}'U_{0;lm}^{(2)}(\boldsymbol{x}_0)|\alpha_{lm}|^2. \tag{4·4}$$

The parts depending on α are to be regarded as an interaction between the extra-particle and the collective motion of the core.

Now, according to our supposition, (II), we regard $H_{1,}$ as a perturbation term, and start from the unperturbed equation in which the interaction between an extra-particle and a core takes place only through the collective oscillation. Using the adiabatic approximation for the core part, the solution of the unperturbed equation,

$$(H_0-E_0)\Psi^0=0, \tag{4·5}$$

where

$$H_0=H_{core}+T_0+U_0(\boldsymbol{x}_0,\ \alpha), \tag{4·6}$$

is expressed by

$$\Psi^0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0\ \alpha)=\varphi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \alpha)\chi(\boldsymbol{x}_0,\ \alpha). \tag{4·7}$$

Using the same procedure as in deriving (3·10), we obtain the equation for $\chi(\boldsymbol{x}_0,\ \alpha)$, namely

$$\Big[m/2\cdot\int(\varphi^{core},\ \rho(\boldsymbol{x})\varphi^{core})\,(grad\,\phi(\boldsymbol{x}))^2 d\boldsymbol{x}+E^{core}(\alpha)+T_0+U_0(\boldsymbol{x}_0,\ \alpha)\Big]\times$$

$$\times\chi(\boldsymbol{x}_0,\ \alpha)=E_0\,\chi(\boldsymbol{x}_0,\ \alpha). \tag{4·8}$$

Assuming that the core deformation is small, (4·8) may be rewritten in the form

$$(H_{surf}+H_{part.}+\bar{H}_{int})\chi(\boldsymbol{x}_0,\ \alpha)=(E_0-E^{(0)})\chi(\boldsymbol{x}_0,\ \alpha), \tag{4·9}$$

where

$$H_{surf.}=\sum_{lm}\left(\frac{1}{2B_{lm}}|\pi_{lm}|^2+E^{(2)}_{lm}|\alpha_{lm}|^2\right),$$

$$H_{part.}=T_0+U^{(0)}_0(\boldsymbol{x}_0), \qquad (4\cdot10)^{*)}$$

and

$$\bar{H}_{int}=\sum_{lm}(E^{(1)}_{lm}+U^{(1)}_{0;lm}(\boldsymbol{x}_0))\,\alpha_{lm}+\sum U^{(2)}_{0;lm}(\boldsymbol{x}_0)\,|\alpha_{lm}|^2.$$

Here $E^{(0)}$, $E^{(1)}$, and $E^{(2)}$ are given by (3·15a), (3·15b) and (3·15c) respectively. It should be noted that $\sum E^{(1)}_{lm}\,\alpha_{lm}$ vanishes actually as discussed in the preceding section.

Now we shall solve eq. (4·9) regarding $\bar{H}_{int}$ as a perturbation term. Then we obtain

$$\chi(\boldsymbol{x}_0,\ \alpha)=\varphi(\boldsymbol{x}_0)\,\chi_0(\alpha)+\sum_{lm}\frac{(\varphi,\ U^{(1)}_{0;lm}\,\varphi)<\chi_{lm},\ \alpha_{lm}\,\chi_0>}{-\hbar\omega_l}\ \varphi(\boldsymbol{x}_0)\,\chi_{lm}(\alpha), \qquad (4\cdot11)$$

where $\omega_l=(2E^{(2)}_{lm}/B_{lm})^{1/2}$. Here, the extra-particle is assumed to stay in the ground state, $\varphi(\boldsymbol{x}_0)$, thus preserving the correspondence to the method of configurational mixing. The energy eigenvalue is

$$(E_0-E^{(0)})=E_p+\sum_{lm}\frac{|(\varphi,\ U^{(1)}_{0;lm}\varphi)\langle\chi_{lm},\ \alpha_{lm}\ \chi_0\rangle|^2}{-\hbar\omega_l}$$
$$+(\varphi,\ U^{(2)}_{0;lm}\,\varphi)\cdot<\chi_0,|\alpha_{lm}|^2\chi_0>, \qquad (4\cdot12)$$

where E_p is the energy eigenvalue of the equation

$$H_{part.}\ \varphi(\boldsymbol{x}_0)=E_p\,\varphi(\boldsymbol{x}_0).$$

We have omitted the zero point energy of the surface oscillation.

The second and third terms of (4·12) arising from the coupling with the extra-particle may also be amalgamated into $\sum_{lm} E^{(2)}_{lm}|\alpha_{lm}|^2$ in $H_{surf.}$ provided that the extra-particle is not excited, and should contribute to the surface rigidity even though this effect is neglected in M-S-Y as well as in A. Bohr's model. Indeed, if we define the quantities

$$\frac{1}{2}\sum_{lm}C_{lm}|\alpha_{lm}|^2=\sum_{lm}\left[E^{(2)}_{lm}+\frac{|(\varphi,\ U^{(1)}_{0;lm}\,\varphi)|^2}{-\hbar\omega_l}+(\varphi,\ U^{(1)}_{0;lm}\,\varphi)\right]\cdot|\alpha_{lm}|^2 \qquad (4\cdot13)$$

and replace $\bar{H}_{surf.}$ and H_{int} in (4·10) by

$$H_{surf.}=\sum_{lm}\left\{\frac{1}{2B_{lm}}|\pi_{lm}|^2+\frac{1}{2}C_{lm}|\alpha_{lm}|^2\right\}, \qquad (4\cdot14a)$$

$$\underline{\bar{H}_{int}}=\sum_{lm}(E^{(1)}_{lm}+U^{(1)}_{0;lm}(\boldsymbol{x}_0))\,\alpha_{lm}$$
$$+\sum_{lm}(U^{(2)}_{0;lm}(\boldsymbol{x}_0)+E^{(2)}_{lm}-1/2\cdot C_{lm})\times|\alpha_{lm}|^2. \qquad (4\cdot14b)$$

respectively, the second and third terms of (4·12) vanish. This corresponds to the renormalization of surfon frequencies,

*) This is nothing but the Hamiltonian of the collective model currently employed.

$$\omega_l \longrightarrow \omega_l, \tag{4·15}$$

where ω_l is given by $(C_{lm}/B_{lm})^{1/2}$, C_{lm} being defined in (4·13). Hereafter we shall use the renormalized surfon frequencies, ω_l, so that $H_{surf.}$ and H_{int} of (4·9) should be read as (4·14).

Up to this point, we have dropped the term, $\sum E^{(1)}_{lm}\alpha_{lm}$ in (4·10) or (4·14b) because this term vanishes as discussed in the preceding section. If we write formally the effect of this term to the first order of $E^{(1)}_{lm}$, (4·12) is expressed, using the renormalized surfon frequency, as

$$(E_0 - E^{(0)}) = E_p + 2\sum_{lm} \frac{(\varphi,\, U^{(1)}_{0;lm}(\boldsymbol{x}_0)\varphi)\cdot E^{(1)}_{lm}}{-\hbar\omega_l} < \chi_0,\ \alpha^2_{lm}\,\chi_0 > . \tag{4·16}$$

Using

$$E^{(1)}_{lm} = (\psi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N), \sum_{i=1}^{N} U^{(1)}_{lm}(\boldsymbol{x}_i)\psi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N)),$$

the second term of (4·16) may be rewritten as

$$(\Psi^s_0,\ 2\sum_{lm} \frac{\sum_{i=1}^{N} U^{(1)}_{lm}(\boldsymbol{x}_i)\cdot U^{(1)}_{0;lm}(\boldsymbol{x}_0)}{-\hbar\omega_l} < \chi_0,\ \alpha^2_{lm}\,\chi_0 > \Psi^s_0), \tag{4·17}$$

where Ψ^s_0 means

$$\Psi^s_0 = \varphi(\boldsymbol{x}_0)\psi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N). \tag{4·18}$$

As is readily seen from its form this arises from the exchange of surfons between an extra-particle and particles in the core. If the collective motion is eliminated, we obtain the interaction,

$$V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0) = 2\sum_{lm} \frac{(\sum_{i=1}^{N} U^{(1)}_{lm}(\boldsymbol{x}_i)\cdot U^{(1)}_{0;lm}(\boldsymbol{x}_0))}{-\hbar\omega_l} < \chi_0,\ \alpha^2_{lm}\,\chi_0 > . \tag{4·19}$$

In fact, (4·16) with (4·17) shows that in so far as the energy eigenvalue of the ground state is concerned, we can reduce the Schrödinger equation, (4·5), to that for the individual motions of the particles:

$$\{\sum_{i=1}^{N}(T_i + U^{(0)}(\boldsymbol{x}_i)) + T_0 + U^{(0)}_0(\boldsymbol{x}_0) + V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)\}\Psi^s(\boldsymbol{x}_0, \boldsymbol{x}_1\cdots\boldsymbol{x}_N)$$
$$= E_0\Psi^s(\boldsymbol{x}_0, \boldsymbol{x}_1\cdots\boldsymbol{x}_N). \tag{4·20}$$

Although $V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)$ has no effect upon the energy calculation, it may become important for the calculation of the expectation values of physical operators such as the nuclear moments.

Thus we are led to an approximate solution of unperturbed equation (4·5). For the real case we must take account of $H_{s.e.}$ which represents a very complicated interaction owing to the fact that all variables are contained. In the decomposition:

$$H_{0.c}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0,\ \alpha)=H^{(0)}_{0.c}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)+\sum_{lm}H^{(1)}_{0.c;lm}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)\alpha_{lm}$$

$$+\cdots\cdots\cdots,$$

we presume that $H^{(1)}_{0.c}$ is the most important one, because the nuclear deformation, α, is supposed to be small in our weak coupling case. Neglecting the effect of the α-dependent part of $H_{0.c}$, the Schrödinger equation (4·1) can be reduced effectively into

$$\{\sum_{i=1}^{N}(T_i+U^{(0)}(\boldsymbol{x}_i))+T_0+U_0{}^{(0)}(\boldsymbol{x}_0)+H^{(0)}_{0.c}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)$$

$$+V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)\}\,\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N)=E\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N),\qquad(4\cdot21)$$

provided that the adiabatic and weak coupling approximations are valid. This is nothing but the Schrödinger equation currently employed for the method of configurational mixing, and $\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N)$ is given by

$$\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N)=\Psi^s_0(\boldsymbol{x}_0,\boldsymbol{x}_1\cdots\boldsymbol{x}_N)+$$

$$+\sum_n\frac{(\Psi^s_n,\{H^{(0)}_{0.c}+V^{(ex)}\}\Psi^s_0)}{E_0-E_n}\Psi^s_n(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N),\quad(4\cdot22)$$

where

$$\Psi^s_n=\varphi(\boldsymbol{x}_0)\,\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_N)\ ^{*)}.$$

The two interactions, $V^{(ex)}$ and $H^{(0)}_{0.c}$, correspond to the "effective inter-particle interaction" which cause the configurational mixing. In M-S-Y, only the effect corresponding to $V^{(ex)}$ is taken into account, while in the current method of configurational mixing the sum of these two effects is regarded as though it were identical with the force between two nucleons.

Thus one can see the correspondence between the method of configurational mixing and the collective model.

§ 5. Expectation values of operators

As mentioned in Section 2 an operator, O, in the original representation undergoes a transformation

$$MOM^{-1}=O+[M,\ O]M^{-1}\qquad(5\cdot1)$$

in the collective representation. M is given by (3·1). Since M depends only on the positions of the nucleons, but not on their momenta, the operator involving only positions will commute with M and only the first term in the right hand side of (5·1) will remain. This is the case for the quadrupole moment, but not the case for the magnetic

*) It should be noted that the averaged potential $U_0(\boldsymbol{x}_0,\ \alpha)$ in (4·1) and (4·3) must be chosen so as to satisfy

$$(\Psi_0{}^s,\ H_{0.c}\ \Psi_0{}^s)=0,$$

moreover because of the condition (4·2), the shell model wave function $\Psi^s{}_n$ must satisfy a condition, which will be discussed in a later section.

moment.

For use in later applications we shall calculate the commutator. In most cases the operators of interest are expressed as a sum of those, each of which involves coordinates of each nucleon.

$$O=\sum_i O_i(\boldsymbol{x}_i, \boldsymbol{p}_i). \tag{5·2}$$

We assume that in O_i the momentum $\boldsymbol{p}_i$ appears linearly. Hence we have only to know the commutators for $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$:

$$[M, \boldsymbol{x}_i]M^{-1}=0$$

$$[M, \boldsymbol{p}_i]M^{-1}=m\int\delta(\boldsymbol{x}-\boldsymbol{x}_i)\,grad\,\phi(\boldsymbol{x})\,d\boldsymbol{x}=m\,grad\,\phi(\boldsymbol{x}_i). \tag{5·3}$$

Thus we obtain the commutator for O_i as

$$[M, O_i(\boldsymbol{x}_i, \boldsymbol{p}_i)]M^{-1}=O_i(\boldsymbol{x}_i, m\,grad\,\phi(\boldsymbol{x}_i))=\bar{O}_i(\boldsymbol{x}_i, \pi^*_{lm}). \tag{5·4}$$

Thus, one has only to replace $\boldsymbol{p}_i$ by $m\,grad\,\phi(\boldsymbol{x}_i)$ or $m\sum_{lm}\pi^*_{lm}\cdot grad\,F_{lm}(\boldsymbol{x})$.

Thus we arrive at

$$[M, O]M^{-1}=\sum_i O_i(\boldsymbol{x}_i, m\,grad\,\phi(\boldsymbol{x}_i))=\int\rho(\boldsymbol{x})\bar{O}(\boldsymbol{x}, \pi^*_{lm})\,d\boldsymbol{x}. \tag{5·5}$$

Since π_{lm} are the collective momenta conjugate to α_{lm}, this part can be regarded as representing the dynamical effect of the collective motion. This effect cannot arise from the shell model, in which the core is assumed to behave as if static. Therefore, the correspondence between the method of configurational mixing and the collective model is strictly true only for the case

$$MOM^{-1}=O. \tag{5·6}$$

As discussed in Section 2, in the method of configurational mixing currently employed the expectation value of an operator, O, is given by

$$(\Psi_{mixed}, O\ \Psi_{mixed})=(\Psi^s_0, O\ \Psi^s_0)$$
$$+[\sum_n\beta_n(\Psi^s_n, O\ \Psi^s_0)+\text{comp. conj.}], \tag{5·7}$$

$$\beta_n=(\Psi^s_n, V_{eff}(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots\boldsymbol{x}_N)\Psi^s_0)/(E_0-E_n).$$

On the other hand, using the adiabatic approximation, the expectation value in the collective representation is given by

$$(\chi(\boldsymbol{x}_0, \alpha), <\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N; \alpha), MOM^{-1}\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N; \alpha)>\chi(\boldsymbol{x}_0, \alpha)). \tag{5·8}$$

If O satisfies (5·6), and if we drop the direct interactions H_{int} and $H_{c.m.}$ discussed in Sections 3 and 4, (5·8) is expressed as

$$(\chi, <\psi^{core}, MOM^{-1}\psi^{core}>\chi)=(\chi, <\psi^{core}, O\psi^{core}>\chi)$$

$$= (\Psi_0^s,\ O\ \Psi_0^s) + \Big[\sum_n \frac{1}{E_0 - E_n} \Big\{ \Big(\Psi_n^s,\ 2\sum_{lm} \frac{\sum_{i=1}^{N} U_{lm}^{(1)}(\boldsymbol{x}_i) \cdot U_{0;lm}^{(1)}(\boldsymbol{x}_0)}{-\hbar\omega_l} \times$$

$$\times (<\chi_{lm},\ \alpha_{lm}\ \chi_0>)^2\ \Psi_0^s \Big) \Big\} (\Psi_n^s\ O\ \Psi_0^s) + \text{comp. conj.} \Big] \qquad (5\cdot 9)$$

by using (3·12), (4·11) and (4·18). Comparing (5·9) with (5·7) these two are equivalent if one chooses the effective interaction to be

$$V_{eff}(\boldsymbol{x}_0,\ \boldsymbol{x}_1 \cdots \boldsymbol{x}_N) = 2\sum_{lm} \frac{\sum_{i=1}^{N} U_{lm}^{(1)}(\boldsymbol{x}_i) \cdot U_{0;lm}^{(1)}(\boldsymbol{x}_0)}{-\hbar\omega_l} < \chi_0,\ \alpha_{lm}^{\dagger}\ \chi_0 > . \qquad (5\cdot 10)$$

(5·10) is nothing but $V^{(ex)}(\boldsymbol{x}_0\,;\ \boldsymbol{x}_1 \cdots \boldsymbol{x}_N)$ given by (4·19).

§ 6. Electric quadrupole moments

In the original representation, the electric quadrupole moment operator is expressed by

$$Q = \sum_{i=0}^{A} g_i (3z_i^2 - r_i^2) = \sum_{i=1}^{N} g_i (3z_i^2 - r_i^2) + g_0 (3z_0^2 - r_0^2), \qquad (6\cdot 1)$$

where

$$g_i = 0 \quad \text{for a neutron,}$$

and

$$g_i = 1 \quad \text{for a proton.}$$

Since this operator does not involve momenta, the last term of (5·1) vanishes and formulas (5·7) and (5·9) can be used.

In the collective model of Bohr and Mottelson,[17] the quadrupole moment operator is given by

$$Q^{B\text{-}M} = Q_{core} + Q_0 = (3/\sqrt{5\pi})\, \mathcal{Z} R_0^2\, \alpha_{20} + g_0 (3z_0^2 - r_0^2), \qquad (6\cdot 2)$$

and the expectation value by

$$(\chi(\boldsymbol{x}_0,\ \alpha),\ Q^{B\text{-}M}\chi(\boldsymbol{x}_0,\ \alpha)) = (3/\sqrt{5\pi})\ \mathcal{Z} R_0^2 (\chi(\boldsymbol{x}_0,\ \alpha),\ \alpha_{20}\ \chi(\boldsymbol{x}_0,\ \alpha))$$
$$+ (\chi(\boldsymbol{x}_0,\ \alpha),\ Q_0\ \chi(\boldsymbol{x}_0,\ \alpha)). \qquad (6\cdot 3)$$

In order to justify the equality of (5·9) and (6·3), we rewrite (6·1) as

$$Q = (3/\sqrt{5\pi})\, N R_0^2 \sum_{i=1}^{N} g_i\, F_{20}(\boldsymbol{x}_i) + g_0 (3z_0^2 - r_0^2), \qquad (6\cdot 4)$$

by taking into account the fact that

$$\sum_{i=1}^{N} F_{lm}(\boldsymbol{x}_i) = (4\pi/3N) \sum_{i=1}^{N} (r/R_0)^l Y_{lm}(\theta_i\ \varphi_i).$$

According to the condition, (3·9), the expectation value of (6·4) in the core particle state, $\psi^{core}(\boldsymbol{x}_1 \cdots \boldsymbol{x}_N\,;\ \alpha)$, is given by

$$(\psi^{core},\ Q\ \psi^{core}) = (3/\sqrt{5\pi})\ \mathcal{Z} R_0^2\ \alpha_{20} + Q_0 = Q^{B\text{-}M}, \qquad (6\cdot 5)$$

so that

$$(\chi, \psi^{core}, Q\psi^{core} > \chi) = (\chi, Q^{H\text{-}M}\chi).$$

This demonstrates the equality of (5·9) and (6·4). In fact, it should be noted that if we use the expression (3·12) for ψ^{core}, but neglecting H_{int}, the condition (3·9) becomes

$$\sum_n \frac{(\psi_n, \sum_{i=1}^{N} U_{lm}^{(1)}(\boldsymbol{x}_i)\psi_0)}{E_0 - E_n}(\psi_n, \sum_{i=1}^{N} F_{lm}(\boldsymbol{x}_i)\psi_0) + \text{comp. conj.} = 1. \tag{6·6}$$

Inserting (6·6) into (5·9) and using the expression (4·11) for $\chi(\boldsymbol{x}_0, \alpha)$, we can easily see that (5·9) reduces to (6·3).

Using the expression on the right hand side of (5·9), the expectation value in the collective model is found to correspond to that in the method of configurational mixing as given by (5·7), provided that V_{eff} used in (5·7) is taken to be that given by (5·10). The similarity which is shown in Table I between the values of the quadrupole moment calculated by the configurational mixing method[5] and by the collective model[6] is based upon such a procedure.

Although these two ways of calculation are similar in forms, there exist some discrepancies between them. Firstly, in the configurational mixing method the perturbation interaction, V_{eff}, is assumed to be independent of the structure of the core, the shell structure being implied only in the energy denominator of β_n. In the collective model however, V_{eff} is dependent on the structure of the core through the collective eigenfrequencies, ω_l. This provides us with an explanation for the differences in the quadrupole moments of some isotopes.[6]

Secondly, the summation over n in (5·7) is usually carried out only for the main terms, and the applicability of configurational mixing is restricted to the case of small perturbation. The same restriction holds in the collective model, in so far as the weak coupling approximation is employed. However, the adiabatic and weak coupling approximations we have used in the foregoing discussions are only for the purpose of showing the correspondence to the configurational mixing method. One may forego such approximations and choose suitable ways for the consideration of real nuclei. In this respect the collective model has wider applicability than the method of configurational mixing. In fact, we have shown that the configurational mixing is valid as a special case of the collective model.

Thirdly, our method is different from the current collective model of A. Bohr.[17] In the latter case ω_l is determined on the basis of the hydrodynamical model, whereas in our case ω_l is obtained from the collective motion for a given rigidity which depends strongly upon the structure of the core.[6]

§ 7. Magnetic dipole moments

In the original representation the magnetic dipole moment operator is expressed as

$$\mu_z = (e\hbar/2mc)\sum_{i=0}^{A}(g_i\, \boldsymbol{l}_i + \gamma_i\, \boldsymbol{s}_i)_z, \tag{7·1}$$

where

$$\gamma_i = -3.826 \qquad \text{for neutron,}$$

$$\gamma_i = \ \ 5.587 \qquad \text{for proton.}$$

In the collective representation this is transformed into

$$\begin{aligned} M\mu_z M^{-1} &= \mu_z + [M,\ \mu_z]M^{-1} \\ &= \mu_z + \frac{e\hbar}{2mc}\int \rho_p(\boldsymbol{x})[\boldsymbol{r}\times m\ grad\ \phi(\boldsymbol{x})]_z\ d\boldsymbol{x} \qquad (7·2) \\ &= \mu_z + \frac{e\hbar}{2mc}\int \rho_p(\boldsymbol{x})\,\boldsymbol{R}_z(\boldsymbol{x})\,d\boldsymbol{x}, \end{aligned}$$

where $\rho_p(\boldsymbol{x})$ is the density of protons and

$$\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{r}\times m\ grad\ \phi(\boldsymbol{x}) \tag{7·3}$$

represents the angular momentum arising from the collective motion of the core.

The first term in the right hand side of (7·2) represents the magnetic moment due to individual nucleons, so that its expectation value can be evaluated by (5·9). However, the Ψ_n^s which give non-vanishing matrix elements of $(\Psi_n^s,\ \mu_z\ \Psi_0^s)$, do not satisfy the collective deformation condition; if V_{eff} in $(\Psi_n^s,\ V_{eff}\ \Psi_0^s)$ is expressed by (5·10), these matrix elements are found to vanish for such Ψ_n^s. The non-vanishing elements appear in the next order, if Ψ_n^s are restricted to that owing to the collective deformation. Nevertheless, the first order term in (5·7) has an effect owing to the direct interaction $H_{0.c}$. This is what is considered in the current method of configurational mixing.

We shall now discuss the dynamical effect implied in the last term of (7·2), which can not be taken into account in the configurational mixing method. In order to obtain the expectation value of the last term of (7·2), we first note that by using the density operator of the core particles the condition (3·9) may be rewritten as follows:

$$(\psi^{core},\ \rho(\boldsymbol{x})\psi^{core}) = (1/2)\rho_0\{1+\epsilon(R(\theta,\varphi)-r)\}. \tag{7·4}$$

Here $\rho_0 = 3N/4\pi R_0^3$ and

$$R(\theta,\ \varphi) = R_0[1+\sum_{lm}\alpha_{lm}\ Y_{lm}(\theta,\ \varphi)]$$

represents the nuclear surface. $\epsilon(x)$ is expressed by

$$\epsilon(x) = \frac{x}{|x|} = \begin{cases} +1, & x>0 \\ -1, & x<0. \end{cases}$$

Therefore, the dynamical part is reduced to

$$(\chi, \cdot\ \zeta^{(\nu)\dagger}, \frac{e\hbar}{2mc}\int \rho_p(\boldsymbol{x})\boldsymbol{R}_z(\boldsymbol{x})\,d\boldsymbol{x}\ \zeta^{(\nu)}\cdot\chi)$$

$$=\frac{e\hbar}{2mc}(\chi,\ (1/2)\rho_{0,p}\int R_0\,\delta(R_0-r)\sum_{lm}\alpha_{lm}\,Y_{lm}\boldsymbol{R}_z(\boldsymbol{x})\,d\boldsymbol{x}\,\chi). \tag{7·6}$$

This is simply the expression derived by Bohr and Mottelson.[17] If this is to be compared with the method of configurational mixing, it will be necessary to retain at least the second order terms in $V^{(\nu\nu)}$. Therefore, the equivalence between the collective model and the configurational mixing holds only to first order in $V^{(\nu\nu)}$. In the collective model currently employed, if higher oder effects are neglected, not only the dynamical part, but also the static part arising from the mixing of single particle states will disappear, leaving only the magnetic moment due to the unperturbed extra-nucleon. The magnetic moment thus calculated is the same as in the single particle model.

The above circumstances are in contrast to the case of the quadrupole moment discussed in the preceding section. In the latter case the angular momentum states of the core particles to be mixed are the $J=2$ states, both in the configurational mixing method and the collective model, while in the case of magnetic moments the states to be mixed in first order are of $J=1$, which cannot occur by the collective deformation. Thus the configurational mixing due to the direct interaction, $H_{0,\nu}$, seems to play a significant role in accounting for magnetic moments. Therefore, we may infer the possibility of analyzing the contributions from $V^{(\nu\nu)}$ and $H_{0,\nu}$ by the quantitative studies of quadrupole moments and magnetic moments. For this purpose it is advisable to refer to as many nuclear properties as possible, such as the pairing energy, the optical potential and so forth. Studies along this line will be made in subsequent papers.

§ 8. Concluding remarks

In the above discussion we have considered the relation between the collective model and the individual particle model with configurational mixing on the basis of the supposition (II). The effective inter-particle interaction is, according to our supposition, shown to be of collective origin in part. Such an effective interaction is presumed to be weak enough to justify the success of the shell model. Leaving this essential problem unsolved, we reach the following conclusions concerning the intimate relationship between the collective and the shell models of nuclei.

i) The individual particle model with configurational mixing does in fact hold in the case of small nuclear deformation, provided that the single particle motion in the collective representation can be treated by means of the adiabatic approximation.

ii) The effective inter-particle force responsible for the configurational mixing consists of two parts, one arising from the exchange of surfons between an extra-particle and particles in the core and the other from a direct inter-particle interaction.

iii) As far as the quadrupole moment is concerned, the collective model and the configurational mixing method do not show any essential difference under the conditions

adopted in i). The difference lies in the effective interaction responsible for mixing single particle states. In the collective model the interaction is due to the exchange of surfons, while in the method of configurational mixing it is due to the direct internucleon force. The former depends strongly on the structure of the core, but the latter does not.

iv) There exists an essential difference in these two methods concerning the interpretation of the magnetic moment. In the collective model the dynamical effect of the core rotation is important, while in the configurational mixing formalism the direct interaction plays a significant role. The latter effect can also be taken into consideration in the collective model, but the dynamical effect cannot appear in the configurational mixing formalism.

v) For large nuclear deformations the configurational mixing method is no longer valid, but the collective model remains tenable, provided that the strong coupling approximation is used.

The above conclusions seem to provide the basis of the results obtained by Blin-Stoyle[3] through his extensive analysis of empirical evidence. Namely, the configurational mixing method is successful in interpreting the nuclear moments of nuclei except for those in which $150 < A < 190$ or $A > 225$. In the latter case the nuclear deformation is believed to be considerable.

As an example in which the current collective model fails, we mention the magnetic moment of ^{209}Bi. This nucleus is composed of a closed shell and an extra-proton. Hence the collective deformation is not so large that the deviation from the Schmidt line may be accounted for only in terms of the dynamical effect. Thus one must take account of the configurational mixing due to the direct interaction. On the contrary, the magnetic moments of ^{151}Eu and ^{153}Eu cannot be explained in terms of the configurational mixing method, but the collective model predicts the moments in good agreement with the observed values. This is because these nuclei are deformed strongly, so that the dynamical effect alone play a dominant role in the deviation from the Schmidt line.

In conclusion we should like to notice that our method of dealing with the coupling of the collective motion to the shell structure is similar to the method developed by Inglis[18] and Araújo[20]. Inglis treated the collective coordinates as time dependent. This can be obtained from our formalism by transforming our Schrödinger representation of the collective coordinates into the Heisenberg representation. Inglis, Araújo and several workers in Copenhagen noticed that their theory of collective motion can describe the moment of inertia in relation to the shell structure. This effect is dropped in our theory because of the adiabatic approximation.* In the present paper we are rather interested in the relation between the collective and shell models and leave the discussions of such problem to a forthcoming paper.

*) The implication of the adiabatic approximation was carefully examined by Tomonaga,[21] who succeeded in describing the moment of inertia in relation to shell structure by means of a method similar to ours.

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On the Bethe-Salpeter Equation and the Excited States of the Hydrogen Atom

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It is shown that the exact Bethe-Salpeter equation does not determine higher excited levels of the hydrogen atom in principle, by analyzing the asymptotic behaviour of two-particle amplitudes associated with photon co-ordinates at large distances. In connection with this analysis, however, an approximate method of treating the problem of excited states is given. Some related problems are also discussed briefly.

§ 1. Introduction and summary

It is the purpose of the present paper to see whether the Bethe-Salpeter equation[1)2)3)4)5)] is able to determine higher excited levels of the hydrogen atom *in principle*.

As Gell-Mann and Low[2)] have shown field-theoretically, the Bethe-Salpeter equation is the eigen-value equation for the whole rest mass of the composite particle. The whole rest mass, $M=(-P_\mu P_\mu)^{1/2}$, is *real* by the definition that $P_i (i=1, 2, 3)$ and $P_0=-iP_4$ are eigenvalues of hermitian operators, the momentum and energy of the system. When the Bethe-Salpeter equation is applied to the problem of the hydrogen atom, it is expected that it will at least determine the ground level.* It is not self-evident, however, whether it determines also higher excited levels. Even if it could determine excited levels, they would not directly correspond to the actual excited states of the hydrogen atom, *metastable* states. So it will be conceivable that such levels would rather correspond to the resonance scattering states of photon by the normal hydrogen atom. Such a conjecture is, however, denied in this paper and it is concluded that *the exact Bethe-Salpeter equation cannot give any higher excited levels of the hydrogen atom.*

After some preliminaries (§ 2), we assume in § 3, first, that the exact Bethe-Salpeter equation can give some higher excited level. Under this assumption the covariant Fock amplitudes, containing as arguments not only co-ordinates of the two particles but also photon co-ordinates, are obtained with the aid of Gell-Mann and Low's limiting process.

* According to reference[5)], the Bethe-Salpeter equation for two fermions have not, in the ladder approximation, any solution which can be connected with the non-relativistic one. This is due to the highly singular character of the kernel of the integral equation. It is an open question whether such a singular character of the kernel is weakened in the exact Bethe-Salpeter equation by virtue of the higher radiative corrections. If it is not weakened by such an effect, some modification of the kernel, corresponding to some cut-off procedure at high virtual momenta, will be needed.

Then it follows that the asymptotic expressions of these amplitudes are composed of *only outgoing* photon waves leaving the hydrogen atom in the lower states in spite of the fact that *stationary* states are being considered. This means that the assumption taken at the very beginning is incorrect. Thus it turns out that the exact Bethe-Salpeter equation can determine only the ground level of the hydrogen atom. However *the Bethe-Salpeter equation with a suitably approximated kernel may give the corresponding approximate levels of the excited hydrogen atom* in a similar fashion to the case treated by Wick[6] and Cutkosky[7] who solved the two-boson system interacting through a scalar photon field in the ladder approximation. They showed also that such an approximate equation has some extra solutions that have no correspondence to the non-relativistic ones. It seems to the authors that the appearance of such extra solutions is due to the change of the kernel by the ladder approximation. In § 4 it is shown that *the coefficients of the asymptotic expressions of the two-particle amplitudes associated with photon co-ordinates, when good solutions of the approximated Bethe-Salpeter equation are substituted in those expressions, give the matrix elements of the spontaneous optical transitions.* In the last section a few problems are discussed briefly in relation to the problem treated in the preceding sections.

§ 2. Preliminaries

1) *Definition of Covariant Fock amplitudes*

We define, after many authors[8)9)10)], a set of covariant Fock amplitudes as follows:

$$\begin{aligned} &F_{\mu\nu\cdots}(x, x', \cdots | y, y', \cdots | \bar{x}, \bar{x}', \cdots | \bar{y}, \bar{y}', \cdots | z, z', \cdots) \\ &\quad \equiv \langle \Omega | N(\zeta^{le}(x)\zeta^{le}(x')\cdots\zeta^{lp}(y)\zeta^{lp}(y')\cdots\bar{\zeta}^{le}(\bar{x})\bar{\zeta}^{le}(\bar{x}') \\ &\qquad \cdots\bar{\zeta}^{lp}(\bar{y})\bar{\zeta}^{lp}(\bar{y}')\cdots A_\mu(z)A_\nu(z')\cdots)|\Psi\rangle \qquad (1) \end{aligned}$$

Here $|\Omega\rangle$ is the vacuum state and $|\Psi\rangle$ is the state to be considered. N means the normal product of operators to be taken in the parenthesis. We adopt as the definition of the normal products of the Heisenberg operators the one proposed by Freese[11], Polkinghorne[12], Kristensen[13] and Maki[14]:

$$T(ABC\cdots Z) = N(ABC\cdots Z) + \sum \epsilon \langle \Omega | T(AB\cdots K) | \Omega \rangle N(L\cdots Z), \qquad (2)$$

where T is the time-ordering operator in Wick's sense[15]. The sum on the right-hand side of equation (2) is taken over all ways of dividing the factors between the T-products and N-products, and ϵ is ± 1 according to whether the permutation of the fermion factors induced by T is even or odd. In equation (1) ζ^{le} and ζ^{lp} are the electron and the proton field respectively and $\bar{\zeta}^{le}$ and $\bar{\zeta}^{lp}$ are their adjoint operators. A_μ is the electromagnetic four potential.

2) *Bethe-Salpeter equation for the hydrogen atom*

As is well known the Bethe-Salpeter equation can be derived, by applying the Gell-Mann and Low's limiting process[9], from the equation of the two-particle (an electron and a proton) Green function $K(x, y; \bar{x}, \bar{y})$:

$$K(x, y; \bar{x}, \bar{y}) = -S_F^e(x, \bar{x}) S_F^p(y, \bar{y})$$

$$+\int\cdot\int d^4x'\cdots d^4\bar{y}'\, S_F^e(x,\bar{x})\, S_F^p(x,\bar{y})\, G(\bar{x}',\bar{y}';x',y')\, K(x',y';\bar{x},\bar{y}). \tag{3}$$

Here

$$\begin{aligned} K(x,y;\bar{x},\bar{y}) &= \langle \Omega | T(\psi^e(x)\psi^p(y)\bar{\psi}^e(\bar{x})\bar{\psi}^p(\bar{y})) | \Omega \rangle, \\ S_F^e(x,\bar{x}) &= \langle \Omega | T(\psi^e(x)\bar{\psi}^e(\bar{x}) | \Omega \rangle, \\ S_F^p(y,\bar{y}) &= \langle \Omega | T(\psi^p(y), \bar{\psi}^p(\bar{y})) | \Omega \rangle, \end{aligned} \tag{4}$$

and $G(\bar{x},\bar{y};x,y)$ is the function* describing the interaction between an electron and a proton, the lowest approximation (ladder approximation) of which is given by the following :

$$G(\bar{x},\bar{y};x,y)_{\text{lowest}} = \eta_e \eta_p \gamma_\mu^e \gamma_\mu^p D_F^{(0)}(\bar{x},\bar{y})\,\delta(\bar{x}-x)\,\delta(\bar{y}-y).$$

Here η_e and η_p are the charge of the electron and the proton respectively. $D_F^{(0)}$ is the photon Green function in the system without interaction and is discriminated by a superscript (0) from $D_{F\mu\nu}$

$$D_{F\mu\nu}(z,z') \equiv \langle \Omega | T(A_\mu(z) A_\nu(z')) | \Omega \rangle. \tag{5}$$

Now by applying the Gell-Mann and Low's limiting process to equation (3), the Bethe-Salpeter equation can be derived :

$$F^{(n)}(x|y|||) = \int\cdot\int d^4x'\cdots d^4\bar{y}'\, S_F^e(x,\bar{x})\, S_F^p(y,\bar{y}')\, G(\bar{x}',\bar{y}';x',y')\, F^{(n)}(x'|y'|||). \tag{6}$$

$F^{(n)}(x|y|||)$ is the two-particle Fock amplitude of a stationary state with the energy-momentum four vector, $P_\mu^{(n)}$. In equation (6) was used the following relation,

$$F^{(n)}(x|y|||) \equiv \langle \Omega | N(\psi^e(x)\psi^p(y) | \Psi^{(n)} \rangle = \langle \Omega | T(\psi^e(x)\psi^p(y)) | \Psi^{(n)} \rangle, \tag{7}$$

which is valid since $\langle \Omega | T(\psi^e(x)\psi^p(y)) | \Omega \rangle = 0$.

When there occurs degeneracy, we must take additional care. We use a superscript $\alpha\,(\alpha=1, 2, \cdots)$ to discriminate those degenerate states. Then we have in general, instead of equation (6), the following equation,

$$\begin{aligned} &\Big(\sum_\alpha F^{(n,\alpha)}(x|y|||) D_{\alpha\beta}^{(n)}\Big) \\ &= \int\cdot\int d^4x'\cdots d^4\bar{y}'\, S_F^e(x,\bar{x})\, S_F^p(y,\bar{y}')\, G(\bar{x}',\bar{y}';x',y') \Big(\sum_\alpha F^{(n,\alpha)}(x'|y'|||) D_{\alpha\beta}^{(n)}\Big), \end{aligned} \tag{6'}$$

where $(D_{\alpha\beta}^{(n)})$ is a hermitian matrix given by

$$D_{\alpha\beta}^{(n)} = \int d^3\vec{r}\, f^{*(n,\alpha)}(\vec{r}\,o) f^{(n,\beta)}(\vec{r}\,o). \tag{8}$$

* Equation (3) can be obtained from the corresponding integro-differential equation in Schwinger's formalism[3] by making use of the positive frequency condition implied in the definition of the Green function.

Here

$$f^{(n,\alpha)}(r)=\exp(-iP^{(n)}X)\cdot F^{(n,\alpha)}(x|y|||), \tag{9}$$

and

$$X_\mu=1/2\cdot(x_\mu+y_\mu), \qquad r_\mu=x_\mu-y_\mu. \tag{10}$$

In general $D_{\alpha,\beta}^{(n)}$ may be non-diagonal in spite of the ortho-normal condition of $\Psi^{(n,\alpha)}$, since this condition can be represented only by the relation involving many other amplitudes as well as two-particle amplitudes.[10] We can find, however, a unitary matrix, $U_{\alpha,\beta}^{(n)}$, which transforms $D_{\alpha,\beta}^{(n)}$ into a diagonal matrix $\delta_{\alpha\beta}D_\alpha^{(n)}$, since $D_{\alpha,\beta}^{(n)}$ is hermitic. Thus equation (6) is valid for those states which are obtained by applying this unitary transformation to $\Psi^{(n,\alpha)}$. For the sake of brevity, we drop from now on the superscript α.

3) *Expressions of the Green functions* $K_{\mu\nu}\cdots(x, y; \bar{x}, \bar{y}; z, z', \cdots)$

In this subsection expressions are given of the two-particle Green functions which involve photon co-ordinates also. The method developed by Nishijima[10] is very useful for this purpose.

We consider for convenience a system laid in the classical external source fields, $J_\mu(z)$, and discriminate the quantities of such a system from those of our system without external sources by the superscript J. Then the following relations are valid between the Green functions $K_{\mu\nu}\cdots(x, y; \bar{x}, \bar{y}; z, z', \cdots)$.

$$CK_{\mu\nu}\cdots(x, y; \bar{x}, \bar{y}; z, z', \cdots)^J=i\delta[CK_\nu\cdots(x, y; \bar{x}, \bar{y}; z', \cdots)^J]/\delta J_\mu(z) \tag{11}$$

where

$$K_{\mu\nu}\cdots(x, y; \bar{x}, \bar{y}; z, z', \cdots)^J\equiv\langle\Omega|T(\zeta^{Je}(x)\zeta^{Jp}(y)\bar{\zeta}^{Je}(\bar{x})\bar{\zeta}^{Jp}(\bar{y})A_\mu(z)A_\nu(z')\cdots)^J|\Omega\rangle, \tag{12}$$

$$C\equiv\langle\Phi_0|T[\exp(-i\int_0^\infty dx\,\boldsymbol{H}_{\mathrm{int}}^J(x))]|\Phi_0\rangle, \tag{13}$$

and

$$\boldsymbol{H}_{\mathrm{int}}^J=i\gamma_e N(\bar{\boldsymbol{\psi}}^e\gamma_\mu^e\boldsymbol{\psi})\boldsymbol{A}_\mu+i\gamma_p N(\bar{\boldsymbol{\psi}}^p\gamma_\mu^p\boldsymbol{\psi}^p)\boldsymbol{A}_\mu+J_\mu\boldsymbol{A}_\mu-\boldsymbol{H}_{\text{self-energy}}. \tag{14}$$

Here $|\Phi_0\rangle$ is the bare vacuum and $\boldsymbol{H}_{\mathrm{int}}^J$ is the interaction Hamiltonian density of the J-system in the interaction representation. Now in the J-system the two-particle Green function satisfies the equation

$$K(x, y; \bar{x}, \bar{y})^J=-S_F^e(x, x)^J S_F^p(y, \bar{y})^J$$
$$+\int\cdot\int d^4x'\cdots d^4\bar{y}'\,S_F^e(x, \bar{x}')^J S_F^p(y, \bar{y}')^J G(\bar{x}', \bar{y}'; x', y')^J K(x', y'; \bar{x}, \bar{y})^J. \tag{3}'$$

By multiplying the equation (3)′ by C, differentiating with respect to J_μ and then putting J_μ to be zero, we obtain the following equation;

$$K_\mu(x, y; \bar{x}, \bar{y}; z)-\int\cdot\int d^4x'\cdots d^4\bar{y}'\,S_F^e(x, \bar{x}')S_F^p(y, \bar{y}')G(\bar{x}', \bar{y}'; x' y')K_\mu(x', y'; \bar{x}, \bar{y}; z)$$

$$
\begin{aligned}
=&-\eta_e\int\cdot\int d^4x'\cdots d^4z'\,S_F^e(x,\bar{x}')\,\Gamma_\nu^e(\bar{x}',x';z')\,S_F^e(x',\bar{x})\,D_{F\nu\mu}(z',z)\,S_F^p(y,\bar{y})\\
&-\eta_p\int\cdot\int d^4y'\cdots d^4z'\,S_F^p(y,\bar{y}')\,\Gamma_\nu^p(\bar{y}',y';z')\,S_F^p(y',\bar{y})\,D_{F\nu\mu}(z',z)\,S_F^e(x,\bar{x})\\
&+\eta_e\int\cdot\int d^4x'\cdots d^4z'\,S_F^e(x,\bar{x}')\,\Gamma_\nu^e(\bar{x}',x';z')\,S_F^e(x',\bar{x}'')\\
&\qquad\times D_{F\nu\mu}(z',z)\,S_F^p(y,\bar{y}')\,G(\bar{x}'',\bar{y}'';x'',y')\,K(x'',y';\bar{x},\bar{y})\\
&+\eta_p\int\cdot\int d^4x'\cdots d^4z'\,S_F^p(y,\bar{y}')\,\Gamma_\nu^p(\bar{y}',y';z')\,S_F^p(y',y'')\\
&\qquad\times D_{F\nu\mu}(z',z)\,S_F^e(x,\bar{x}')\,G(\bar{x}',\bar{y}'';x',y'')\,K(x',y'';\bar{x},\bar{y})\\
&+\int\cdot\int d^4x'\cdots d^4z'\,S_F^e(x,\bar{x}')\,S_F^p(y,\bar{y}')\,G_\nu(\bar{x}',\bar{y}';x',y';z')\,D_{F\nu\mu}(z',z)\,K(x',y';\bar{x},\bar{y}).
\end{aligned}
\tag{15}
$$

The vertex functions Γ_μ^e, Γ_μ^p and G_μ are defined by the equations

$$
\begin{aligned}
&i\partial S_F^e(x,\bar{x})^J/\partial J_\mu(z)\,|_{J=0}\\
&\qquad=\eta_e\int\cdot\int d^4x'\cdots d^4z'\,S_F^e(x,\bar{x}')\,\Gamma_\nu^e(\bar{x}',x;z')\,S_F^e(x',\bar{x})\,D_{F\nu\mu}(z',z),\\
&i\partial S_F^p(y,\bar{y})^J/\partial J_\mu(z)\,|_{J=0}\\
&\qquad=\eta_p\int\cdot\int d^4y'\cdots d^4z'\,S_F^p(y,y')\,\Gamma_\nu^p(\bar{y}'\,y';z')\,S_F^p(y',\bar{y})\,D_{F\nu\mu}(z',z),
\end{aligned}
\tag{16}
$$

$$
i\partial G(\bar{x},\bar{y};x,y)^J/\partial J_\mu(z)\,|_{J=0}=\int d^4z'\,G_\nu(\bar{x},\bar{y};x,y;z')\,D_{F\nu\mu}(z',z).
\tag{17}
$$

From the equation (15) we have, by making use of the equation (3),

$$
\begin{aligned}
&K_\mu(x,y;\bar{x},\bar{y};z)\\
&=\eta_e\int\cdot\int d^4x'\cdots d^4z'\,K(x,y;\bar{x}',\bar{y}')\,\Gamma_\nu^e(\bar{x}',x';z')\,S_F^e(x',\bar{x})\,D_{F\nu\mu}(z',z)\\
&+\eta_p\int\cdot\int d^4y'\cdots d^4z'\,K(x,y;\bar{x},\bar{y}')\,\Gamma_\nu^p(\bar{y}',y';z')\,S_F^p(y',\bar{y})\,D_{F\nu\mu}(z',z)\\
&-\int\cdot\int d^4x'\cdots d^4z'\,K(x,y;\bar{x}',\bar{y}')\,Q_\nu(\bar{x}',\bar{y}';x',y';z')\,D_{F\nu\mu}(z',z)\,K(x',y';\bar{x},\bar{y})
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
Q_\nu(\bar{x},\bar{y};x,y;z)=&\eta_e\int\cdot\int d^4x'\,d^4\bar{x}'\,\Gamma_\nu^e(\bar{x},x';z)\,S_F^e(x',\bar{x}')\,G(\bar{x}',\bar{y};x,y)\\
&+\eta_p\iint d^4y'\,d^4\bar{y}'\,\Gamma_\nu^p(\bar{y},y';z')\,S_F^p(y',\bar{y}')\,G(\bar{x},\bar{y}';x,y)\\
&+G_\nu(\bar{x},\bar{y};x,y;z).
\end{aligned}
\tag{19}
$$

It should be noted that the arbitrariness of adding any solution of the equation,

$$X_{\mu\nu}(x, y; x, \bar{y}; z) - \int\cdot\int d^4x'\cdots d^4\bar{y}'\, S_F^e(x, \bar{x}')\, S_F^p(y, \bar{y}')\, G(\bar{x}', \bar{y}'; x', y')\, X_\mu(x', y'; \bar{x}, \bar{y}; z),$$

to the right-hand side of the equation (18) is eliminated by the positive frequency condition implied in the definition of K_μ.

In the similar fashion we can obtain an expression of $K_{\mu\nu}$ and so forth,

$$\begin{aligned}
&K_{\nu\mu}(x, y; \bar{x}, \bar{y}; z, z') - K(x, y; \bar{x}, \bar{y})\, D_{F\mu\nu}(z, z') \\
&= \eta_e \int\cdot\int d^4x'\cdots d^4z''\, K_\nu(x, y; \bar{x}', \bar{y}'; z')\, \Gamma'^e_\rho(\bar{x}', x'; z'')\, S_F^e(x', \bar{x})\, D_{F\rho\mu}(z'', z) \\
&\quad + \eta_e \int\cdot\int d^4x'\cdots d^4z'''\, K(x, y; \bar{x}', \bar{y})\, \Gamma'^e_{\rho\sigma}(\bar{x}', x'; z'', z''') \\
&\qquad\qquad \times S_F^e(x', \bar{x})\, D_{F\rho\mu}(z'', z)\, D_{F\sigma\nu}(z''', z') \\
&\quad + \eta_e^2 \int\cdot\int d^4x'\cdots d^4z'''\, K(x, y; \bar{x}', \bar{y})\, \Gamma'^e_\rho(\bar{x}', x'; z'')\, S_F^e(x', \bar{x}'')\, \Gamma'^e_\sigma(\bar{x}'', x''; z''') \\
&\qquad\qquad \times S_F^e(x'', \bar{x})\, D_{F\rho\mu}(z'', z)\, D_{F\sigma\nu}(z''', z') \\
&\quad + \eta_p \int\cdot\int d^4y'\cdots d^4z''\, K_\nu(x, y; \bar{x}, \bar{y}'; z')\, \Gamma'^p_\rho(\bar{y}', y'; z'')\, S_F^p(y', \bar{y})\, D_{F\rho\mu}(z'', z) \\
&\quad + \eta_p \int\cdot\int d^4y'\cdots d^4z'''\, K(x, y; \bar{x}, \bar{y}')\, \Gamma'^p_{\mu\sigma}(\bar{y}', y'; z'', z''') \\
&\qquad\qquad \times S_F^p(y', \bar{y})\, D_{F\mu\nu}(z'', z)\, D_{F\sigma\nu}(z''', z') \\
&\quad + \eta_p^2 \int\cdot\int d^4y'\cdots d^4z'''\, K(x, y; \bar{x}, \bar{y}')\, \Gamma'^p_\rho(\bar{y}', y'; z'')\, S_F^p(y', \bar{y}'')\, \Gamma'^p_\sigma(\bar{y}'', y''; z'') \\
&\qquad\qquad \times S_F^p(y'', y)\, D_{F\rho\mu}(z'', z)\, D_{F\sigma\nu}(z''', z') \\
&\quad - \int\cdot\int d^4x'\cdots d^4z''\, K_\nu(x, y; \bar{x}', \bar{y}'; z')\, Q_\rho(\bar{x}', \bar{y}'; x', y'; z'') \\
&\qquad\qquad \times D_{F\rho\mu}(z'', z)\, K(x', y'; x, y) \\
&\quad - \int\cdot\int d^4x'\cdots d^4z'''\, K(x, y; \bar{x}', \bar{y}')\, Q_{\rho\sigma}(\bar{x}', \bar{y}'; x', y'; z'', z''') \\
&\qquad\qquad \times D_{F\rho\mu}(z'', z)\, D_{F\sigma\nu}(z''', z')\, K(x', y'; x, y) \\
&\quad - \int\cdot\int d^4x'\cdots d^4z''\, K(x, y; \bar{x}', \bar{y}')\, Q_\rho(\bar{x}', \bar{y}'; x', y'; z'') \\
&\qquad\qquad \times D_{F\rho\mu}(z'', z)\, K_\nu(x', y'; \bar{x}, \bar{y}, ; z'), \qquad (20)
\end{aligned}$$

where $\Gamma'^e_{\rho\sigma}$, $\Gamma'^p_{\rho\sigma}$ and $Q_{\rho\sigma}$ are defined respectively by the equations,

$$i\delta\, \Gamma'^e_\rho(\bar{x}, x; z)\cdot/\delta J_\nu(z')\,|_{J=0} = \int d^4z''\, \Gamma'^e_{\rho\sigma}(\bar{x}, x; z, z'')\, D_{F\sigma\nu}(z'', z')$$

$$i\partial\Gamma^{\rho}_{\mu}(\bar{y}, y; z)\cdot/\partial J_{\nu}(z')|_{J=0}=\int d^4z''\,\Gamma^{\rho}_{\mu\sigma}(\bar{y}, y; z, z'')\,D_{F\sigma\nu}(z'', z'), \tag{21}$$

$$i\partial Q_{\mu}(\bar{x}, \bar{y}; x, y; z)\cdot/\partial J_{\nu}(z')|_{J=0}=\int d^4z''\,Q_{\mu\sigma}(\bar{x}, y; x, y; z, z')\,D_{F\sigma\nu}(z'', z').$$

§ 3. Two-particle amplitudes associated with photon co-ordinates

In this section we assume firstly that the Bethe-Salpeter equation (6) gives not only the ground level $M_{(0)}$ but also excited levels $M_{(n)}$ $(n=1, 2, \cdots)$. We refer this assumption to (A). The corresponding stationary states with energy-momentum $P^{(n)}_{\mu}$ $(P^{(n)}_{\mu}P^{(n)}_{\mu}=-M^2_{(n)})$ we denote by $\Psi^{(n)}$. Under the assumption (A) we shall obtain the expressions of $F^{(n)}_{\mu\nu}\cdots(x|y|||z, z', \cdots)$ of these states $\Psi^{(n)}$ and then examine the validity of this assumption, (A), by analysing their asymptotic behaviour at large distances.

1) $F^{(n)}_{\mu}(x|y|||z)$

The assumption (A) means that the Bethe-Salpeter equation has non-zero solutions $F^{(n)}(x|y|||)$ $(n=1, 2, \cdots)$. Accordingly we can make use of the equation (18) in order to get an expression of $F^{(n)}_{\mu}(x|y|||z)$ by Gell-Mann and Low's procedure. Consider the equation (18) in the case when $x_0, y_0, z_0 > \bar{x}_0=\bar{y}_0(\equiv t)$, multiply it by $\gamma^e_4\gamma^p_4F^{(n)}(\bar{x}|\bar{y}|||)$, perform the integration over $\vec{\bar{x}}$ and $\vec{\bar{y}}$, and take the limit of $t\to-\infty$ in the Gell-Mann and Low's sense. As $t\to-\infty$, the regions of integration in which time variables of integration in equation (18) are earlier than t give vanishingly small contributions to the integrals. Then we have

$$\begin{aligned}D^{(n)}F^{(n)}(x|y|||z)=&I^{(n)}_{\mu}(x|y|||z)+\\&-D^{(n)}\int\cdots\int d^4x'\cdots d^4z'\,K(x, y; x', y')\,Q_{\nu}(\bar{x}', \bar{y}'; x', y'; z')\\&\times D_{F\nu\mu}(z', z)\,F^{(n)}(x'|y'|||),\end{aligned} \tag{22}$$

where

$$\begin{aligned}&I^{(n)}_{\mu}(x|y|||z)\\&=\underset{t\to-\infty}{L}\eta_e\int\cdots\int d^4x'\cdots d^4z'\,d^3\bar{x}\,d^3\bar{y}\sum_{l,m}\langle\Omega|T(\psi^e(x)\psi^p(y)\bar{\psi}^e(\bar{x}'))|m\rangle\langle m|\bar{\psi}^p(\bar{y})|\Omega\rangle\\&\quad\times\Gamma^{\rho e}_{\nu}(\bar{x}', x'; z')\langle\Omega|\psi^{\rho e}(x')|l\rangle\langle l|\bar{\psi}^{e}(\bar{x})|\Omega\rangle D_{F\nu\mu}(z', z)\gamma^e_4\gamma^p_4F^{(n)}(\bar{x}|\bar{y}|||)\\&-\underset{t\to-\infty}{L}\eta_p\int\cdots\int d^4y'\cdots d^4z'\,d^3\bar{x}\,d^3\bar{y}\sum_{l,m}\langle\Omega|T(\psi^e(x)\psi^p(y)\bar{\psi}^p(\bar{y}'))|l\rangle\langle l|\bar{\psi}^e(\bar{x})|\Omega\rangle\\&\quad\times\Gamma^{ep}_{\nu}(\bar{y}', y'; z')\langle\Omega|\psi^{ep}(y')|m\rangle\langle m|\bar{\psi}^p(\bar{y})|\Omega\rangle D_{F\nu\mu}(z', z)\gamma^e_4\gamma^p_4F^{(n)}(\bar{x}|\bar{y}|||).\end{aligned} \tag{23}$$

The sums on the right-hand side of (23) are taken over a complete set $\{|l\rangle\}$ and $\{|m\rangle\}$. $I^{(n)}(x|y|||z)$ vanishes, however, in the limit $t\to-\infty$, since the sum of energies of states $|l\rangle$ and $|m\rangle$ for which $\langle l|\bar{\psi}^e(\bar{x})|\Omega\rangle$ and $\langle m|\bar{\psi}^p(\bar{y})|\Omega\rangle$ do not vanish is always larger than the energy of $\Psi^{(n)}$. Thus we have the expression

$$F_{\mu}^{(n)}(x|y|||z) = -\int\cdot\int d^4x'\cdots d^4z' K(x, y; \vec{x}', \vec{y}') Q_{\nu}(\vec{x}', \vec{y}'; x', y'; z') \times D_{F\nu\mu}(z', z) F^{(n)}(x'|y'|||). \tag{24}$$

$F_{\mu}^{(n)}$ can be written in the form

$$F_{\mu}^{(n)}(x|y|||z) = \exp(iP^{(n)}Z)\cdot f_{\mu}^{(n)}(r, \vec{\xi}) \tag{25}$$

by using the center of mass co-ordinates and the relative co-ordinates,

$$Z_{\nu} = 1/3\cdot(x_{\nu} + y_{\mu} + z_{\lambda}), \qquad \vec{\xi}_{\nu} = z_{\nu} - 1/2\cdot(x_{\nu} + y_{\lambda}). \tag{26}$$

We are interested in $F_{\mu}^{(n)}$ at an equal time $x_0 = y_0 = z_0$, and accordingly in $f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)$. In the center of mass system ($\vec{P}^{(n)} = 0$, $P_0^{(n)} = M_{(n)}$),

$$f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)^{\text{C.M.S.}} = -\int\cdot\int d^4x'\cdots d^4z' K(\vec{x}o, \vec{y}o; \vec{x}', \vec{y}') Q_{\nu}(\vec{x}', \vec{y}'; x', y'; z') \times D_{F\nu\mu}(z', \vec{z}o) F^{(n)}(x'|y'|||)^{\text{C.M.S.}}. \tag{27}$$

The integral on the right-hand side of (27) can be divided into eight terms,

$$f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)^{\text{C.M.S.}} = \sum_{i=1}^{\text{VIII}} f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_i^{\text{C.M.S.}}, \tag{28}$$

according to the division of the integration over time variables $\bar{x}_0'$, $\bar{y}_0'$ and z_0',

$$\int_{-\infty}^{\infty} d\bar{x}_0' \int_{-\infty}^{\infty} d\bar{y}_0' \int_{-\infty}^{\infty} dz_0' = \int_{-\infty}^{0} d\bar{x}_0' \int_{-\infty}^{0} d\bar{y}_0' \int_{-\infty}^{0} dz_0' + \cdots + \int_{0}^{\infty} d\bar{x}_0' \int_{0}^{\infty} d\bar{y}_0' \int_{0}^{\infty} dz_0' = (\text{I}) + \cdots + (\text{VIII}). \tag{29}$$

We consider firstly the first term of (28). For the specified order of times,

$$K(\vec{x}o, \vec{y}o; \vec{x}', \vec{y}') = \sum_{n'}' \langle \Omega | T(\varphi^{\rho}(\vec{x}o)\varphi^{\sigma}(\vec{y}o)|n'\rangle \langle n'|T(\bar{\varphi}^{\rho}(\vec{x}')\bar{\varphi}^{\sigma}(\vec{y}'))|\Omega\rangle \tag{30}$$

and

$$D_{F\nu\mu}(z', \vec{z}o) = \sum_{m}' \langle \Omega | A_{\mu}(\vec{z}o)|m\rangle\langle m|A_{\nu}(z')|\Omega\rangle. \tag{31}$$

Since Q_{ν} is translation invariant, it can be written in the form:

$$Q_{\nu}(\vec{x}, \vec{y}; x, y; z) = (2\pi)^{-16}\int\cdot\int d^4k\, d^4k'\cdots d^4k''''\, \delta^4(k + k' - k'') \times Q_{\nu}(k, k'\cdots k'''') \exp\{i(k\bar{X} + k'z - k''X + k'''r + k''''r)\}. \tag{32}$$

Furthermore $f_{\mu}^{(n)}$ can be divided into two terms,

$$f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_{\text{I}}^{\text{C.M.S.}} = f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_{\text{I}A}^{\text{C.M.S.}} + f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_{\text{I}B}^{\text{C.M.S.}}, \tag{33}$$

according to

$$\int_{-\infty}^{0} d\bar{x}_0' \int_{-\infty}^{0} d\bar{y}' = \int_{-\infty}^{\infty} dr_0' \int_{-\infty}^{0} d\bar{X}_0' + \int_{-\infty}^{\infty} d\bar{r}_0' \int_{0}^{-(1/2)|\bar{r}_0'|} d\bar{X}_0'$$

$$= (A) + (B). \tag{34}$$

Then from equations (30), (31) and (32) it follows that

$$f_{\mu}^{(n)}(\vec{r}\,o, \vec{\xi}\,o)_{IA}^{\text{C.M.S.}} = -(2\pi)^{-4} \sum_{M_{(n')},\, M_{(m)}}{}' \int_{-\infty}^{\infty} dk_0' \int d^3\vec{P}^{(m)}\, \delta_-(\sqrt{M_{(n')}^2 + |\vec{P}^{(m)}|^2} - M_{(n)} + k_0')$$

$$\times \delta_-(\sqrt{M_{(m)}^2 + |\vec{P}^{(m)}|^2} - k_0') \exp(i\vec{P}^{(m)}\vec{\xi})\, a_{\mu}^{(m)}[P^{(m)}]$$

$$\times f^{(n')}(\vec{r}\,o)\, S^{(n'm<n)}[P^{(n')}, P^{(m)}; P^{(n)}], \tag{35}$$

where

$$\begin{aligned}
a_{\mu}^{(m)}[P^{(m)}] &= \exp(-iP^{(m)}z)\langle \Omega | A_{\mu}(z) | m\rangle, \\
a_{\mu}^{(m')}[P^{(m')}] &= \exp(iP^{(m')}z)\langle m | A_{\mu}(z) | \Omega\rangle, \\
f^{(n')}(r) &= \exp(-iP^{(n')}X)\langle \Omega | T(\varphi^{I^e}(x)\varphi^{I^p}(y)) | n'\rangle, \\
\bar{f}^{(n')}(r) &= \exp(iP^{(n')}X)\langle n' | T(\bar{\varphi}^{I^e}(x)\bar{\varphi}^{I^p}(y)) | \Omega\rangle
\end{aligned} \tag{36}$$

and

$$S^{(n'm<n)}[P^{(n')}, P^{(m)}; P^{(n)}]$$

$$= \int\cdot\int d^4k'''\, d^4k''''\, d^4r\, d^4r\, f^{(n')}(\bar{r})\, \bar{a}_{\nu}^{(m)}[P^{(m)}] Q_{\nu}(P^{(n')}, P^{(m)}, P^{(n)}, k''', k'''')$$

$$\times \exp\{i(k'''r + k''''r)\cdot\} f^{(n)}(r) \tag{37}$$

with

$$\begin{aligned}
P_{\mu}^{(n')} P_{\mu}^{(n')} &= -M_{(n')}^2, \qquad \vec{P}^{(n')} = -\vec{P}^{(m)}, \qquad P_0^{(n')} = M_{(n)} - k_0', \\
P_{\mu}^{(m)} P_{\mu}^{(m)} &= -P_{(m)}^2, \qquad P_0^{(m)} = k_0'.
\end{aligned} \tag{38}$$

In equation (35) the sum $\sum_{M_{(n')},\, M_{(m)}}{}'$ means to sum over degenerate states with fixed $P^{(n')}$ and $P^{(m)}$ and then to sum over $M_{(n')}$ and $M_{(m)}$. The asymptotic form of $\exp(i\vec{P}^{(m)}\vec{\xi})$ at a large $|\vec{\xi}|$ is given by the formula:

$$\exp(i\vec{P}^{(m)}\vec{\xi}) \sim (2\pi)\, |\vec{P}^{(m)}|^{-1} |\vec{\xi}|^{-1} \sum_{j,k}{}' (i)^j Y_{j,k}(\Omega(\vec{\xi}))\, Y_{j,k}^{*}(\Omega(\vec{P}^{(m)}))$$

$$\times [\exp\{i(|\vec{P}^{(m)}||\vec{\xi}| - 1/2\cdot(j+1)\pi)\} + \exp\{-i(|\vec{P}^{(m)}||\vec{\xi}| - 1/2\cdot(j+1)\pi)\}], \tag{39}$$

where $\Omega(\vec{v})$ is the solid angle of a vector $\vec{v}$. If (39) is substituted in (35) and the integration over $|\vec{P}^{(m)}|$ is performed, then we have for a large $|\vec{\xi}|$

$$f_{\mu}^{(n)}(\vec{r}\,o, \vec{\xi}\,o)_{IA}^{\text{C.M.S.}} \sim -(2\pi)^{-3} \sum_{j,k}{}' \sum_{M_{(n')},\, M_{(m)}}{}'$$

$$\times \int_{M_{(m)}}^{\infty} dk_0' \cdot k_0'\, \delta_-(\sqrt{M_{(n')}^2 + (k_0')^2} - M_{(m)}^2 - M_{(n)} + k_0')$$

$$\times |\vec{\xi}|^{-1} \exp\{i(\sqrt{(k_0')^2 - M_{(m)}^2}\,|\vec{\xi}| - 1/2\cdot(j+1)\pi\}\, (i)^j Y_{j,k}(\Omega(\vec{\xi})) \int d\Omega(\vec{P}^{(m)})$$

$$\times a_{\mu}^{(m)}[P^{(m)}]f^{(n')}(\vec{r}o)S^{(n'm\leftarrow n)}[P^{(n')}, P^{(m)}; P^{(n)}]Y_{j,k}^{*}(\Omega(\vec{P}^{(m)})), \tag{40}$$

with

$$P_{\mu}^{(n')}P_{\mu}^{(n')} = -M_{(n')}^{2}, \qquad \vec{P}^{(n')} = -\vec{P}^{(m)}, \qquad P_{0}^{(n')} = M_{(n)} - k_{0}',$$

$$P_{\mu}^{(m)}P_{\mu}^{(m)} = -M_{(m)}^{2}, \qquad P_{0}^{(m)} = k_{0}', \qquad |\vec{P}^{(m)}| = \sqrt{(k_{0}')^{2} - M_{(m)}^{2}}. \tag{41}$$

The expression (40) arises from the contribution of $|\vec{P}^{(m)}|$ for which the argument of $\delta(\sqrt{M_{(m)}^{2} + |\vec{P}^{(m)}|^{2}} - k_{0}')$ in the integrand of (35) vanishes. In order that this argument vanishes, k_{0}' must not be smaller than $M_{(m)}$. Next performing the k_{0}'-integration, we have for large $|\vec{\xi}|$

$$f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_{1}^{\mathrm{C.M.S.}} \sim -(2\pi)^{-1} \sum_{j,k}' \sum_{M_{(n')}, M_{(m)}}' (M_{(n)})^{-1}(M_{(n)} - k_{0}')k_{0}'|\vec{\xi}|^{-1}$$

$$\times \exp\{i(\sqrt{(k_{0}')^{2} - M_{(m)}^{2}}|\vec{\xi}| - 1/2\cdot(j+1)\pi)\} \cdot (i)^{j}Y_{j,k}(\Omega(\vec{\xi}))$$

$$\times \int d\Omega(\vec{P}^{(m)})\, a_{\mu}^{(m)}[P^{(m)}]f^{(n')}(\vec{r}o)S^{(n'm\leftarrow n)}[P^{(n')}, P^{(m)}; P^{(n)}]Y_{j,k}^{*}(\Omega(\vec{P}^{(m)})), \tag{42}$$

with the condition (41) and

$$k_{0}' = (2M_{(n)})^{-1}(M_{(n)}^{2} + M_{(m)}^{2} - M_{(n')}^{2}). \tag{43}$$

In the expression (42) the prime attached to $\sum'$ means to sum over those restricted $M_{(n')}$ and $M_{(n)}$ which satisfy the condition

$$M_{(n')} + M_{(m)} \leq M_{(n)}.$$

This restriction arises from the requirement that the value of k_{0}', for which the argument $\delta(\sqrt{M_{(n')}^{2} - M_{(m)}^{2} + (k_{0}')^{2}} - M_{(n)} + k_{0}')$ vanishes, must not be smaller than $M_{(m)}$. Those states of which the value of $M_{(m)}$ satisfies this condition are restricted to photon states, since any difference between two levels of the hydrogen atom, $M_{(n)} - M_{(n')}$, is smaller than the mass of the electron. Furthermore only the contributions from one-photon states can give the terms of the order $|\vec{\xi}|^{-1}$ at a large $|\vec{\xi}|$, because for many-photon states the sum over $M_{(m)}$ must be in fact replaced by the integration over $M_{(m)}$ and accordingly this integration gives the terms of the order higher than $|\vec{\xi}|^{-1}$ at a large $|\vec{\xi}|$.

In a similar way we have analysed $f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_{1\mathrm{R}}^{\mathrm{C.M.S.}}$ and $\sum_{i=\mathrm{II}}^{\mathrm{VIII}}{}' f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)_{i}^{\mathrm{C.M.S.}}$, with the result that they are of the order higher than $|\vec{\xi}|^{-1}$ at a large $|\vec{\xi}|$. (For $n=0$ they describe a part of the virtual photon clouds surrounding the hydrogen atom in the ground state.) Thus we finally get the asymptotic expression,

$$f_{\mu}^{(n)}(\vec{r}o, \vec{\xi}o)^{\mathrm{C.M.S.}} \sim -(2\pi)^{-3} \sum_{j,k}' \sum_{M_{(n')} < M_{(n)}}' \sum_{\substack{\text{photon} \\ \text{polarization}}}' (M_{(n)})^{-1}(M_{(n)} - w_{n'\leftarrow n})w_{n'\leftarrow n}$$

$$\times |\vec{\xi}|^{-1} \exp\{i(w_{n'\leftarrow n}|\vec{\xi}| - 1/2\cdot(j+1)\pi)\}(i)^{j}Y_{j,k}(\Omega(\vec{\xi})) \int d\Omega(\vec{P}^{(m)})$$

$$\times a_{\mu}^{(m)}[P^{(m)}]f^{(n')}(\vec{r}o)S^{(n'm\leftarrow n)}[P^{(n')}, P^{(m)}; P^{(n)}]Y_{j,k}^{*}(\Omega(\vec{P}^{(m)})), \tag{44}$$

where

$$P_\mu^{(n')}P_\mu^{(n')}=-M_{(n')}^2 \qquad \vec{P}^{(n')}=-\vec{P}^{(n)}, \qquad P_0^{(n')}=M_{(n)}-w_{n'\leftarrow n},$$
$$P_\mu^{(n)}P_\mu^{(n)}=0, \qquad P_0^{(n)}=w_{n'\leftarrow n}, \qquad w_{n'\leftarrow n}=(2M_{(n)})^{-1}(M_{(n)}^2-M_{(n')}^2). \tag{45}$$

The asymptotic expression (44) is composed only of outgoing photon waves leaving the hydrogen atom in the lower states.

2) $F_{\mu\nu}^{(n)}(x|y|||z, z')$

By noting the relation

$$\langle \Omega|N(\psi^e(x)\psi^p(y)A_\mu(z)A_\nu(z'))|\Psi^{(n)}\rangle$$
$$=\langle \Omega|T(\psi^e(x)\psi^p(y)A_\mu(z)A_\nu(z'))|\Psi^{(n)}\rangle-\langle \Omega|T(\psi^e(x)\psi^p(y))|\Psi^{(n)}\rangle D_{F\mu\nu}(z, z')$$

and by applying the Gell-Mann and Low's procedure to equation (20), we have

$$F_{\mu\nu}^{(n)}(x|y|||z, z')=\sum_{i=1}^{III}F_{\mu\nu}^{(n)}(x|y|||z, z')_i. \tag{46}$$

Here $F_{\mu\nu}^{(n)}(x|y|||z, z')_i\,(i=\mathrm{I, II, III})$ are given as follows;

$$F_{\mu\nu}^{(n)}(x|y|||z, z')_{\mathrm{I}}=F_{\nu\mu}^{(n)}(x|y|||z', z)_{\mathrm{II}}$$
$$=\int\cdot\int d^4x'\cdots d^4z'''K(x, y; \bar{x}', \bar{y}')Q_\rho(\bar{x}', \bar{y}'; x' y'; z'')K(x', y'; \bar{x}'', \bar{y}'')D_{F\rho\nu}(z'', z')$$
$$\times Q_\sigma(\bar{x}'', y''; x'', y''; z''')D_{F\sigma\mu}(z''', z)F^{(n)}(x'' y''|||), \tag{47}$$

$$F_{\mu\nu}^{(n)}(x|y|||z, z')_{\mathrm{III}}$$
$$=-\int\cdot\int d^4x'\cdots d^4z'''K(x, y; \bar{x}', \bar{y}')R_{\rho\sigma}(\bar{x}', \bar{y}'; x', y'; z'', z''')$$
$$\times D_{F\rho\mu}(z'', z)D_{F\sigma\nu}(z''', z')F^{(n)}(x'|y'|||), \tag{48}$$

where

$$R_{\rho\sigma}(\bar{x}', \bar{y}'; x', y'; z'', z''')$$
$$=\eta_e\iint d^4x''\,d^4\bar{x}''\,\Gamma_\sigma^e(\bar{x}', x''; z''')S_F^e(x'', \bar{x}')Q_\rho(\bar{x}'', \bar{y}'; x', y'; z'')$$
$$+\eta_p\iint d^4y''\,d^4\bar{y}''\,\Gamma_\sigma^p(\bar{y}', y''; z''')S_F^p(y'', \bar{y}'')Q_\rho(\bar{x}', \bar{y}''; x', y'; z'')$$
$$+Q_{\rho\sigma}(\bar{x}', \bar{y}'; x', y'; z'', z'''). \tag{49}$$

By using the co-ordinates

$$Z_\mu'=1/4\cdot(x_\mu+y_\mu+z_\mu+z_\mu'),$$
$$\xi_\mu=z_\mu-1/2\cdot(x_\mu+y_\mu), \qquad \xi_\mu'=z_\mu'-1/2\cdot(x_\mu+y_\mu), \tag{50}$$
$$r_\mu=x_\mu-y_\mu,$$

$F_{\mu\nu}^{(n)}$ can be written in the form

$$F_{\mu\nu}^{(n)}(x|y|||z, z')=\exp(iP^{(n)}Z')\cdot\sum_{i=1}^{III}f_{\mu\nu}^{(n)}(r, \xi, \xi')_i. \tag{51}$$

In a similar manner to $f_{\mu}^{(n)}$, we get for $f_{\mu\nu}^{(n)}(\overset{*}{r}o, \vec{\xi}o, \vec{\xi}'o)^{\mathrm{C.M.S.}}$ the asymptotic expression at large $|\vec{\xi}|$ and $|\vec{\xi}'|$,

$$f_{\mu\nu}^{(n)}(\overset{*}{r}o, \vec{\xi}o, \vec{\xi}'o)_{\mathrm{I}}^{\mathrm{C.M.S.}}\,(=f_{\nu\mu}^{(n)}(\overset{*}{r}o, \vec{\xi}'o, \vec{\xi}o)_{\mathrm{II}}^{\mathrm{C.M.S.}})$$

$$\sim (2\pi)^{-6}\sum_{j',k}{}'\sum_{j,k}{}'\sum_{M_{(n'l')}<M_{(n'')}}{}'\sum_{M_{(n'l')}<M_{(n)}}{}'\sum_{\substack{\text{photons}\\ \text{polarization}}}\int d\Omega(\vec{p}^{(m'')})\int d\Omega(\vec{P}^{(m)})$$

$$\times (w_{n'\leftarrow n}\cos\chi)^{-1}\left(\sqrt{M_{(n'')}^{2}+(w_{n''\leftarrow n})^{2}}-w_{n''\leftarrow n'}\right)(w_{n''\leftarrow n'})\,M_{(n)}^{-1}(M_{(n)}-w_{n''\leftarrow})\,w_{n''\leftarrow n}$$

$$\times|\vec{\xi}|^{-1}|\vec{\xi}'|^{-1}\exp\{i(w_{n''\leftarrow n'}|\vec{\xi}'|-1/2\cdot(j'+1)\pi\}\cdot\exp\{i(w_{n''\leftarrow n}|\vec{\xi}'|-1/2\cdot(j+1)\pi)\}$$

$$\times(i)^{j'+j}Y_{j',k'}(\Omega(\vec{\xi}'))\,Y_{j,k}(\Omega(\vec{\xi}))\,a_{\nu}^{(m')}[P^{(m')}]\,a_{\mu}^{(m)}[P^{(m)}]\,f^{(n'')}(\overset{*}{r}o)$$

$$\times Y_{j',k'}^{*}(\Omega(\vec{P}^{(m')}))\,Y_{j,k}^{*}(\Omega(\vec{P}^{(m)}))\,S^{(n''m'\leftarrow n')}[P^{(n'')},P^{(m')};P^{(n')}]$$

$$\times S^{(n'm\leftarrow n)}[P^{(n')},P^{(m)};P^{(n)}],\qquad(52)$$

where

$$P_{\mu}^{(n')}P_{\mu}^{(n')}=-M_{(n)}^{2},\quad \vec{P}^{(n')}=-\vec{P}^{(m)},\quad P_{0}^{(n')}=M_{(n)}-w_{n'\leftarrow n},$$

$$P_{\mu}^{(m)}P_{\mu}^{(m)}=0,\quad P_{0}^{(m)}=w_{n'\leftarrow n},\quad w_{n'\leftarrow n}=(2M_{(n)})^{-1}(M_{(n)}^{2}-M_{(n')}^{2}),$$

$$P_{\mu}^{(n'')}P_{\mu}^{(n'')}=-M_{(n'')}^{2},\quad \vec{P}^{(n'')}=-(\vec{P}^{(m)}+\vec{P}^{(m')}),\quad P_{0}^{(n'')}=\sqrt{M_{(n')}^{2}+(w_{n'\leftarrow n})^{2}}-w_{n''\leftarrow n'},$$

$$P_{\mu}^{(m')}P_{\mu}^{(m')}=0,\quad P_{0}^{(m')}=w_{n''\leftarrow n'},\quad w_{n''\leftarrow n'}=[2(w_{n'\leftarrow n}\cos\chi$$

$$+\sqrt{M_{(n')}^{2}+(w_{n'\leftarrow n})^{2}})]^{-1}(M_{(n')}^{2}-M_{(n'')}^{2}),\qquad(35)$$

$\chi=$ (angle between $\vec{P}^{(m)}$ and $\vec{P}^{(m')}$).

$$f_{\mu\nu}^{(n)}(\overset{*}{r}o, \vec{\xi}o, \vec{\xi}'o)_{\mathrm{III}}^{\mathrm{C.M.S.}}$$

$$\sim -(2\pi)^{-5}\sum_{j,k}{}'\sum_{j'k'}{}'\sum_{M_{(n'l')}<M_{(n)}}\sum_{\text{polarizations}}\int d\Omega(\vec{P}^{(m)})\int d\Omega(\vec{P}^{(m')})\int_{0}^{(M_{(n)}^{2}-M_{(n'')}^{2})/2M_{(n)})^{-1}}dP_{0}^{(m)}$$

$$\times P_{0}^{(m)}(M_{(n)}-P_{0}^{(m)}-P_{0}^{(m')})^{-1}(M_{(n)}-P_{0}^{(m)}+P_{0}^{(m)}\cos\chi)\,P_{0}^{(m')}$$

$$\times|\vec{\xi}|^{-1}|\vec{\xi}'|^{-1}\exp\{i(P_{0}^{(m)}|\vec{\xi}|-1/2\cdot(j+1)\pi\}\exp\{i(P_{0}^{(m')}|\vec{\xi}'|-1/2\cdot(j'+1)\pi)\}$$

$$\times(i)^{j+j'}Y_{j,k}(\Omega(\vec{\xi}))\,Y_{j',k'}(\Omega(\vec{\xi}'))\,a_{\mu}^{(m)}[P^{(m)}]\,a_{\nu}^{(m')}[P^{(m')}]\,f^{(n')}(\overset{*}{r}o)$$

$$\times Y_{j,k}^{*}(\Omega(\vec{P}^{(m)}))\,Y_{j',k'}(\Omega(\vec{P}^{(m')}))\,S^{(n'm_{1}m'\leftarrow n)}[P^{(n')},P^{(m)},P^{(m')};P^{(n)}],\qquad(54)$$

where

$$S^{(n'm_{1}m'\leftarrow n)}[P^{(n')},P^{(m)},P^{(m')};P^{(n)}]$$

$$=\int\cdot\int d^{4}\bar{r}\,d^{4}r\,d^{4}k^{(5)}\,d^{4}k^{(6)}\,f^{(n')}(r)\,a_{\rho}^{(m)}[P^{(m)}]\,a_{\sigma}^{(m')}[P^{(m')}]\cdot\exp\{i(k^{(5)}\bar{r}+k^{(6)}r)\}$$

$$\times R_{\rho\sigma}[P^{(n')},P^{(m)},P^{(m')},P^{(n)},k^{(5)},k^{(6)}]\cdot f^{(n)}(r),\qquad(55)$$

$$R_{\rho\sigma}(\bar{x},\bar{y};x,y;z,z')\equiv 2(\pi)^{-20}\int\cdot\int d^{4}k^{(1)}\cdots d^{4}k^{(6)}\,\delta^{4}(k^{(1)}+k^{(2)}+k^{(3)}-k^{(4)})$$

$$\times R_{\rho\sigma}(k^{(1)}, \cdots, k^{(6)}) \exp\{i(k^{(1)}\overline{X}+k^{(2)}z+k^{(3)}z'-k^{(4)}X+k^{(5)}\bar{r}+k^{(6)}r)\}, \tag{56}$$

and

$$P_\mu^{(n')}P_\mu^{(n')}=-M_{(n')}^2, \quad \vec{P}^{(n')}=-(\vec{P}^{(m)}+\vec{P}^{(m')}), \quad P_0^{(n')}=M_{(n)}-P_0^{(m)}-P_0^{(m')},$$
$$P_\mu^{(m)}P_\mu^{(m)}=P_\mu^{(m')}P_\mu^{(m')}=0, \quad P_0^{(m')}=[2(M_{(n)}-P_0^{(m)}+P_0^{(m)}\cos\chi)]^{-1}$$
$$\times(M_{(n)}^2-M_{(n')}^2-2P_0^{(m)}M_{(n)}), \tag{57}$$

$\chi=$ (angle between $\vec{P}^{(m)}$ and $\vec{P}^{(m')}$).

Thus $f_{\mu\nu}^{(n)}(\dot{r}\,o, \dot{\xi}\,o, \dot{\xi}'\,o)^{\text{C.M.S.}}$ are also composed only of outgoing photon waves leaving the hydrogen atom in the lower states.

3) We can continue such an analysis to the two-particle amplitudes associated with more and more photon co-ordinates $F_{\mu\nu\sigma\cdots}^{(n)}(x|\gamma||z, z', z''\cdots)$, with the consequence that they also contain only outgoing photon waves and no ingoing photon wave at large distances. It is essential here that they contain no incident wave, while the appearance of outgoing waves is natural since we are dealing with causal functions. Such a consequence, however, contradicts with the fact that those states we have considered are stationary states. This contradiction originates from the assumption (A) taken at the biginning of this section. In fact, if we abandon this assumption, (A), at the outset we have only an identity $0=0$ instead of (23), (46) and so forth, and have no contradiction. Thus we are led to the conclusion that the Bethe-Salpeter equation in its strict sense is unable to give any higher excited level of the hydrogen atom.

§ 4. Approximation

Since it is very difficult in practice to solve the Bethe-Salpeter equation (6) exactly, its approximate equation has been treated by many authors.[5)6)7)16)17)] Such an approximate equation may give the corresponding approximate excited levels in spite of the conclusion stated above. In this section it is shown that the coefficients of the asymptotic forms, (44), (52), (53), etc., when such an approximate two-particle amplitude given by such approximate equation are substituted in them, give an approximate knowledge on the spontaneous optical transition probabilities.

As an example we consider the ladder approximation and then take the limit of the proton mass $M_p\to\infty$. In the ladder approximation equation (6) is reduced to

$$F^{(n)}(x|\gamma|||)_{(\text{ladder})}=\gamma_e\gamma_p\int\!\!\int d^4x'\,d^4y'\,\overset{(0)}{S}{}_F^e(x, x')\overset{(0)}{S}{}_F^p(y, y')\tilde{\gamma}_\mu^e\tilde{\gamma}_\mu^p\overset{(0)}{D}{}_F(x', y')F^{(n)}(x'|\gamma'|||)_{(\text{ladder})}. \tag{58}$$

Furthermore in the limit $M_p\to\infty$ we may, according to Hayashi and Munakata[5], put

$$F^{(n)}(x|\gamma|||)_{\left(\substack{\text{ladder}\\ M_p\to\infty}\right)}=\varphi^{(n)}(x)\varphi^p(\gamma). \tag{59}$$

Then $\varphi^{(n)}(x)$ satisfies the equation

$$\varphi^{(n)}(x) = -\eta_e \int d^4x' \overset{(n)}{S}{}^e_F(x, x') \gamma_\mu^e A_\mu^{(p)}(x') \varphi^{(n)}(x'), \tag{60}$$

where

$$A_\mu^{(p)}(x) = i \int d^4y \overset{(0)}{D}{}_F(x, y) j_\mu^{(p)}(y), \tag{61}$$

$$j_\mu^{(p)}(x) = i \eta_p \overline{\varphi}^p(y) \gamma_\mu^p \varphi^p(y). \tag{62}$$

In the limit of $M_p \to \infty$, we may take the static approximation to

$$j_4^{(p)}(y) = i \eta_p \delta(\vec{y}), \qquad \vec{j}^{(p)}(y) = 0. \tag{63}$$

Then it follows that

$$A_4^{(p)}(x) = i \eta_e / (4\pi |\vec{x}|), \qquad \vec{A}^{(p)}(x) = 0 \tag{64}$$

and $\varphi^{(n)}(x)$ is an eigen-solution of the Dirac equation of the electron in the Coulomb field (64). Now let us consider the coefficients $S^{(n'm \leftarrow n)}[P^{(n')}, P^{(m)}; P^{(n)}]$ in the expression (44). It is easily shown from (32) and (37) that

$$\begin{aligned}
&(2\pi)^{-4} \delta^4(P^{(n')} + P^{(m)} - P^{(n)}) S^{(n'm \leftarrow n)}[P^{(n')}, P^{(m)}; P^{(n)}] \\
&= \int \cdot \int d^4x \cdots d^4z \bar{F}^{(n')}(\vec{x}|\vec{y}|\,|\,) \langle m | A_\mu(z) | \Omega \rangle Q_\mu(\vec{x}, \vec{y}; x, y; z) F^{(n)}(x y | \,) \\
&\equiv S.
\end{aligned} \tag{65}$$

Here Q_μ is the function defined by (19). The lowest approximation of Q_μ is given by

$$\begin{aligned}
Q_\mu(\vec{x}, \vec{y}; x, y; z)_{(\text{lowest})} &= \eta_e^2 \eta_p \delta(\vec{x} - z) \delta(\vec{y} - y) \gamma_\mu^e \overset{(n)}{S}{}^e_F(\vec{x}, x) \gamma_\nu^e \gamma_\nu^p \overset{(n)}{D}{}_F(x, y) \\
&+ \eta_e \eta_p^2 \delta(y - z) \delta(x - x) \gamma_\mu^p \overset{(0)}{S}{}^p_F(y, y) \gamma_\nu^e \gamma_\nu^p \overset{(n)}{D}{}_F(x, y) \\
&\equiv Q_\mu^e(\vec{x}, \vec{y}; x, y; z)_{(\text{lowest})} + Q_\mu^p(\vec{x}, \vec{y}; x, y; z)_{(\text{lowest})}.
\end{aligned} \tag{66}$$

Accordingly we get for the lowest approximation of S

$$\begin{aligned}
S_{(\text{ladder})} = \int \cdot \int d^4x \cdots d^4z \bar{F}^{(n')}(\vec{x}|\vec{y}|\,|\,)_{(\text{ladder})} \boldsymbol{a}_\mu^{(m)}(z) \\
\times Q_\mu(\vec{x}, \vec{y}; x, y; z)_{(\text{lowest})} F^{(n)}(x|y|\,|\,)_{(\text{ladder})},
\end{aligned}$$

where

$$\boldsymbol{a}_\mu^{(m)}(z) = \langle \Phi^{(m)}_{\text{photon}} | \boldsymbol{A}_\mu(z) | \Phi_0 \rangle. \tag{67}$$

In the limit of $M_p \to \infty$ we may neglect the contribution from $Q^p_{\mu\,(\text{lowest})}$. Then we have

$$\begin{aligned}
S_{\left(\substack{\text{ladder} \\ M_p \to \infty}\right)} = \int \cdot \int d^4x \cdots d^4z F^{(n')}(x|y|\,|\,)_{\left(\substack{\text{ladder} \\ M_p \to \infty}\right)} \boldsymbol{a}_\mu^{(m)}(z) \\
\times Q_\mu^e(\vec{x}, \vec{y}; x, y; z)_{(\text{lowest})} F^{(n)}(x|y|\,|\,)_{\left(\substack{\text{ladder} \\ M_p \to \infty}\right)}.
\end{aligned} \tag{68}$$

By virtue of (59), (62), (61) and (60) the expression (68) can be simplified as follows :

$$S_{\left(\substack{\text{ladder}\\ M_p\to\infty}\right)} = -\eta_e{}^2\int\!\!\int d^4x\,d^4x'\,\overline{\varphi}^{(n')}(x')\,\bar{a}_\mu^{(m)}(x')\,\gamma_\mu^{\,e}\overset{(\mathrm{II})}{S}{}_F^{\,e}(x',x)\,\gamma_\nu^{\,e}A_\nu^{(p)}(x)\,\varphi^{(n)}(x)$$

$$= \eta_e\int d^4x'\,\overline{\varphi}^{(n')}(x')\,\bar{a}_\mu^{(m)}(x')\,\gamma_\mu^{\,e}\varphi^{(n)}(x'). \qquad (69)$$

This expession (69) is just the matrix element of the spontaneous optical transition from the level (n) to the lower level (n') in the ordinary treatment.

Thus we can read the asymptotic expression (44) of $F_\mu^{(n)}(x|y|||)$ as follows : It is the sum of terms, each of which is a product of a two-particle amplitude $f^{(n')}$, an outgoing photon wave $a_\mu^{(m)}$ emitted on the occasion of the transition from the level (n) to a lower level (n') and the matrix element $S^{(n'm\leftarrow n)}$ of such a spontaneous emission. We can read the asymptotic expression of $F_{\mu\nu}^{(n)}(x|y|||z,z')$ also in a similar fashion. It is composed of two terms of different types, (52) and (54). First the expression (52) corresponds to the transitions in which two photons are *emitted stepwise* on the occasion of the transition of the hydrogen atom from the level (n) *through an intermediate* level (n') to a lower level (n''). On the other hand the expression (54) corresponds to the transitions in which two photons are *emitted at one act*[18] on the occasion of the transition of the hydrogen atom from the level (n) to a lower level (n') *without passing through* any intermediate level. $S^{(n'm'm\leftarrow n)}$ is the matrix element of such a transition.

§ 5. Concluding remarks

1) *The level shift and the life of the excited level*

From the preceding arguments, especially from (24), (44), (65), (47), (52), (48), (55) and (56), we can see that the function $Q_\mu(\bar{x},\bar{y};x,y;z)$ plays the role of the interaction between the hydrogen atom and a photon, $R_{\mu\nu}(\bar{x},\bar{y};x,y;z,z')$ plays the role of the direct interaction between the hydrogen atom and two photons, and so forth. Accordingly we can calculate the "self-energy of the hydrogen atom" $\Delta E_{(n)}$ by the expression

$$\begin{aligned}
&\delta(P^{(n')}-P^{(n)})\,\Delta E_{(n)}\\
&= -\int\cdot\int d^4x\cdots d^4z'\,\overline{F}^{(n')}(\bar{x}|\bar{y}|||)\,Q_\mu(\bar{x},\bar{y};x,y;z)\,D_{F\mu\nu}(z,z')\\
&\quad\times K(x,y;x',y')\,Q_\nu(x',\bar{y}';x',y';z')\,F^{(n)}(x'|y'|||)\\
&\quad+\int\cdot\int d^4x\cdots d^4z'''\,\overline{F}^{(n')}(\bar{x}|\bar{y}|||)\,Q_\mu(\bar{x},\bar{y};x,y;z)\,D_{F\mu\nu}(z,z')\\
&\quad\times K(x,y;\bar{x}',\bar{y}')\,Q_\nu(\bar{x}',\bar{y}';x',y';z')\,K(x',y';\bar{x}'',\bar{y}'')\,Q_\rho(\bar{x}'',\bar{y}'';x'',y'')\\
&\quad\times D_{F\rho\sigma}(z'',z''')\,K(x'',y'';\bar{x}''',\bar{y}''')\,Q_\sigma(\bar{x}''',\bar{y}''';x''',y''';z''')\,F^{(n)}(x'''|y'''|||)
\end{aligned}$$

$$+\int\cdots\int d^4x\cdots d^4z'''\,\overline{F}^{(n')}(\bar{x}|\bar{y}|||)\,Q_\mu(\bar{x},\bar{y};x,y;z)\,D_{F\mu\rho}(z,z'')$$
$$\times K(x,y;x',y')\,Q_\nu(x',y';x',y';z')\,K(x',y';\bar{x}'',\bar{y}'')\,Q_\rho(\bar{x}'',\bar{y}'';x'',y'';z'')$$
$$\times D_{F\nu\sigma}(z',z''')\,K(x'',y'';\bar{x}''',\bar{y}''')\,Q_\sigma(\bar{x}''',\bar{y}''';x''',y''';z''')\,F^{(n)}(x'''|y'''|||)$$
$$-\int\cdots\int d^4x\cdots d^4z'''\,\overline{F}^{(n')}(\bar{x}|\bar{y}|||)\,R_{\mu\nu}(\bar{x},\bar{y};x,y;z,z')\,D_{F\mu\rho}(z,z'')\,D_{F\nu\sigma}(z',z''')$$
$$\times K(x,y;x',y')\,R_{\rho\sigma}(x',y';x',y';z'',z''')\,F^{(n)}(x'|y'|||)$$
$$+\cdots. \qquad (70)$$

The level shift and the life of the approximate level (n) are given by the real and imaginary parts of $\Delta E_{(n)}$ respectively. This problem will be discussed more fully elsewhere.

2) *Heisenberg's theory*

Recently Heisenberg[10] has proposed an interesting attempt in which various elementary particles including a photon are derived from a single primary interaction of Fermi type by a new quantization method of non-linear field. This theory involves many interesting ideas on the structure of various interactions of elementary particles, though it is questionable whether the mathematical scheme of the theory has inner consistency.[2] In this theory mass spectra of elementary particles are determined from Bethe-Salpeter type equations which can be obtained in the "N-approximation", a sort of a modified Tamm-Dancoff approximation. If viewed from our present understanding, higher mass levels thus obtained must correspond, unless forbidden by selection rules, to unstable excited particles which decay into lighter ones. The structure of decay interactions in Heisenberg's theory may be able to be treated along the line of the thought described in the present paper.

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Electromagnetic Properties of Deuteron, II

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Electric quadrupole moment of the deuteron is calculated by the formalism proposed in the first paper[1] of this series and with the meson theoretical wave function of the deuteron. The result obtained agrees with the experimental evidence within the inevitable ambiguity involved in the present meson theory. The mesonic correction part of the quantity is found to be essentially due to the recoil effect, and therefore the calculated value is reliable only in a qualitative sense. However, the numerical calculation shows that this part amounts to only a few percent of the total value. This seems to be a further support to the discussions of Iwadare et al.[2] on the applicability of the meson theory of the nuclear force in the low energy region.

§ 1. Introduction

Recently, Fukuda, Sawada, and Taketani,[3] and Sato and the present author[4] have pointed out the importance of the normalization condition for the wave function in the meson theoretical treatment of the two-nucleon problems. For example, when one investigates the problem using the Tamm-Dancoff amplitudes, he must modify the customary formalism so as to take account of the normalization condition. In the calculation of the nuclear potential operator, etc., such a modification yields a new multiplicative factor corresponding to the dissociation of the constituent nucleons into the bare ones and the meson cloud. The nuclear potential operator based on such an idea (F-S-T potential) has already been derived.[3),5] In the paper I, Sato and the present author have also considered the interaction of the deuteron with the external electromagnetic field along the same idea and given the formulas for the magnetic dipole and the electric quadrupole moments of the deuteron. According to these formulas, the deuteron moments are given as the sum of the "customary" and the "mesonic correction" parts. The latter part of the quadrupole moment has been calculated in that paper using the T-D amplitudes as the correction to the value given by Brueckner and Watson.[6] However, those calculations were carried out taking into account only the *S*-state contribution, since the reliable information on the *D*-state amplitude of the deuteron was not available in those days. Therefore, the calculated value of the paper I cannot be taken seriously, but only as an example of the calculation. But the further developments have since been achieved in the domain of the meson theoretical analysis of the two-nucleon problems in the low energy region and nowadays it seems possible to draw out rather concrete information on the quadrupole moment of the deuteron from the meson theory.

In connection with the motivation of this research, we want further to notice the

following situation. In spite of the various defects of the present meson theory, it can also be expected for several reasons that this is still useful for certain problems. In fact, it proves to be sufficiently useful for the two-nucleon problems in the low energy region. It has been pointed out by Iwadare et al.[2] that if the hard core in the nuclear potential is assumed at $\mu r \approx 0.2$ and if the coupling constant is chosen as $f^2/4\pi \approx 0.08$, the nuclear force derived from the meson theory is capable to explain the features of the various experiments on the low energy nucleon-nucleon collisions, or, at least, any fact which seems to be inconsistent with the meson theoretical nuclear force has not yet been found experimentally in this energy region. Especially, they have shown that the numerical solution for the deuteron wave function can actually be constructed in this case and it leads to the quadrupole moment value which agrees excellently with the experimental data. However, it must be noticed here that, in their calculation of the quadrupole moment, the customary formula has been used so that the "mesonic correction" part was entirely omitted. Thus, the smallness of the latter part is necessary for the validity of their conclusions on the usefulness of the meson theoretical nuclear force.

On such a stand-point we calculate, in the present paper, the mesonic correction part of the quadrupole moment of the deuteron using the wave function of Iwadare et al.. As will be shown later, the mesonic correction part is essentially due to the recoil effect, so the result cannot be reliable in its quantitative details. However, the result of the calculation seems to show, at least, that the mesonic correction part is very small (only a few percent of the "customary" part) and accordingly it supports the conclusions of Iwadare et al.

In the paper I, there have been derived two equivalent formulas for the energy shift ϵ_1 of the deuteron due to the external electromagnetic field, namely,

$$\epsilon_1 = (\chi_0,\ U\ \chi_0)/(\chi_0,\ P\ \chi_0), \qquad (I\text{–}2.10)$$

and

$$\epsilon_1 = (\varphi_0,\ P^{-1/2}\ U\ P^{-1/2}\ \varphi_0) \qquad (I\text{–}2\cdot13)$$

with

$$\varphi_0 = P^{1/2}\ \chi_0.$$

In these two formulas, χ_0 and φ_0 have quite different physical contents: χ_0 is the T-D amplitude corresponding to the (unphysical) state where only two bare nucleons exist and is not normalized to unity, while φ_0 is normalized to unity and can be properly interpreted as the wave function. Thus, the quantity

$$(\chi_0,\ \chi_0) = (\varphi_0,\ P^{-1}\ \varphi_0)$$

is the probability for finding only two bare nucleons in the physical two-nucleon state described by φ_0. Since the aim of the calculation in I was to obtain the correction to the value of Brueckner and Watson for the quadrupole moment, the formula (I–2·10) [with the amplitude of B-W substituted for χ_0] has been used there. However, in the present paper we treat the wave function corresponding to F-S-T potential, hence we

must do with φ_0, and therefore the formula (I–2.13).

In the following sections, we shall present the outline of the calculations and their results (§ 2) and several comments on our subject (§ 3). Notations are the same as in the papers I and/or ref. 5), in which the reader will find also the details of the derivation and the calculations of the basic equations (10)∼(13) of the present paper.

§ 2. Outline of the calculations and their results

When an external electromagnetic field acts on our system, the Schroedinger equation to be considered is

$$(\epsilon - K - H_0 - H - He)\,\Psi = 0, \tag{1}$$

where ϵ and H_0 are the energy eigenvalue of the system and the Hamiltonian of the free pion field, respectively. As for the interaction Hamiltonian He between the system and the external field, it is sufficient for our purpose to take only the one between the nucleons and the electromagnetic field, and treat both of the proton and the neutron as the particles possessing the electric charge $e/2$[1),7)]. On reducing the Hamiltonian to the non-relativistic form by the Tani-Foldy transformations, we get the following expressions for the interaction Hamiltonian H between the nucleons and the pion field and the kinetic energy operator K of the nucleons:

$$H = H_\sigma + H_p + H_\pi \tag{2}$$

and

$$K = (\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2)/2M = (\boldsymbol{P}^2/4M) + (\boldsymbol{p}^2/M), \tag{3}$$

where

$$H_\sigma = -(f/\mu)\sum_{i=1,2}\tau_i\,\sigma_i\,\Gamma\varphi(\boldsymbol{r}_i), \tag{4}$$

$$H_p = -(1/2M)(f/\mu)\sum_i \tau_i\{(\boldsymbol{p}_i\cdot\sigma_i),\ \pi(\boldsymbol{r}_i)\}, \tag{5}$$

and

$$H_\pi = (1/2M)(f/\mu)^2\sum_i \pi(\boldsymbol{r}_i)^2. \tag{6}$$

For the calculation of the quadrupole moment, an inhomogeneous electric field is introduced as the external one, so that

$$He = -(e\alpha/4)(3r_z^2 - \boldsymbol{r}^2), \tag{7}$$

where $\boldsymbol{r}$ is the relative coordinate vector between two nucleons.

From the results of many investigations on the pion-nucleon problems, it seems now reasonable to introduce the extended model for the nucleon as the substitute for the various unknown effects such as those of the new unstable particles, etc. As the most reasonable model we choose here the source function of Gaussian type which has also been adopted in the derivation of the F-S-T potential by Inoue et al..[4] That is, in

the actual calculations, we introduce the following cut-off factor $C(k)$ [in the momentum representation] into the interaction Hamiltonian H between the nucleons and the pion field:

$$C(k)=\exp(-k^2/2k^2_{max}), \qquad k_{max}=6\mu. \tag{8}$$

Let us assume that, corresponding to eq. (1), the Schroedinger equation for our system in absence of the external field is

$$(\epsilon_0-K-H_0-H)\Psi_0=0, \tag{9}$$

then the energy shift ϵ_1 of our system due to the external field is given by[1]

$$\epsilon_1=\epsilon-\epsilon_0=(\varphi_0,\ P^{-1/2}\ U\ P^{-1/2}\ \varphi_0), \tag{10}$$

where φ_0 is the correctly normalized wave function of our system (deuteron) in the case of no external field. The operators P and U are respectively given by

$$P=<b^*(\epsilon_0)\,b(\epsilon_0)> \tag{11}$$

and

$$U=<b^*(\epsilon_0)\,Heb(\epsilon_0)> \tag{12}$$

where the notation $<\ >$ means the expectation value in the meson vacuum, and

$$b(\epsilon_0)=[1-(\epsilon_0-H_0-K)^{-1}(1-v)\,H]^{-1}, \tag{13}$$

v denoting the projection operator to the meson vacuum. The conventional formula for the quadrupole moment of the deuteron,

$$Q_0=(1/\mu^2)[(\sqrt{2}/10)\int_0^\infty x^2\ uwdx-(1/20)\int_0^\infty x^2\ w^2\ dx], \tag{14}$$

corresponds to eq. (10) with its right-hand side replaced by

$$(\varphi_0,\ He\ \varphi_0). \tag{14'}$$

(14) is the equation derived by taking into account merely the existence of the D-state admixture in the deuteron ground state, and the difference between (10) and (14′) can be interpreted as the mesonic correction.

Now, it can easily be seen that this mesonic correction part is essentially due to the recoil effect: When the recoils of the nucleons are entirely neglected, H reduces to H_0 only and also K is omitted. Therefore, $b(\epsilon_0)$ becomes commutable with He and thus

$$U=He\ P=P\ He \quad \therefore\ \epsilon_1=(\varphi_0,\ He\ \varphi_0).$$

That is, in the static approximation, the mesonic correction to the quadrupole moment vanishes independently of the power series expansion in the coupling constant.*

*) For the case of the magnetic moment, this does not generally hold, since He is then spin-dependent. Thus, also in the static approximation, the spin-flip effects still remain. However, in the approximation, where the fourth and higher order terms in the coupling constant are neglected, the static approximation yields eventually the vanishing correction to the magnetic moment. At any rate, the mesonic correction parts are due to the higher order and/or the recoil effects.

Because of such a circumstance, it is not permitted to neglect entirely the recoils of the nucleons in our calculation. In this paper, following I, we shall regard the order of the magnitude of K as corresponding to f^2 and therefore H_ν and H_π to f^3 and f^4, respectively. In the actual calculations, we shall expand U and P up to the terms whose orders of magnitudes are regarded, in such a sense, as corresponding to ef^4 and f^4, respectively*).

Though the operators U and P do not involve any divergences owing to the introduction of the cut-off factor $C(k)$, it would be yet reasonable to perform the renormalization procedures. In this paper, we shall follow the prescription proposed in the papers I and ref. 5)**).

The calculations are quite straightforward. The expression for P has already been given in ref. 4). The resulting form of U is

$$U=(1/2)\{He,\ P\}+U_1 \qquad ([He,\ P]=0), \tag{15}$$

and therefore, we get for the quadrupole moment Q of the deuteron

$$Q=Q_0+Q_1 \tag{16}$$

with

$$Q_0=-(1/e\alpha)(\varphi_0,\ He\ \varphi_0) \tag{17}$$

and

$$Q_1=-(1/e\alpha)(\varphi_0,\ P^{-1/2}\ U_1\ P^{-1/2}\ \varphi_0). \tag{18}$$

The operator $P^{-1/2}\ U_1\ P^{-1/2}$ is calculated by the method of the paper I, and the expression of U_1 will be given in Appendix.

According to ref. 2), the numerical wave function of Iwadare et al. (which corresponds to the F-S-T potential) can be excellently approximated by the functions

$$\begin{aligned} u&=1.039\exp(-0.328\,x)-1.392\exp(-2\cdot360\,x),\\ w&=0.02624\{1+3/(0.328\,x)+3/(0.328\,x)^2\}\exp(-0.328\,x)\\ &\quad-(1.298/x^2)\exp(-0.962\,x) \qquad \text{for the region } x\geq 1 \end{aligned} \tag{19}$$

the quadratic functions which join smoothly to the ones given above at $x=1$, and vanish at $x=0.2$ for the region $0.2\leq x\leq 1$. $(x=\mu|\boldsymbol{r}|)$

*) There would be another method of approximation where, for example, the operators are expanded first up to the terms of the second order in the coupling constant, and next up to the first order in μ/M. However, this approximation method also yields almost the same numerical result as ours. The formulas (15) and (A·1) of the present paper still hold formally for this case.

**) Provided it concerns only the renormalization of the coupling constant, the prescription $(1+2\lambda-P_2')\times f^2=f_r^2$ of the paper I is effectively the same as that of Chew[8]. Also, it must be noticed that the mass renormalization procedure of I has no effect on the present problem. However, as a whole, these two methods are slightly different from each other, and the method of Chew leads to $Q_1\approx 3\times10^{-29}\,cm^2$ instead of (21) of the text. But, such a small difference between the results of these two methods is quite unimportant for the present problem, and our conclusion is unaltered at all.

Here, u and w are the S- and D-radial parts, respectively:

$$\varphi_0(x) = (1/\sqrt{4\pi})\,(1/x)\,[u(x) + (1/\sqrt{8})\,S_{12}\,w(x)]\chi_{11}, \tag{20}$$

χ_{11} being the spin function. This is normalized to unity within an error of the order $< 1\%$, and leads to

$$Q_0 \approx 2.70 \times 10^{-27}\ \text{cm}^2.$$

(The normalization integral

$$\int |\varphi_0(x)|^2\ d^3x$$

is somewhat larger than unity.) We use this analytic form for the wave function φ_0 and obtain the following result:

$$Q_1 \approx 4 \times 10^{-29}\ \text{cm}^2. \tag{21}$$

Thus, the "mesonic correction" part, Q_1, amounts to only a few percent of the "customary" part, Q_0.

§ 3. Discussions

First, we should notice the difference of the T-D amplitude and the wave function in the usual sense. As F-S-T[3] have pointed out, the "potential operator" term in the Schroedinger equation for the T-D amplitude χ_0 in eq. (I–2·10) is not guaranteed to be hermitian, while this is the case for φ_0. Hence, it is not reasonable to substitute the usually accepted wave function for the T-D amplitude χ_0. When the calculation is carried out aiming to test the meson theoretical wave function corresponding to F-S-T potential or the phenomenological one corresponding to the hermitian nuclear potential (in its proper meaning), not the quantity χ_0 but φ_0 must be used for the sake of consistency.

Next, we have seen that the mesonic correction part Q_1 of the quadrupole moment is essentially the recoil correction. In the problem of the recoil effect, the behavior of the meson cloud, etc., near the nucleon core plays an important role, which cannot yet be treated properly by the present meson theory. As for the nuclear potential or the wave function, no device is known more reasonable than representing phenomenologically the behavior at small distances by the hypothetical hard core, while, in the strict sense, the calculated operators such as dissociation probability, etc., are valid only for $x \cdot 0.5 \sim 0.7$. Therefore, we cannot assert the quantitative validity of our result for Q_1. However, in so far as it concerns a subject such as the quadrupole moment in which the effects from comparatively large distances would contribute to a tolerable extent, our calculation would be useful for the qualitative discussions or estimation of the order of magnitude of the quantity in question. Hence, our very small value for Q_1 would be satisfactory in the sense that it supports the validity of the analysis in ref. 2) on the usefulness of the meson theoretical nuclear potential in the low energy region.

For checking of the meson theory of nuclear force, it would also be interesting to

investigate the magnetic moment of the deuteron. Thus far, the D-state probability p_D of the deuteron ground state was taken to be about 4%. This value is, however, the one derived to explain the magnetic moment anomaly of the deuteron by supposing it entirely due to the existence of the D-state admixture. Therefore, it cannot be taken seriously and would change to a certain extent by introducing the mesonic correction term. For example, in the wave function of Iwadare et al.[2], p_D is 6.4% and this of course does not necessarily mean the break-down of their wave function. For completeness, it is necessary to test whether or not their wave function gives the correct value for the deuteron magnetic moment. However, since the considerably large part of the magnetic moment would be affected by the behavior of the wave function at small distances, the situation is much more complicated and ambiguous than in the case of the quadrupole moment. It would be impossible or insignificant even to estimate the order of magnitude of the magnetic moment at the present stage of the meson theory, and accordingly we did not touch upon this problem.

The problem on the magnetic dipole and electric quadrupole moments of the deuteron has been investigated by several authors.[7),9),10),11)] Among them, the works by Deser and Villars[9)] will not be mentioned here, since they have already been criticized in the paper I. Sugawara[10)] has investigated the magnetic moment using the T–D amplitude and concluded that the D-state probability of the deuteron ground state is less than 4%. However, for the reasons mentioned above, his conclusion can hardly be accepted. The present meson theory is too helpless about the small distance effects to draw out the reliable conclusions for such a delicate problem. Furthermore, it should be noted that he has used the phenomenological wave function for the T-D amplitude. Hence, even if this were eventually to be permitted, the quantity calculated by him is the D-state probability of the T–D amplitude, which is never the same as that of the proper wave function. In the paper of Sessler[11)], the quadrupole moment is calculated up to the second order by means of the various meson theories and using the T-D amplitude. However, he has taken the conventional expression (14) for the quadrupole moment as corresponding to $(\chi_0, He\ \chi_0)$ in our notations, and thus substituted the usual wave function for χ_0*. Such an interpretation of Q_0 would be inconsistent in the connection of the meson theoretical wave function with the phenomenological one for the reasons mentioned in the beginning of this section.

It would be interesting to take account of the effects of the new unstable particles, such as K-mesons, to the quadrupole moment. However, in the present paper, we have supposed that these virtual effects are included phenomenologically in the hard core and the source function (cut-off factor inserted to the interaction Hamiltonian), and have not explicitly considered them.

The author wishes to express his cordial thanks to Professor K. Nakabayasi for his kind interest shown to this work, and to Professor I. Sato for his discussions all over

*) Because of such an interpretation of Q_0 [Q_{phen} in his notation], his definition of the "correction" to the quadrupole moment is different from ours. This is the reason for that, in spite of the static approximation, his "correction" did not vanish.

this research. The author is also indebted very much to Drs. S. Otsuki, W. Watari, J. Iwadare, and R. Tamagaki as for the wave function of the deuteron, and to Professor S. Machida and Mr. K. Inoue for sending to him the numerical tables of the operator $P[(P_0 J^+ JP_0)$ in ref. 4)], which was used in the numerical computations of this paper.

Appendix

U_1 in eq. (15) of the text is given by

$$\begin{aligned}U_1=&[(3r_z^2-\boldsymbol{r}^2)/\boldsymbol{r}^2](\mu/2M)(e\alpha/\mu^2)(f_\nu^2/4\pi)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\\&\times[(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\{(1/x)f_{34}+(3/x^2)g_{24}-(3/x^3)f_{14}\}\\&+\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})/\boldsymbol{r}^2\}\{(9/x^3)f_{14}-(9/x^2)g_{24}-(4/x)f_{34}+g_{44}\}]\\&-(\mu/2M)(e\alpha/\mu^2)(f_\nu^2/4\pi)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})/\boldsymbol{r}^2\}\\&\times[2\{(1/x)f_{13}+(3/x^2)g_{24}-(3/x^3)f_{11}\}(3r_z^2/\boldsymbol{r}^2)\\&\qquad-\{g_{22}-(1/x)f_{12}\}]\\&+(\mu/4M)(e\alpha/\mu^2)(f_\nu^2/4\pi)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)[(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})\sigma_{2z}/|\boldsymbol{r}|+(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})\sigma_{1z}/|\boldsymbol{r}|]\\&\times(3r_z/|\boldsymbol{r}|)[2\{(1/x)f_{34}+(3/x^2)g_{24}-(3/x^3)f_{14}\}-\{g_{22}-(1/x)f_{12}\}],\end{aligned}\tag{A·1}$$

with

$$f_{jm}(x)=(2/\pi)\mu^{m-(j+1)}\int_0^\infty dk\,\frac{k^j\sin k|\boldsymbol{r}|}{(k^2+\mu^2)^{m/2}}C^2(k)$$

and

$$g_{jm}(x)=(2/\pi)\mu^{m-(j+1)}\int_0^\infty dk\,\frac{k^j\cos k|\boldsymbol{r}|}{(k^2+\mu^2)^{m/2}}C^2(k).\tag{A·2}$$

As the spin factor in the last term of eq. (A·1) is commutable with $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, the existence of such a term does not cause any difficulty in rationalizing[15] the operator $P^{-1/2}$. In the case of absence of the cut-off factor [i.e., the case $C(k)=1$], this last term vanishes identically and the second term reduces to

$$\begin{aligned}&(\mu/2M)(e\alpha/\mu^2)(f_\nu^2/4\pi)\{(3r_z^2-\boldsymbol{r}^2)/\boldsymbol{r}^2\}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\\&\qquad\times[(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})/\boldsymbol{r}^2]\{g_{22}-(1/x)f_{12}\},\end{aligned}$$

with the factors $C(k)$ in g_{22} and f_{12} replaced by unity.

U_1 involves only the functions f_{jm} and g_{jm} with even m. [P does involve those with odd m.] These functions with even m can be obtained by adequate differentiations of the formula

$$\begin{aligned}\int_0^\infty\frac{\cos kr}{k^2+\mu^2}\exp(-k^2/k_{max}^2)\,dk=&(\sqrt{\pi}/2\mu)\exp(\mu^2/k_{max}^2)\\\times&\left[\exp(-\mu r)\int_\alpha^\infty\exp(-v^2)\,dv+\exp(\mu r)\int_\beta^\infty\exp(-v^2)\,dv\right],\end{aligned}\tag{A·3}$$

with

$$\alpha = (\mu/k_{max}) - (1/2)k_{max}\, r$$

and

$$\beta = (\mu/k_{max}) + (1/2)k_{max}\, r.$$

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On the Spins and Decay Interactions of $K_{\mu 3}$ and K_{e3}[1)]

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The decay events $K_{\mu 3}$ and K_{e3} are investigated using the Lorentz invariance requirements as the restriction for the transition matrix, assuming that the weak Fermi interactions are participating in the decays. Fermi interaction concerned here is that realized among $\Lambda(\Sigma)$, N, $\mu(e)$ and ν. The study of the experimental data in the light of the present analysis shows that K_{e3} seems to have spin 0 and the tensor type of Fermi interaction is necessary, while for $K_{\mu 3}$ we can not draw any definite conclusion at present.

§ 1. Introduction

Amongst the newly discovered particles, there is a group of particles characterized as K-particles. These particles have peculiar common features; i. e., their masses and life-times are almost equal to each other. Nothing is known of their genealogy, for example it is not known which of these particles are indeed identical. Hence it would be necessary to systematize the K-particles in order to go further investigation of the new particles.

The K-particles have a colorful variety of decay modes, and moreover some of them decay into leptons. They are christened according to each decay mode, but it is improbable that there exist so many different particles as decay modes. It may be more reasonable, considering the observed equality of K-particles in life-time and mass, to think that only one or two different particles exist and that they decay in various ways.

As to the weak decay interaction there have been observed several weak interactions, some of which possess a universal character to some extent.[2),3)] They are phenomenologically termed weak Fermi interaction, weak Boson-Fermion interaction and weak Boson-Boson interaction. A definite conclusion has not yet been reached concerning which is the primary interaction in these weak interactions, or whether all or some of them coexist in nature as primary interactions.[4)] Yet the weak Fermi interaction is the most promising one if we aim at a unified description of the weak interactions in terms of only one interaction.[5),6)]

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For exploring the mutual relations among various K-particles, and also for the investigation of the whole scheme of the interaction of the elementary particles, it is important to determine the spin, parity and other attributes of each K-particle and to make clear the interaction of these K-particles.

As for the spins and parities of the K-particles it is supposed at present that those of the τ-meson are 0^- or 2^-,[7] 0^- being more likely than 2^-, and those of θ^0 particle are probably 0^+ but could be 1^-.[8] As to other K-particles ($K_{\mu 2}$, $K_{\mu 3}$, and K_{e3}) no attempts to examine their spins or parities have been presented so far.

For the purpose of the spin and parity determination it may be desirable to use only kinematics in the Lorentz space and/or the isobaric spin space irrespective of the details of the dynamics. The investigation of the properties of K-particles through the strong interactions seems not so fruitful at present, partly because of the scantiness of experiments and partly because of the difficulties in the treatment of the strong interaction. Rather there happen the cases in which the study of weak interactions are suitable for such purpose. The study of weak interactions themselves is also a task which remains to be done. In the investigation of the interactions, weak or strong, it is desirable to use only the well established frameworks such as the Lorentz invariance requirement. This is just the way taken by Dalitz[9] in his analysis of $\tau(\rightarrow 3\pi)$.

However, the Lorentz invariance requirements are generally not so restrictive for the determination of the interaction, and some view points or some models of the interaction are required to make the complicated situations simpler.

We show in this paper that it is possible to obtain some information about spins and decay interactions of $K_{\mu 3}$ and K_{e3} by an analysis using the Lorentz invariance requirements under the assumption of *primary Fermi interaction** for the decay interaction. $K_{\mu 3}$ and K_{e3} have been left untouched up to now, while the investigation of $K_{\mu 3}$ and K_{e3} might give the clues to surmount the present Nishijima, Gell-Mann and Pais[10] scheme of elementary particles which does not include the lepton family.[11]

We investigate in the following the decay of $K_{\mu 3}$ and K_{e3} through the energy spectra of their charged secondaries. We can reasonably assume the decay modes: $K_{\mu 3}\rightarrow\mu+\nu+\pi^0$ [12] and $K_{e3}\rightarrow e+\nu+\pi^0$ for $K_{\mu 3}$ and K_{e3} respectively. Our interest here is in their spins and types of interactions participating in the decays.

The starting-points which we take in this analysis are summarized as:

(I) We use only the Lorentz invariance requirements as the restriction for the transition matrix (and gauge invariance if necessary).

(II) For the decay modes of $K_{\mu 3}$ and K_{e3} we assume $K_{\mu 3}\rightarrow\mu+\nu+\pi^0$ and $K_{e3}\rightarrow e+\nu+\pi^0$ respectively.

(III) Regarding the mechanism of the decay we take the standpoint of the weak Fermi interaction,* that is, K-particles decay via baryon loops by the weak Fermi interaction (See Fig. 1).

* From the standpoint of the primary Fermi interaction for the decay, a particle decays by the cooperation of strong interactions and weak Fermi interaction which acts once for all.

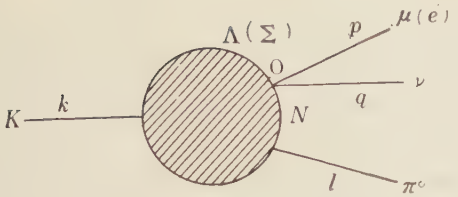

Fig. 1a. The diagram for the decay of $K_{\mu}(e)_3$. Fermi interaction is acting at O.

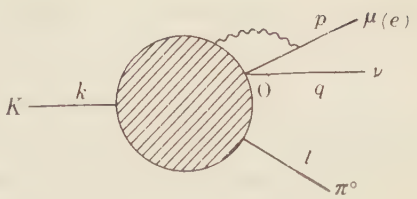

Fig. 1b. One of the higher order corrections which are omitted in the present consideration. The waved line denotes the photon.

Under the above three assumptions we intend to decide the types of the interactions at point O in Fig. 1 as well as the spins of the K-particles by computing the energy spectra of the secondary particles. (The present analysis gives no information about the parity of the K–particles owing to the vanishingly small mass of the neutrino.) In the next section we derive the transition matrix for the decay process by using only the Lorentz invariance requirements, and in section 3 we give the energy spectra of the secondary particles calculated with the matrix elements given in section 2. Section 4 is devoted to discussions of the experimental results in the light of the present analysis. In section 5 we examine the validity of our analysis developed in this paper.

§ 2. Derivation of matrix elements

In this section we construct the transition matrix for the decay process $K_{\mu(e)3} \rightarrow \mu(e)+\nu+\pi^0$ depicted in Fig. 1. In general the transition matrix for this process can be written with the covariance requirements as

$$\boldsymbol{M}=(2\pi)^4\delta^4(k-p-q-l)\bar{\psi}^{\mu}(p)O_j\psi^{\nu}(q)\phi^*(l)\Psi_i(k)A^{i(j)}, \tag{1}$$

where $\Psi_i(k)$, $\psi^{\mu}(p)$, $\psi^{\nu}(q)$ and $\phi(l)$ are the wave functions of K-meson, muon (electron), neutrino, and pion respectively. The O_j's are expressed in terms of ordinary Dirac matrices. Here the $A^{i(j)}$'s are certain functions of the four-momenta p, q and k with appropriate transformation property depending on the matrices O_j's and the types of wave functions of the fields under consideration. The $A^{i(j)}$'s are the contribution from all the possible virtual states. We use the Fermi interaction to restrict the form of A. Then the four-momenta p and q of muon (electron) and neutrino appear in A, only in the combination $p+q$ if we omit the contribution from some types of higher order radiative corrections such as shown in Fig. 1b, which gives small correction at least of the order of $(e^2/\hbar c=1/137)$, since there is no strong interaction for the lepton family.

For the expression A we can write as follows, taking $O_j=\gamma_5\gamma_{\mu}$ for the vector K–meson as an example :

$$\begin{aligned} A^{i(j)} \equiv A^{\Gamma(P\Gamma)} &= \delta_{\mu\nu}F^{\Gamma(P\Gamma)}+k_{\mu}(p+q)_{\nu}F^{\Gamma(P\Gamma)'} \\ &= K_{\Gamma(P\Gamma)}F^{\Gamma(P\Gamma)}+K_{\Gamma(P\Gamma)'}F^{\Gamma(P\Gamma)'}. \end{aligned} \tag{2}$$

In equation (2), the F's must be the scalar functions of the four-momenta p, q, and k, and

we call them effective coupling constants. They may be written as

$$F=a+(k_\mu(p+q)_\mu/M^2)b+\cdots\cdots \tag{3}$$

where a, b,$\cdots$ are constants, and M is the baryon mass which appears in the Feynman diagram of decay via baryon loops.

In the following calculations we approximate the expression (2) by taking only the terms of the lowest order in the momenta. This involves two kinds of approximation.

a) F is approximated by a constant.

b) When K_α and K_α' depend on momenta in different manners, and K_α is of lower order in momenta than K_α', only K_α is retained. About the validity of these approximations we shall discuss in section 5, so we merely note that our interest in this analysis is not the absolute value of the matrix elements, but the energy distribution of the secondary particles, and this fact must always be kept in mind in discussing the validity of our analysis. In Table I we list the possible forms of $K_\alpha F^\alpha$ which are of the lowest orders in the four momenta p, q and k, assuming spin 0 and 1 for the K-particles. We omit the possibility of higher spins for K-particle from our aesthetics on nature,* although some comments will be given in section 5.

Table I. Possible form of $K_\alpha F^\alpha$

Type of Fermi int. (O_j) / Type of K-meson	$S(I)$	$V(\gamma_\mu)$	$T(\gamma_\mu\gamma_\nu)$	$A(\gamma_5\gamma_\mu)$	$P(\gamma_5)$
S	forbidden	forbidden	$k_\mu(p+q)_\nu F^{S(T)}$	$k_\mu F^{S(A)}$ $(p+q)_\mu F^{S(A)'}$	$F^{S(P)}$
V	forbidden	$\varepsilon_{\mu\nu\lambda\rho}k_\mu(p+q)_\nu$ $\times F^{V(1)}$	$\delta_{\mu\lambda}k_\nu F^{V(T)}$ $\delta_{\mu\lambda}(p+q)_\nu F^{V(T)'}$	$\delta_{\mu\nu}F^{V(A)}$	$(p+q)_\mu F^{V(P)}$
PV	$(p+q)_\mu F^{PV(S)}$	$\delta_{\mu\nu}F^{PV(V)}$	$\delta_{\mu\lambda}k_\nu F^{PV(T)}$ $\delta_{\mu\lambda}(p+q)_\nu F^{PV(T)'}$	$\varepsilon_{\mu\nu\lambda\rho}k_\mu(p+q)_\nu$ $\times F^{PV(A)}$	forbidden
PS	$F^{PS(S)}$	$k_\mu F^{PS(V)}$ $(p+q)_\mu F^{PS(V)'}$	$k_\mu(p+q)_\nu F^{PS(T)}$	forbidden	forbidden

§ 3. Energy spectrum of the secondary particles

Throughout this paper we use the units $\hbar=c=1$. The transition probability per unit energy range is easily obtained from the matrix elements by the well-known formula,

$$P(E)\,dE=|\boldsymbol{M}_{if}|^2\rho'(E),$$

where $\rho'(E)$ is the final state density. In this section we give the expressions for the energy spectrum calculated by the matrix elements in the preceding section. Generally

* If we made a substantial consideration about the structure of the K-particles such as composite model,[13] it would appear to be unphysical that the K-particle with higher spin exists more stably than the one with 0 or 1 spin, and there is a report[8] which does not favor higher spin for K-particle.

speaking, the transition matrix may be a linear combination of the various terms tabulated in Table I, but for the convenience of analysis we give the spectrum for each of the terms in Table I. The spectra are given for scalar and vector K-meson. The pseudoscalar and pseudovector K-meson give the same spectra as those of scalar K and the vector K respectively, but the types of the Fermi interaction must be altered to the corresponding allowed types, and also a similar situation occurs if we take the parity of the muon (electron) opposite to that of the neutrino.

The calculated spectra are:
for scalar K-meson,

$$\left.\begin{aligned}
P^{S(F)}(E) &= (F^{S(F)2}/16\pi^3)\, m_k (m_k I_3 - I_2)\,(W-E)^2 \\
P^{S(A)}(E) &= (F^{S(A)2}/16\pi^3)\, m_k^3 \{-(m_k-2E) I_3 + I_2\}\,(W-E)^2 \\
P^{S(A)'}(E) &= (F^{S(A)'2}/16\pi^3)\, m_k m_\mu^2 (m_k I_3 - I_2)\,(W-E)^2 \\
P^{S(T)}(E) &= (F^{S(T)2}/16\pi^3)\, m_k^3 [m_k^3 I_5 (W-E)^4 \\
&\quad - m_k \{m_k(W+E) - 2m_\mu^2\} I_4 (W-E)^3 \\
&\quad + m_k \{2EW - E^2 - m_\mu^2\} I_3 (W-E)^2 - (E^2-m_\mu^2) I_2 (W-E)^2]
\end{aligned}\right\}, \tag{4}$$

and for vector K-meson,

$$\left.\begin{aligned}
P^{V(F)}(E) &= (F^{V(F)2}/48\pi^3)\, m_k [m_k^3 I_5 (W-E)^4 - m_k^2 \{W-3E+2m_k\} I_4 \times \\
&\quad \times (W-E)^3 + m_k \{E^2 - m_\mu^2 + (2m_k - E)(W-E)\} I_3 \times \\
&\quad \times (W-E)^2 - \{E^2 - m_\mu^2 + m_k (W-E)\} I_2 (W-E)^2] \\
P^{V(A)}(E) &= (F^{V(A)2}/48\pi^3)\, m_k \{(m_k+2E) I_3 - I_2\}\,(W-E)^2 \\
P^{V(T)}(E) &= (F^{V(T)2}/48\pi^3)\, m_k^3 \{(4E-m_k) I_3 + I_2\}\,(W-E)^2 \\
P^{V(T)'}(E) &= (F^{V(T)'2}/48\pi^3)\, m_k [m_k^3 I_5 (W-E)^4 \\
&\quad + m_k \{2(m_k^2 + m_\mu^2) - m_k (W+E)\} I_4 (W-E)^3 \\
&\quad + m_k \{4m_k E + 2WE - 4m_k W - E^2 + 2m_\mu^2\} I_3 \times \\
&\quad \times (W-E)^2 + \{-E^2 - 2m_\mu^2 + 2m_k (W-E)\} I_2 (W-E)^2] \\
P^{V(V)}(E) &= (F^{V(V)2}/48\pi^3)\, m_k^3 [m_k^3 I_5 (W-E)^4 - m_k^2 \{W-3E+2m_k\} I_4 (W-E)^3 \\
&\quad + m_k \{E^2 - m_\mu^2 + (2m_k - E)(W-E)\} I_3 (W-E)^2 \\
&\quad - \{E^2 - m_\mu^2 + m_k (W-E)\} I_2 (W-E)^2]
\end{aligned}\right\}. \tag{5}$$

As is seen in eqs. (4) ~ (5), $P^{S(A)'}$ and $P^{V(T)'}$ can be given only by altering effective coupling constants of $P^{S(T)}$ and $P^{V(T)}$ respectively, and we omit them in the following discussions. We shall also give here the spectra due to the interference terms among the interactions in Table I for further study.

They are, for scalar K-meson,

$$
\begin{aligned}
P^{S(P,A)} &= (Re[iF^{S(P)}F^{S(A)*}]/8\pi^3)\, m_k^2 m_\mu I_3 (W-E)^2 \\
P^{S(A,T)} &= (Re[iF^{S(A)}F^{S(T)*}]/8\pi^3)\, m_k^3 m_\mu [-m_k I_4 (W-E)^3 \\
&\quad + (m_k - E) I_3 (W-E)^2 - I_2 (W-E)^2] \\
P^{S(P,T)} &= (Re[F^{S(P)}F^{S(T)}]/8\pi^3)\, m_k^2 [m_k^2 I_4 (W-E)^3 \\
&\quad + (-m_k E + m_\mu^2) I_3 (W-E)^2 + E I_2 (W-E)^2]
\end{aligned}
\qquad (6)
$$

$P^{a(b,c)}$ means the interference between the b and c type interactions for the a type K-meson. For vector K-meson

$$
\begin{aligned}
P^{V(P,A)} &= (Re[iF^{V(P)}F^{V(A)*}]/24\pi^3)\, m_k m_\mu [-m_k I_4 (W-E)^3 \\
&\quad + (m_k - E) I_3 (W-E)^2 - I_2 (W-E)^2] \\
P^{V(P,T)} &= (Re[F^{V(P)}F^{V(T)*}]/24\pi^3)\, m_k^2 [-m_k^2 I_4 (W-E)^3 \\
&\quad + (m_k W - m_\mu^2) I_3 (W-E)^2 - E I_2 (W-E)^2] \\
P^{V(P,T)'} &= (Re[F^{V(P)}F^{V(T)'*}]/24\pi^3)\, m_k [-m_k^3 I_5 (W-E)^4 \\
&\quad + m_k \{m_k (W-E) - m_\mu^2\} I_4 (W-E)^3 \\
&\quad + \{m_k E - m_\mu^2\} E I_3 (W-E)^2 - E^2 I_2 (W-E)^2] \\
P^{V(A,T)} &= (-Re[iF^{V(A)}F^{V(T)*}]/24\pi^3)\, m_k^2 m_\mu I_3 (W-E)^2 \\
P^{V(A,T)'} &= (Re[iF^{V(A)}F^{V(T)'*}]/24\pi^3)\, m_k [-m_k m_\mu I_4 (W-E)^3 \\
&\quad - m_\mu (2m + E) I_3 (W-E)^2 + 2m_\mu I_2 (W-E)^2] \\
P^{V(T,T)'} &= (Re[F^{V(T)}F^{V(T)'*}/24\pi^3)\, m_k^2 [-m_k^2 I_4 (W-E)^3 \\
&\quad + m_k W I_3 (W-E)^2 - E I_2 (W-E)^2]
\end{aligned}
\qquad (7)
$$

In the expressions (4) $\sim$ (7), m_k, m_π and m_μ are the masses of the K-meson, pion and muon (electron) respectively, and I_1, $I_2 \cdots$ are the functions defined by

$$
\begin{aligned}
I_n &= \int_0^\pi (P \sin\theta d\theta / (m_k - E + P\cos\theta)^n), \\
I_2 &= 2(E^2 - m_\mu^2)^{1/2} (m_k^2 + m_\mu^2 - 2m_k E)^{-1}, \\
I_3 &= 2(E^2 - m_\mu^2)^{1/2} (m_k - E)\, (m_k^2 + m_\mu^2 - 2m_k E)^{-2}, \\
I_4 &= \tfrac{2}{3} (E^2 - m_\mu^2)^{1/2} (3m_k^2 + 4E^2 - 6m_k E - m_\mu^2) \times \\
&\quad \times (m_k^2 + m_\mu^2 - 2m_k E)^{-3}, \\
I_5 &= 2(E^2 - m_\mu^2)^{1/2} (m_k - E)\, (m_k^2 + 2E^2 - 2m_k E - m_\mu^2) \times \\
&\quad \times (m_k^2 + m_\mu^2 - 2m_k E)^{-4},
\end{aligned}
$$

and W is given by

$$W=(1/2m_k)\,(m_k^2+m_\mu^2-m_\pi^2)\,.$$

As noted before we cannot infer the parity of K-mesons in this analysis, by virtue

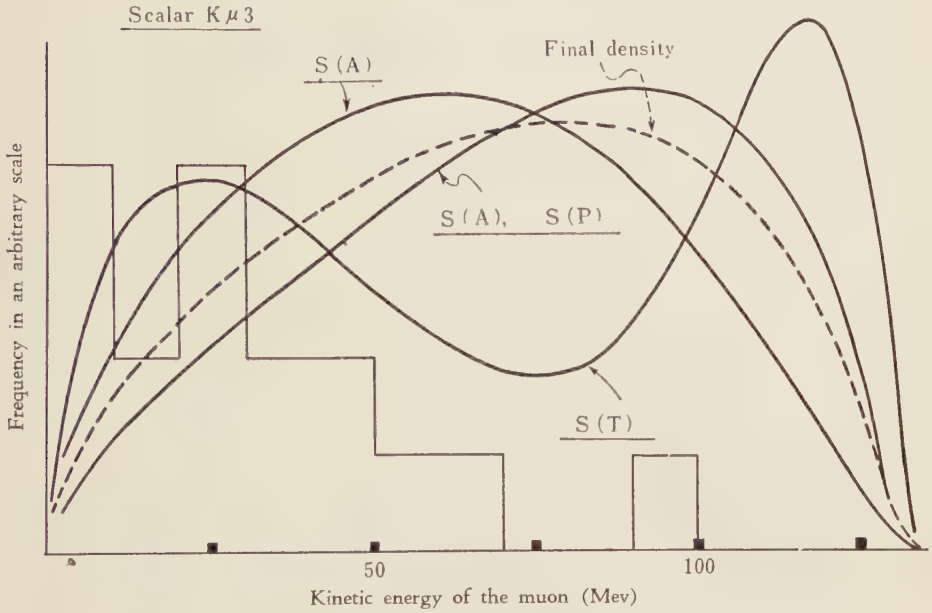

Fig. 2a. Energy spectra of the secondary muons for the scalar $K_{\mu 3}$. The histogram indicates the experimental data by *G-R-C*.

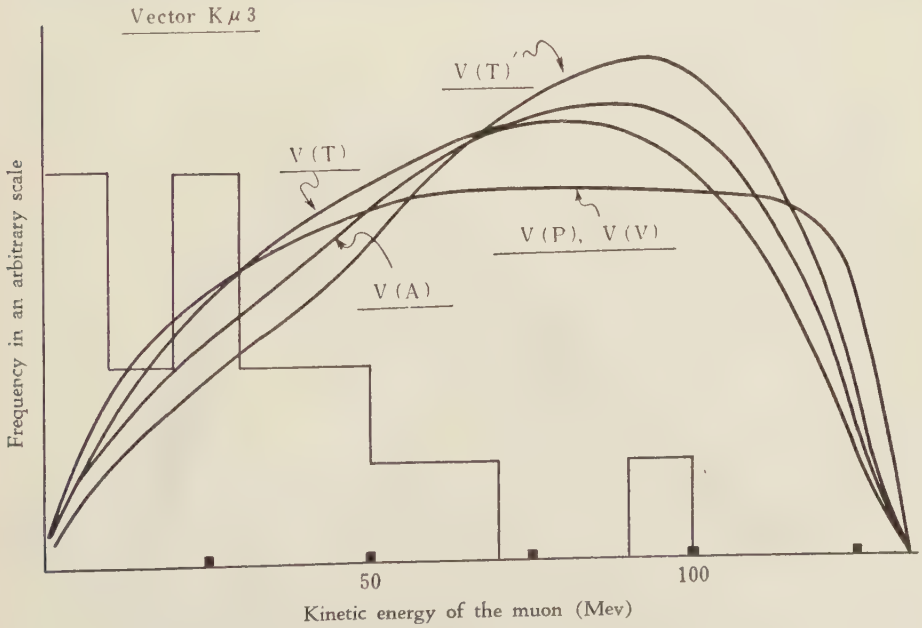

Fig. 2b. Energy spectra of the secondary muons for the vector $K_{\mu 3}$.

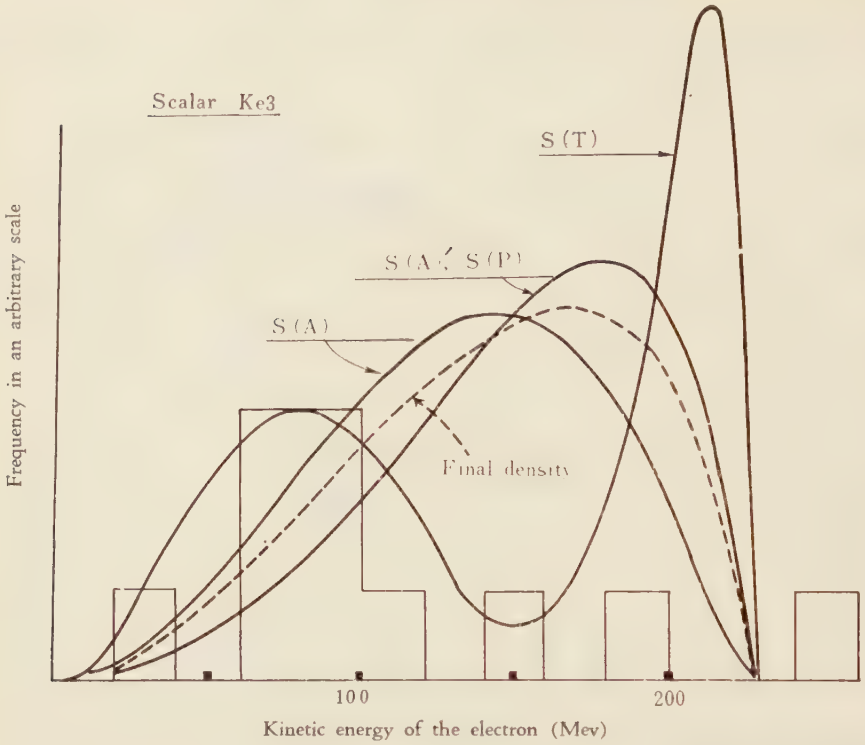

Fig. 3a. Energy spectra of the secondary electrons for the scalar K_{e3}. The histogram indicates the experimental data by *G-R-C*.

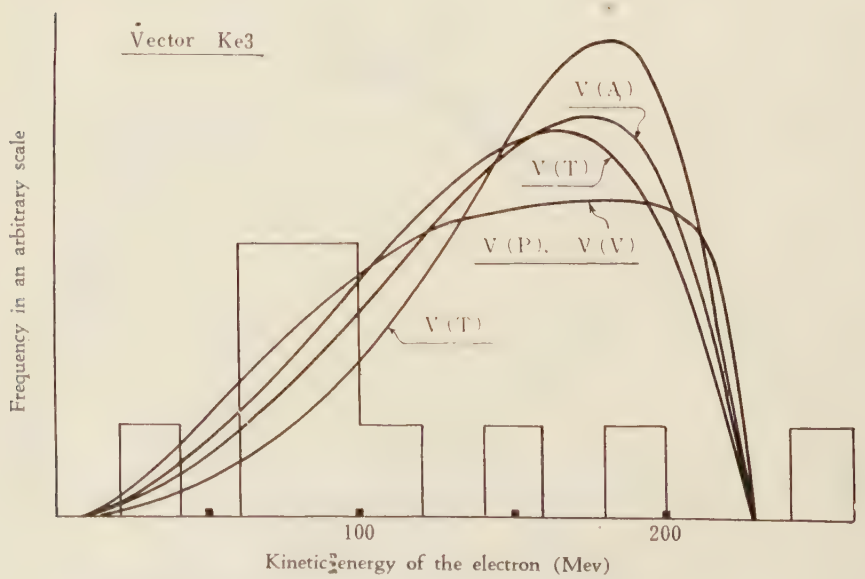

Fig. 3b. Energy spectra of the secondary electrons for the vector K_{e3}.

of the 0 mass of the neutrino. This may be a defect of this analysis. But if the parity would not be conserved in the weak decay process, this analysis still has its workability. The curves of these distributions are given in Figs. 2–3 with the mass of the K-meson 965 m_e.

Table II. Experimental results by G–R–C and expected values from the theoretical distribution

a. $K_{\mu 3}$

	Experimental results							
	G-stack		Ritson et al.		Crussard et al.		Total	
	No. of events	%	No. of events	%	No. of events	%	No. of events	%
I (0–25 Mev)	1	20	6	86	2	40	9	53
II (25–50)	3	60	1	14	1	20	5	29
III (50–80)	1	20	0	0	1	20	2	12
IV (80–134)	0	0	0	0	1	20	1	6

	Expected value from theoretical distribution							
	$S(P)$	$S(A)$	$S(T)$	$V(P)$	$V(A)$	$V(T)$	$V(T)'$	density of the final state
	%	%	%	%	%	%	%	%
I (0–25 Mev)	8	13	18	9	9	10	7	10
II (25–50)	17	24	21	20	18	21	16	20
III (50–80)	30	33	15	28	30	31	30	30
IV (80–134)	45	30	46	43	43	38	47	40

b. K_{e3}

	Experimental results							
	G-stack		Ritson et al.		Crussard et al.*		Total	
	No. of events	%	No. of events	%	No. of events	%	No. of events	%
I (0–50 Mev)	1	20	0	0	0	0	1	10
II (50–120)	3	60	1	100	3	75	7	70
III (120–170)	1	20	0	0	0	0	1	10
IV (170–230)	0	0	0	0	1	25	1	10

	Expected value from theoretical distribution							
	$S(P)$	$S(A)$	$S(T)$	$V(P)$	$V(A)$	$V(T)$	$V(T)'$	density of the final state
	%	%	%	%	%	%	%	%
I (0–50 Mev)	2	4	8	4	2	3	1	3
II (50–120)	22	36	36	30	24	28	18	28
III (120–170)	38	40	8	33	39	39	39	37
IV (170–230)	38	20	48	33	35	30	42	32

* There is one event ($E_e = 250 \pm 60$ Mev) in C. It is not clear whether it belongs to K_{e3} or to K_{e2}.

§ 4. Comparison with the experimental results*

The only systematic experiments available for us at present are those by *G*-stack collaboration (*G*),[11] Ritson et al. (*R*),[15] and Crussard et al. (*C*).[16] Besides these works a few data have been published which we do not use in the present comparison with the theory, for we do not know the conditions of each experiment.** Data by *G*-*R*-*C* are given in Figs. 2–3 and Table II.

We discuss these data in each case of $K_{\mu 3}$ and K_{e3} based on the interactions in section 2.

$K_{\mu 3}$

Recently Crussard et al.[16] have summarized data of $K_{\mu 3}$ from various publications and concluded that the energy distribution of μ-mesons would be given by the final state density of the decay process $K_{\mu 3} \to \pi^0 + \mu + \nu$, if appropriate considerations were made to correct the detection bias of the high energy muon. The feature of the spectrum like this is always expected, because the density of the final state is a main factor for the energy distribution of the secondary particle. However it is premature to draw any definite conclusion about the *detailed form* of the distribution from scanty data accumulated so far under the situations where complicated biases are expected.

For our study it is rather important to know how the actual spectrum deviates from the distribution predicted from the final state density. We should not understand that the role of the final state density is more than a convenient measure of comparison. In this sense, the present $K_{\mu 3}$ data should not be taken literally, although we use them tentatively for the analysis presented later. The above notices should be kept in mind in reading the following.

In analyzing the experimental data it is convenient to use such a quantity as to specify the character of the distribution as a whole. As such a quantity we use the relative expected values in domains I, II, III, and IV. The range of each domain is the same as that of Crussard et al. in their classification of $K_{\mu 3}$ data. The relative expected values with no-bias (in percentage) are listed in Table II for various types of interactions. In Table III we give numerical values of the distributions for various types of interactions for future analyses of the experiments.

In comparison with the present data, we emphasize the importance of the analysis at low energy end of the distribution, because in the region $\lesssim$ 50 Mev we can expect the non-biased data irrespective of conditions under which the experiments are performed.

Now as is seen from Figs. 3–4 and Table II, any one of the cases $S(A)'$, $S(P)$, and $V(T)'$ does not seem to be probable. But since they can be mixed into other types of

* In the following we treat $K_{\mu 3}$ and K_{e3} as if they are scalar or vector particles. This analysis however gives information which can immediately be applied to the case in which those K's happen to be *ps* or *pv*, merely by changing the interaction to an appropriate one with opposite parity as noted in § 2.

** Since we have not been able to get a detailed information about the data by Birge et al., which had been reported at Pisa conference (1955) and listed in the works of Ritson and Crussard, we shall omit them from our present consideration.

interactions by a small portion, the coexistence of these interactions will not be rejected.

If in the future the experiments give the distribution which is in agreement with

Table III.* Numerical values for distribution

a. $K_{\mu 3}$

total energy of muon	$B^{S(P)}$	$B^{S(A)}$	$B^{S(T)}$	$B^{S(P,A)}$**	$B^{S(P,T)}$	$B^{S(A,T)}$
110 Mev	2.0 Mev^{-1}	2.0	0.22	1.9	0.66	−0.65
120	4.2	3.6	0.34	3.4	1.1	−1.1
134	6.7	5.1	0.36	4.7	1.4	−1.3
148	9.1	5.9	0.29	5.4	1.3	−1.3
162	11.	6.4	0.22	5.8	0.92	−1.1
176	13.	6.3	0.17	5.8	0.28	−0.81
190	14.	5.6	0.19	5.5	−0.59	−0.43
204	14.	4.4	0.31	4.7	−1.5	−0.014
218	11.	2.7	0.50	3.3	−2.1	0.31
232	3.6	0.63	0.37	1.0	−1.1	0.23
normalization*** constant	1.	2.1	31.	—	—	—

b. K_{e3}

total energy of electron	$B^{S(P)}$	$B^{S(A)}$	$B^{S(T)}$	$B^{S(P,A)}$**	$B^{S(P,T)}$	$B^{S(A,T)}$
10 Mev	0.087 Mev^{-1}	0.082	0.0084	0.0043	0.032	−0.19
30	0.76	0.67	0.11	0.012	0.29	−0.52
50	2.1	1.7	0.22	0.019	0.67	−0.76
70	4.0	2.9	0.28	0.025	1.1	−0.91
90	6.4	4.1	0.28	0.030	1.3	−0.97
110	9.2	5.1	0.21	0.033	1.3	−0.93
130	12.	5.7	0.10	0.035	1.1	−0.83
150	15.	5.8	0.055	0.035	0.20	−0.62
170	17.	5.2	0.11	0.033	−0.91	−0.30
190	17.	3.8	0.32	0.027	−2.2	0.020
210	11.	1.6	0.71	0.015	−2.7	0.23
220	4.6	0.49	0.53	0.0059	−1.5	0.15
normalization*** constant	1.	2.6	39.	—	—	—

* Here we only give the values for scalar $K_{\mu(e)3}$ particle.

** The distribution for the mixed interaction is given as follows: Taking the case of mixing $S(P)$ and $S(T)$ as an exeample, the energy distribution can be written as

$$P(E) = P^{S(P)}(E) + P^{S(T)}(E) + P^{S(P,T)}(E)$$
$$= (F^{S(P)2}/16\pi^3) m_k B^{S(P)} + (F^{S(T)2}/16\pi^3) m_k^5 B^{S(T)} + (Re(F^{S(P)} F^{S(T)*})/8\pi^3) m_k^3 B^{S(P,T)}.$$

*** In the above table the functions $B^{S(P)}$, $B^{S(A)}$,$\cdots$are given. Multiply this normalization constant to the values in the column when the curve with the normalized area (normalized to $S(P)$) is wanted.

the distribution due only to the volume of the phase space (i. e., the density of the final state), the mixed interaction of $S(P)$ and $S(A)$ is probable for K-meson with spin 0 and $V(T)'$ for K-meson with spin 1. Or when the situation occurs where the experimental distribution has a very steep rise (roughly speaking, steeper than the one for $S(A)$), possibility of spin 1 for $K_{\mu 3}$ is small. In this case the mixed interaction of $S(T)$ with $S(A)$ or $S(P)$ is perhaps the interaction through which the decay process takes place.

It is noted that the existence of tensor interaction is determined by the distribution from 70 to 134 Mev, and if there exists a plateau or a second peak in that region, this will be a strong indication of the existence of the tensor interaction, for any combination of the interactions without tensor type cannot lead to the above characteristic feature of the distribution. It is also noted here that most interactions which contribute to the decay $K_{\mu 2}$ cannot make any contribution to the decay $K_{\mu 3}$ on account of odd parity of pion wave function, if $K_{\mu 2}$ and $K_{\mu 3}$ are the same particles.

In comparing the experimental results with the theoretical predictions it must be remarked that there are two main origins which bias the results. One of these is the scanning bias. However, this kind of bias has little influence at low energy (regions I and II), mainly because of the easiness of the scanning procedures in the analysis, while this kind of bias brings a large effect for the events with the energy ranging from about 50 Mev to maximum energy. And it is the distribution in this region that has a serious significance for our analysis.

Another type of bias, or more strictly speaking, the contamination by the other events is expected. It is the contamination by $K_{\pi 2}$ and radiative $K_{\mu 2}$. The $K_{\pi 2}$ contamination can be readily removed, if we use only the data which are accompanied by the decay events of their charged secondaries. The contamination by radiative $K_{\mu 2}$ cannot be removed in contrast with those of $K_{\pi 2}$, but it may be very small. In the radiative $\pi \to \mu + \nu$ decay which is the analogue of radiative $K_{\mu 2}$, the observed data can be explained as that the process with large contribution is the one with infrared catastrophe[17] (Fig. 4b) and not the one indicated in Fig. 4a, and it is also known that the decay $\pi \to e + \nu + \gamma$ is not yet established, though beta decay of the nucleon allows the latter mode.

These situations are also expected to occur in the case of radiative $K_{\mu 2}$, so we consider the contribution only from the diagram in Fig. 4b. This catastrophic transition process gives a little contribution above the maximum energy of $K_{\mu 2}$, but below it the

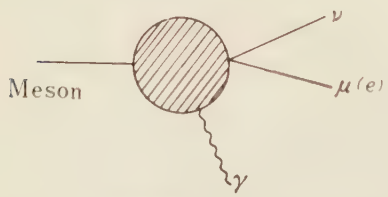

Fig. 4a. Diagram for the radiative meson decay.

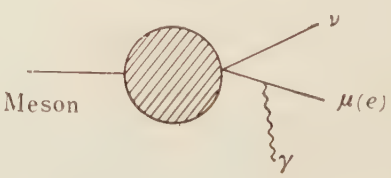

Fig. 4b. Diagram for the radiative meson decay. (Infrared Catastrophy)

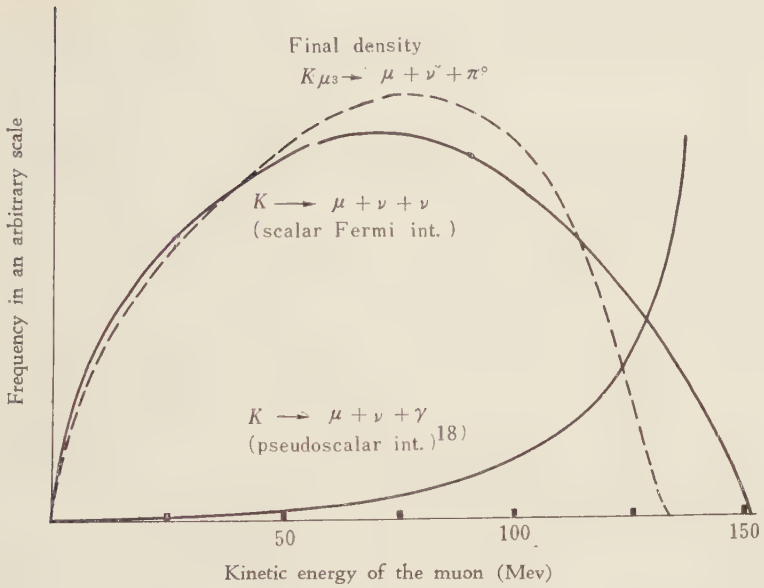

Fig. 5. Enery spectra of the secondary muons for the radiative $K_{\mu 2}$ and fermion $K_{\mu 3}$.

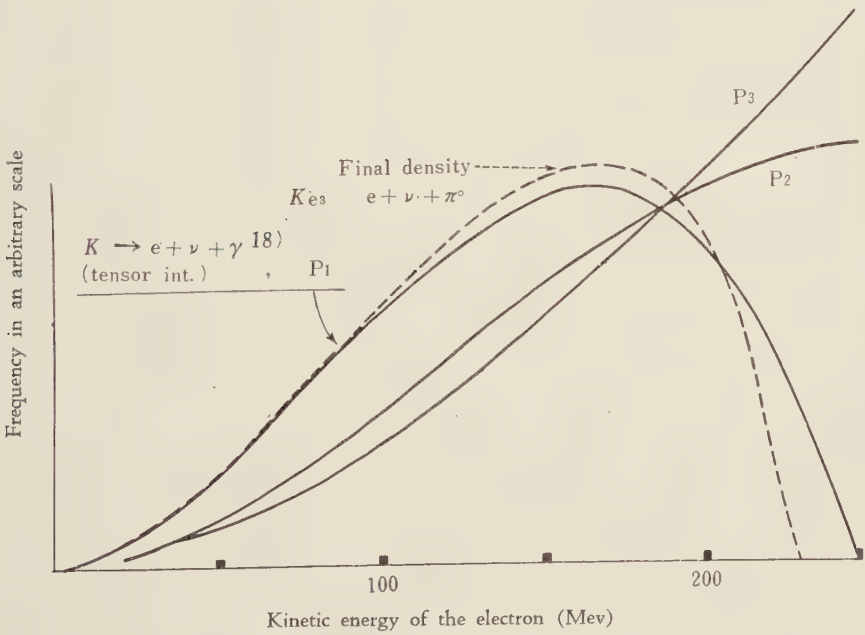

Fig. 6. Energy spectra of the secondary electrons for the radiative K_{e2} and Michel's curves for fermion K_{e3} ($\rightarrow e+\nu+\nu$). Notations are same with those of Michel's.

contribution is very small. Thus this contamination is practically negligible. The curve for this process is given in Fig. 5. With regard to radiative $K_{\mu 2}$ we would like to make

a comment; although radiative $K_{\mu 2}$ may give the information about $K_{\mu 2}$, the spectrum is overwhelmed by the event $K_{\mu 3}$ in the major part of its range, and the analysis of the kind proposed by Eguchi[18] in the case of $\pi \to e+\nu+\gamma$ seems to be very difficult.

K_{e3}

Contrary to the case of $K_{\mu 3}$ the analysis of the data of K_{e3} shows in some degree what is going on. With the present data it is clearly seen that there is a hollow in the middle region of energy distribution. This strongly indicates that the K_{e3} is spin 0 and there is a tensor interaction, and even if mixed with other interaction the main part of the interaction must be occupied by the tensor interaction $S(T)$. It may seem dangerous to emphasize the character shown with so few data. However in the experiment of K_{e3} the bias is very weakly dependent on the energy in the region of importance, although at extremely low and high energy regions experiments may be strongly biased.

The decrease in the number of events above 100 Mev is realized only *by the existence of the tensor interaction in the case of spin 0 K_{e3}, and no other interactions. Even if we take account of their mixing or their higher momentum dependent terms, we cannot obtain the distribution shown in the experimental results.** Not only the distribution as a whole, the analysis of the distribution at the low energy region also gives a strong support to spin 0 and tensor interaction for K_{e3}.

For the contamination from the radiative K_{e2} which might exist, the discussions given for $K_{\mu 3}$ are valid. In this case the process $K \to e+\nu+\gamma$ is expected to occur as the diagram in Fig. 4a, for there is no established evidence for the existence of K_{e2}. The curve of energy distribution for this process is given in Fig. 6.

§ 5. Validity of the present analysis

The validity of the analysis in this paper is dependent on several conditions. In the following we examine each of the conditions.

1) The validity of the assumption of the Fermi interaction.

For this problem no definite arguments are possible, and it depends on how the weak Fermi interaction can correctly describe the phenomena realized in nature in unified manner. The conjecture rests on the possibility that all weak interactions can be described by the universal Fermi interaction, collaborating with the strong interaction on which some characteristic features[19] seem to be settled. We should also note that though the assumption of the Fermi interaction is used in this analysis, it results in that the four momenta p and q always appear in the combination $p+q$, and this assumption merely imply, in this sense, to confine the interaction between muon and neutrino to an one point interaction. The lack of the strong interaction in the final state may favor the present treatment. The same condition for p and q is given under the assumption of the weak Boson-Fermion interaction,[7] but in this assumption some of the types of interactions in Table I are absolutely forbidden. (They are $S(T)$ for spin 0 K-meson and

* The second maximum (at ~200 Mev) of $S(T)$ can be reduced to any height by appropriate mixing with $S(P)$ or $S(A)$ without destroying the feature of the first maximum.

$V(P)$ for spin 1 K-meson.) The fact that experimental results on K_{e3} seem to be explained by $S(T)$ scheme may indicate that the weak Boson-Fermion interaction plays no essential role in K_{e3} decay, if it exists, and gives support to the superiority of the Fermi interaction as a universal interaction.

2) The assumption of the decay modes $K_{\mu 3}\to\mu+\nu+\pi^0$and $K_{e3}\to e+\nu+\pi^0$.

Some events which may confirm the existence of the assumed decay mode are found about $K_{\mu 3}$,[12] and from the observed symmetry of the electron and the muon in the four field interactions (beta decay of nucleon and nuclear absorption of muon, and also beta decay of muon), K-decay with the mode $K\to e+\nu+\pi^0$ seems to be probable. The modes $K\to\mu+\nu+\gamma$ and $K\to\mu+\nu+\nu$ may be enchanting. However, assignment of all of the observed $K_{\mu 3}$ data to the radiative $K_{\mu 2}$ is not possible, because the radiative $K_{\mu 2}$ must be very rare events compared with $K_{\mu 2}$, so the observed ratio $K_{\mu 3}/K_{\mu 2}\sim 1/20$ is inconsistent with the calculated value $\sim 1/1000$. The assumption of fermion $K_{\mu 3}\to\mu+\nu+\nu$ may also be excluded from the analysis of the energy distribution. To check the possibility of this decay mode the energy distribution of the scalar Fermi interaction is given in Fig. 5. (Scalar interaction gives the distribution mostly shifted to the low energy end. This is the reason for taking the scalar interaction to check the possibility). At the low energy this curve is quite well fitted by the final state density distribution curve of $K_{\mu 3}$, so the analysis in this region cannot reject the possibility if $K_{\mu 3}\to\mu+\nu+\nu$ decays by the scalar interaction, but we could expect more events than the distribution due to the final state density in the region IV, and this is inconsistent with the observation. Further if we take the coupling constant between the four fermions equal to that of the beta decay, the calculation gives 10^{-9} sec for its life-time. This is a little shorter than the observed life-time of $K_{\mu 3}$, 10^{-8} sec. However it is not possible to *deny the existence* of fermion $K_{\mu 3}\to\mu+\nu+\nu$ with very small abundance in the observed $K_{\mu 3}$ particle.

For K_{e3} these situations are quite same as in $K_{\mu 3}$, and even the analysis of the distribution at the low energy region rejects the possibility of fermion $K_{e3}\to e+\nu+\nu$. We give the Michel's[20] curve for fermion K_{e3} in Fig. 6.

3) Approximation used in § 3.

This approximation has a direct connection with the final conclusions. To scrutinize the approximation used in calculating the matrix elements is not easy, since we have not yet established the plausible methods to handle the strong interaction. This is also the reason for us to adopt such a treatment as developed in this paper, basing on the covariance requirements. The fact that there is no strong interaction between the particles in the final state may also be favorable for the present treatments. For the contribution from the intermediate state we use the approximation such as

$$F=a+(k_\mu(p+q)_\mu/M^2)\,b+\cdots\cdots,$$

where k, p and q are energy momentum four vectors of the K-meson, muon (electron) and neutrino, and M is baryon mass. We have no way to know the magnitude of each terms of this expansion, hence as a rough estimation of magnitude we have tentatively calculated these terms from the loops by using the conventional perturbation calculation,

taking only the lowest order terms of the Feynman diagram in Fig. 1.

For example the calculation gives for scalar K-meson, except some common factors,

$$a^{S(I)} \simeq (1/60) M^{-1} (30 - 8 (m_k/M)^2 - (m_\pi/M)^2),$$

$$b^{S(I)} \simeq (1/60) M^{-1},$$

$$a^{S(T)} \simeq (1/12) M^{-1},$$

$$b^{S(T)} \simeq 0 .$$

In the calculation we have assumed 1 and γ_5 couplings for ΛNK and $NN\pi$, and made the approximations :

(1) Hyperon mass = Nucleon mass = cut-off parameter.

(2) $t^2 + \Delta$, a denominator of integrand in a parameter integral, is replaced by *some average value* M^2, where $t = k(1-\xi_1) + l\xi_1(1-\xi_2)$, $\Delta = M^2 - k^2 + (k^2 - l^2)\xi_1 + l^2\xi_1\xi_2$ and parameters ξ_1, ξ_2 take the values between 0 and 1.

As is seen in the above the relative magnitude of the first to the second terms is considerably large $(k(p+q)/M^2 \simeq 1/10)$, so the result of perturbation calculation may be regarded as to support our approximations. The appearence of secondary particles with energies higher than their rest masses does not matter to our present expansion, because we have used the expansion by four momentum, while some portions of the higher momentum dependent parts are reduced to constants by virtue of the equation of motion, and this makes us to expect a better situation.

In concluding this section we discuss about the cases of higher spins and higher order terms. They show similar behaviors to the present analysis. Generally speaking those higher order terms give a curve which takes more after the final state density curve than those given the lowest order term. So if the distribution of experimental data deviates largely from the curve of final state density it may indicate the higher order term is not dominant, and this is the same circumstance as in the case of K_{e3}.

Though we have not extensively investigated the case of higher spin for K-particles, a tentative examination of the case of spin 2 and our experience in the study of spin 0 and spin 1 cases seem to indicate that the higher spin K-particles give the distributions which are very much similar to the final state density. The reason is because the interactions are summed and averaged over their each component, and the main feature of the distribution is completely governed by the final state density. The situation is well informed by the comparison of the curve in the case of spin 0 K-meson with that of spin 1 K-meson. If the data deviate largely from the final state density such as in K_{e3}, the possibility of high spin can be rejected.

§ 6. Concluding remarks

We have attempted in this paper to determine the spins and decay interactions of $K_{\mu 3}$ and K_{e3}, and obtained some knowledge about them. K_{e3} particle is likely to be spin 0 particle, and the weak Fermi interaction of tensor type is required to explain the

features shown in the experiments, while it seems a little early to make *any definite conclusion* for $K_{\mu 3}$. We have investigated those K-particles to fill up the blank in our knowledge about them. The methods taken here may be orthodox and elementary, and this treatment of the problem can be applied to other cases, especially to $K^0 \to \mu(e) + \nu + \pi^0$ which may be the neutral counter particle of $K_{\mu(e)3}$.

Besides the analysis of energy spectrum of the charged secondary, the analysis of the angular distribution of the secondary particles $(\mu(e), \pi)$ may also contributive to the affirmation of the conclusion given by the study of energy distribution, though the experiments on these angular distributions seem to be an extremely severe task.

We have investigated one aspect of K-particles, the weak decay interaction, and another important one is the problem of strong interaction of K-particle. There may appear several difficulties in the investigation of strong interaction of K-particles. Apart from theoretical difficulties inherent in the treatment of strong interactions, there is an experimental difficulty in treating the K-particle. It is experimentally impossible to separate each K-particle beam from a flux of K- particles, if there exist more than two kinds of K-particles. And the information of the genealogy of K-particle accumulated from the study of weak interaction may lend some help to simplifying the complicated situations in analysis of the strong interaction.

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Note added in proof

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2) In the eqs. (6) and (7), the relative phases of the effective coupling constants are remained arbitrarily. Some invariance requirments such as time reversibility will fix the above arbitrariness and conversely the study in our course may provide the examination of the validity of the invariance principles.

Recently A. Pais and S. B. Treiman have analysed the angular correlation of the charged secondaries in the neutral $K_{\mu 3}$-and K_{e3}-decay from the view point similar to ours and given a comment on time reversibility.

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On the Universality of the Weak Interactions

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The possibility of extending the concept of iso-spin to leptons produced in the lepton reactions—the K-meson decays ($K_{\mu 2}$, $K_{\mu 3}$ and K_{e3} decays), pion decays, muon decays, β-decays, and nuclear capture of muon—is investigated. If we assign the iso-spin one half for muons and electrons, and zero for the neutrinos produced in the $K_{\mu 2}$ decays, we can find out a suitable selection rule which allows all the well established lepton reactions and forbids all the unwanted ones. This method is applied to the K-meson decays in which the lepton reactions coexist with the boson-boson decays ($\tau \to 3\pi$, $\theta \to 2\pi$).

§ 1. Introduction

By using the concept of iso-spin for strange particles, Gell-Mann and Nishijima[1] have recently proposed indepedently a theory of strange particles which accounts for their stability (weak interactions) and copious production (strong interactions) in terms of a new concept, viz. the conservation of strangeness. In this scheme it is proposed that the strong interactions are invariant with respect to arbitrary rotations in the iso-space (charge independent), while the weak ones are not (charge dependent).

In this paper we investigate a possibility of extending the concept of iso-spin to leptons. As is well known, the charge Q of each nucleon which has a definite iso-spin is connected to the third component of the iso-spin I_3 by the equation

$$Q = I_3 + \tfrac{1}{2}.$$

In the theory of Gell-Mann and Nishijima the above relation between the third component I_3 and charge Q of each nucleon is extended to the following one,

$$Q = I_3 + \tfrac{1}{2}(N+S), \qquad (1\cdot1)$$

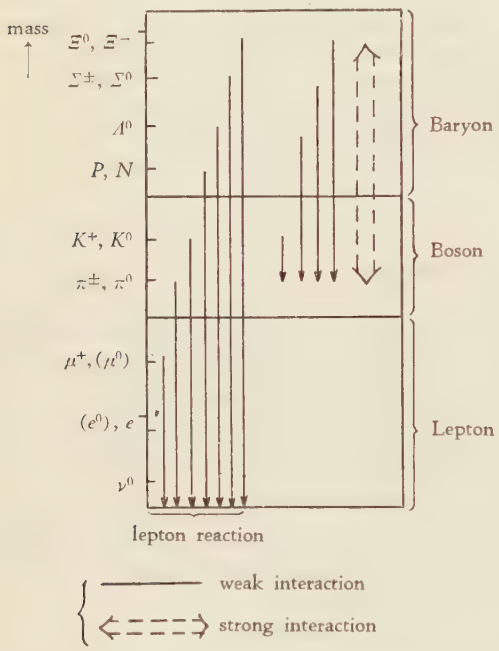
for each strange particle having strangeness quantum number S. N in $(1\cdot1)$ is baryon number.

Now, if we pay our attention to the above relation $(1\cdot1)$, we can see a remarkable fact that the orientation of the third axis in the iso-space (isotopic-spin space) is characterized only by the existence of the electromagnetic field. On the other hand we know that the electromagnetic field has the famous universal interaction not only with the charged baryons and bosons (we call both K and π mesons as bosons) but also with the charged leptons, such as

$$ej_\mu A_\mu (+e^2\text{-term for bosons}),$$

where ej_μ is the current of the charged particle and A_μ the four potentials. It is therefore natural to assign the iso-spin for leptons. In that case we modify (1·1) such as N stands for lepton number. Here we must remember that there is a large difference between the reactions in the baryon-boson system and those accompanied with leptons, viz. the former has strong interaction, while the latter has no strong one. As far as we know from the results of experiments, the lepton process is always accompanied by at least one neutrino. These characteristic features of the lepton interactions give a key to dissolve the universal weak interactions into two groups, viz. one is the weak interaction in the baryon-boson system, the other the weak lepton interaction. In the following sections, we will clarify the relation between the two groups.

Fig. 1.



In this section we will present some outline of our idea concerning the weak interactions. For convenience' sake, we divide our discussion on the practically observed lepton processes without real photons into two steps, (1) we neglect the self energy effects due to the electromagnetic interactions and those caused by a newly introduced "L-field" (which is considered in our model to reduce the rest mass of each neutral lepton to the rest mass of the observed neutrino ——the nature of the "L-field" will be clarified in section 5), (2) we then introduce only the virtual effects of the electromagnetic and L-field interactions which were neglected in (1). Then we can usefully apply the concept of charge independence to the lepton reactions in the first step which in practice do not correspond to the lepton phenomena observed, but taking into consideration the virtual effects of the electromagnetic and L-fields in the second step we shall get the practical reactions which show a large deviation from charge independence. The very nature of this lepton reaction in the second step shows that the lepton interaction is sharply distinguished from the strong interaction in the boson-baryon system where the charge independence holds in a good approximation even when the virtual effects of the electromagnetic field are considered. Thus we can distinguish between the strong and weak interactions in the reactions observed in experiments according to whether charge independence holds in a good approximation or not when the virtual processes are taken into consideration.

If the iso-spins of elementary particles are restricted to be 0, $\frac{1}{2}$ and 1, it can be shown that in the first step in the lepton reactions there is only one charge independent formalism*[1,2,3] which has consistent correspondence with the experiments. The iso-spin assignment in this case is given in Table 1, where iso-spin of baryon and meson is assumed to be the same as that of Nishijima and Gell-Mann.

Table 1. Iso-spin assignment of the elementary particles

Family	Particle	I	I_3	$Q-I+\frac{1}{2}(N+S)$	S	$U-N+S$	Transformation properties
Baryon	Ξ^0, Ξ^-	$\frac{1}{2}$	$\frac{1}{2}$, $-\frac{1}{2}$	0, -1	$I_3-\frac{1}{2}$	-2	-1	2nd kind iso-spinor
	Σ^+, Σ^0, Σ^-	1	1, 0, -1	1, 0, -1	I_3	-1	0	iso-pseudovector
	Λ^0	0	0	0	I_3	-1	0	iso-scalar
	P, N	$\frac{1}{2}$	$\frac{1}{2}$, $-\frac{1}{2}$	1, 0	$I_3+\frac{1}{2}$	0	1	1st kind iso-spinor
Boson	K^+, K^0	$\frac{1}{2}$	$\frac{1}{2}$, $-\frac{1}{2}$	1, 0	$I_3+\frac{1}{2}$	1	1	1st kind iso-spinor
	π^+, π^0, π^-	1	1, 0, -1	1, 0, -1	I_3	0	0	iso-pseudovector
Lepton	μ^+, μ^0	$\frac{1}{2}$	$\frac{1}{2}$, $-\frac{1}{2}$	1, 0	$I_3+\frac{1}{2}$	0	1	1st kind iso-spinor
	e^0, e^-	$\frac{1}{2}$	$\frac{1}{2}$, $-\frac{1}{2}$	0, -1	$I_3-\frac{1}{2}$	-2	-1	2nd kind iso-spinor
	ν^0	0	0	0	I_3	-1	0	iso-scalar
	γ	0	0	0	I_3	0	0	iso-scalar
	L	0	0	0	I_3	0	0	iso-scalar

In Table 1 the positive muon μ^+ and the electron e^- are taken as the normal particles, and μ^0, e^0 are respectively their neutral counter parts. The observed rest masses of μ^0, e^0 and ν^0 are vanishingly small because of the self-energy caused by the "L-field". Therefore we can call all of μ^0, e^0, and ν^0 by the same name "neutrino". Although the transformation properties of μ^0, e^0, and ν^0 in the iso-space are different from each other, the neutral leptons cannot be distinguished by experiments.

Such a characteristic property that there is no process without neutrino in the lepton reactions results at once in the following remarkable fact: in the case of the boson decay which belongs to the lepton reaction, the parity of the decay parent cannot be determined through this process, because the existence of γ_5 gives no effects on the value of the transition probability. It is this circumstance that has inspired us to take the lepton reactions as a starting point of unifying the interactions of elementary particles.

* The iso-spin formalism was first applied to the lepton reactions by M. Cini and A. Gamba.[2] The similar line of investigation has recently been tried in Japan.[3] But their schemes have defects more or less inconsistent with experiments. Cf. section 4 and references (2) and (3).

§ 2. Allowed lepton reactions

First of all we will consider the following well established reactions:

$$\mu^- + P \to N + \tilde{\nu} \qquad (\mu\text{-capture}), \tag{2·1}$$

$$\mu^+ \to e^+ + 2\nu \qquad (\mu\text{-decay}), \tag{2·2}$$

$$N \to P + e^- + \tilde{\nu} \qquad (\beta\text{-decay}), \tag{2·3}$$

$$\pi^+ \to \mu^+ + \tilde{\nu} \qquad (\pi\text{-}\mu \text{ decay}), \tag{2·4}$$

$$K^+ \to \mu^+ + \tilde{\nu} \qquad (K\text{-decay}), \tag{2·5}$$

where the conservation of fermion number is assumed. Since the matter stability assures the conservation of baryon number, this is equivalent to the assumption of the conservation of lepton number. Moreover, the difference of the transformation properties of the neutrino in the iso-space is not considered in (2·1) ~ (2·5).

Now performing the iso-spin assignment according to the table 1, and neglecting the virtual effects which are caused by the electromagnetic- and L-fields, all the reactions (2·1) ~ (2·5) satisfy the following selection rule;

$$\begin{cases} \Delta N(\text{baryon}) = 0, \quad \Delta N(\text{lepton}) = 0, \\ \Delta I = 0, \quad \Delta S = 0, \end{cases} \tag{α}$$

viz. the conservation of baryon number N(baryon), lepton number N(lepton), total iso-spin I, and of strangeness S. It is clear that (α) assures the conservation of charge because of the relation (1·1). By clarifying the difference of the transformation properties of neutrinos in the iso-space, the reactions (2·1) ~ (2·5) which are allowed by the selection rule (α) will be rewritten as follows:

$$\mu^- + P \to N + \tilde{\mu}^0, \tag{2·6}$$

$$\mu^+ \to e^+ + 2\nu^0, \tag{2·7}$$

$$\to e^+ + \mu^0 + e^0, \tag{2·7′}$$

$$N \to P + e^- + \tilde{e}^0, \tag{2·8}$$

$$\pi^+ \to \mu^+ + \tilde{\mu}^0, \tag{2·9}$$

$$K^+ \to \mu^+ + \tilde{\nu}^0. \tag{2·10}$$

For the decay events, $K_{\mu 3}$ and K_{e3}, we assume the following decay modes[4]:

$$K^+ \to \mu^+ + \tilde{\nu}^0 + \pi^0, \tag{2·11}$$

$$K^+ \to e^+ + \nu^0 + \pi^0. \tag{2·12}$$

Then the reactions (2·11) and (2·12) are also allowed by the selection rule (α).

According to the selection rule (α), not only the reactions (2·6) ~ (2·12) are allowed, but also the following reactions are not forbidden:

$$\pi^+ \to e^+ + e^0, \tag{2·9'}$$

$$K^+ \to e^+ + \nu^0. \tag{2·10'}$$

These reactions are derived formally by putting e^+ instead of μ^+ in (2·9) and (2·10). Moreover there are several allowed reactions in which at least a hyperon is contained:

$$\Lambda^0 \to P + e^- + \tilde{\nu}^0, \tag{2·13}$$

$$\Lambda^0 \to P + \mu^- + \nu^0, \tag{2·14}$$

$$\Sigma^+ \to P + e^0 + \tilde{\nu}^0, \tag{2·15}$$

$$\Sigma^+ \to \Lambda^0 + e^+ + e^0, \tag{2·16}$$

$$\Xi^- \to \Lambda^0 + e^- + \tilde{\nu}^0. \tag{2·17}$$

As regards the allowed lepton reactions (2·6) ~ (2·17) we have written only one representative reaction mode in each reaction type.

Next we examine whether the reactions (2·6) ~ (2·17) are consistent with experiments by setting the following assumption:

(I) The strength of weak interactions is universal but the type of coupling is different from process to process.

Under this assumption (I) we will begin with the investigation on the reaction (2·9). It is known experimentally that branching ratio ρ for the alternative decay mode (2·9'), relative to the normal pion decay mode (2·9), is very small ($\rho \lesssim 5\times 10^{-5}$).[5] If we use PV coupling in both (2·9) and (2·9'), we can obtain the following ratio which is consistent with experiments:[6]

$$\rho = \frac{w(\pi \to e+\nu)}{w(\pi \to \mu+\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.3\times 10^{-4}. \tag{2·19}$$

Similarly we get

$$\rho = \frac{w(K \to e+\nu)}{w(K \to \mu+\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_K^2 - m_e^2}{m_K^2 - m_\mu^2}\right)^2 = 2.5\times 10^{-5}, \tag{2·20}$$

if we also assume PV coupling for both (2·10) and (2·10'). This is again not inconsistent with experiment since even one example of the decay mode $K^+ \to e^+ + \nu^0$ has not been found up to now.

Next we consider the reactions (2·13) ~ (2·17). Taking $S-T+P$ combination with the universal coupling[7] $f = 1.4\times 10^{-49}$ erg cm^3 for each reaction (2·13) ~ (2·17), we get the following lifetimes for decay modes $\Lambda^0 \to N+\nu+\tilde{\nu}$, $\Sigma^0 \to N+\nu+\tilde{\nu}$, $\Sigma^0 \to \Lambda^0 + \nu + \tilde{\nu}$, $\Xi^0 \to \Lambda^0 + \nu + \tilde{\nu}$, which have the largest available phase space volume for (2·13) ~ (2·18):

$$\begin{aligned} &\tau(\Lambda^0 \to N+\nu+\tilde{\nu}) \cong 1.6\times 10^{-8} \text{ sec} \\ &\tau(\Sigma^0 \to N+\nu+\tilde{\nu}) \cong 1.3\times 10^{-8} \text{ sec} \\ &\tau(\Sigma^0 \to \Lambda^0 + \nu + \tilde{\nu}) \cong 0.85\times 10^{-8} \text{ sec} \end{aligned} \tag{2·21}$$

$$\tau(\Xi^0 \to \Lambda^0 + \nu + \tilde{\nu}) \cong 0.77 \times 10^{-8} \text{ sec.}$$

If we take for the main decays of hyperons the following lifetimes;

$$\begin{aligned} &\tau(\Lambda^0 \to P + \pi^-) = 3.7 \times 10^{-10} \text{ sec}, \\ &\tau(\Sigma^+ \to N + \pi^+) \sim 10^{-10} \text{ sec}, \\ &\tau(\Xi^- \to \Lambda^0 + \pi^-) \sim 10^{-10} \text{ sec}, \end{aligned} \tag{2·22}$$

we obtain

$$\begin{aligned} &w(\Lambda^0 \to N + \nu + \tilde{\nu}) / w(\Lambda^0 \to P + \pi^-) \sim 2 \times 10^{-2}, \\ &w(\Sigma^0 \to N + \nu + \tilde{\nu}) / (\Sigma^+ \to N + \pi^0) \sim 10^{-2}, \\ &w(\Sigma^0 \to \Lambda^0 + \nu + \tilde{\nu}) / w(\Sigma^+ \to N + \pi^0) \sim 10^{-2}, \\ &w(\Xi^0 \to \Lambda^0 + \nu + \tilde{\nu}) / w(\Xi^- \to \Lambda^0 + \pi^-) \sim 10^{-2}. \end{aligned} \tag{2·23}$$

These results mean that the three body decays of hyperons due to the universal Fermi interaction are infrequent as compared with the respective main hyperon decay modes by a factor less than about 1/100. So we may consider that there is no contradiction with the experimental fact that only one case of β-decay of Σ (perhaps $\Sigma^{\pm} \to e^{\pm} + N + \nu$)[8] has been reported till now.

For the sake of later discussions we list here the interaction Hamiltonians corresponding to the reactions (2·6)$\sim$(2·12) which are all scalar in the iso-space.*

$$H(NN\mu\mu) = g\left[\frac{c_0}{2} \{P^* P + N^* N\} \{\mu^{+*}\mu^{+} + \mu^{0*}\mu^{0}\} + \frac{c_1}{\sqrt{3}} \{P^* N \mu^{0*}\mu^{+} + N^* P \mu^{+*}\mu^{0} - \frac{1}{2}(P^* P - N^* N)(\mu^{+*}\mu^{+} - \mu^{0*}\mu^{0})\} \right], \tag{2·24}$$

$$H(\mu e \nu^0 \nu^0) = g\{\mu^{+} e^{-} - \mu^{0} e^{0}\} \nu^{0*} \nu^{0} + h.c., \tag{2·25a}$$

$$H(\mu\mu ee) = g\left[\frac{c_0}{2} \{\mu^{+*}\mu^{+} + \mu^{0*}\mu^{0}\} \{e^{0*} e^{0} + e^{-*} e^{-}\} + \frac{c_1}{\sqrt{3}} \{\mu^{+*}\mu^{0} e^{-*} e^{0} + \mu^{0*}\mu^{+} e^{0*} e^{-} - \frac{1}{2}(\mu^{+*}\mu^{+} - \mu^{0*}\mu^{0})(e^{0*} e^{0} - e^{-*} e^{-})\} \right], \tag{2·25b}$$

* We can bring the various coupling constants appearing in (2·24)$\sim$(2·30) into the same dimension with the use of the appropriate powers of c, $\hbar$ and $\lambda = \hbar / m_\pi c \sim 10^{-13}$ cm, the Compton wave length for pion. For the Fermi coupling constant $g \sim 10^{-49}$ erg cm^3 in (2·22)$\sim$(2·26), we obtain

$$g^2 (1/\hbar c)^2 \lambda^{-4} \sim 10^{-14}$$

and for the coupling constant $g \sim 10^{-29}$ erg$^{1/2}$ cm$^{3/2}$ in (2·27) and (2·28)

$$g^2 (1/\hbar c) \lambda^{-2} \sim 10^{-14}.$$

Similarly we get in the case of (2·29) and (2·30)

$$g^2 (1/\hbar c)^2 \lambda^{-4} \sim 10^{-28}.$$

$$H(NNee)=g\Big[\frac{c_0}{2}\{P^*P+N^*N\}\ \{e^{0*}e^0+e^{-*}e^-\}$$

$$+\frac{c_1}{\sqrt{3}}\{P^*Ne^{-*}e^0+N^*Pe^{0*}e^- -\frac{1}{2}(P^*P-N^*N)\ (e^{0*}e^0-e^{-*}e^-)\}\Big], \quad (2\cdot26)$$

$$H(\pi\mu\mu)=g\{\mu^{+*}\mu^0\pi^+ +\mu^{0*}\mu^+\pi^- +\frac{1}{\sqrt{2}}(\mu^{+*}\mu^+ -\mu^{0*}\mu^0)\pi^0\}\ , \quad (2\cdot27)$$

$$H(K\mu\nu^0)=g\{\mu^{+*}K^+ +\mu^{0*}K^0\}\nu^0+h.c.\ , \quad (2\cdot28)$$

$$H(K\mu\pi\nu^0)=g\{K^{+*}\mu^0\pi^+ +K^{0*}\mu^+\pi^- +\frac{1}{\sqrt{2}}(K^{+*}\mu^+ -K^{0*}\mu^0)\pi^0\}\nu^{0*}+h.c.\ , \quad (2\cdot29)$$

$$H(Ke\pi\nu^0)=g\{K^+e^0\pi^{+*}-K^0e^-\pi^{-*}+\frac{1}{\sqrt{2}}(K^0e^0+K^+e^-)\pi^{0*}\}\nu^{0*}+h.c.\ . \quad (2\cdot30)$$

Corresponding to the decay modes $(2\cdot13)\sim(2\cdot17)$ we can also easily write down the interaction Hamiltonians which are scalar in the iso-space, but all of them are omitted here since they are not needed for our later discussions.

From the iso-scalar interaction Hamiltonians, we can see at once all the processes included in them, for example, the Hamiltonian $(2\cdot24)$ contains, besides the normal process of μ^--capture by proton

$$\mu^- +P\to N+\tilde{\mu}_0, \quad (2\cdot31)$$

the following new types of reactions:

$$\mu^+ +\tilde{P}\to\tilde{N}+\mu_0 \quad (2\cdot32)$$

and

$$\begin{aligned}&\mu^+ +P\ \to P+\mu^+,\\ &\mu^+ +N\ \to N+\mu^+,\\ &\mu^0 +P\to P+\mu^0,\\ &\mu^0 +N\to N+\mu^0.\end{aligned} \quad (2\cdot33)$$

All of them may be at present far beyond the possibility of experimental detection except the well established μ^--capture (2.31).

Similarly we can easily see that although numerous modes are included in each of the interaction Hamiltonians $(2\cdot25)\sim(2\cdot30)$, the superfluous modes except the well established ones are far beyond the possibility of experimental detection at present. Finally we will make a few remarks on the $K_{\mu3}$- and K_{e3}-decays, which are contained in the interaction Hamiltonian $(2\cdot29)$ and $(2\cdot30)$ respectively. The reaction modes included in $(2\cdot29)$ and $(2\cdot30)$ are as follows:

$$\left.\begin{array}{ll} K^+ \to \mu^+ + \pi^0 + \tilde{\nu}^0 & K^0 \to \mu^+ + \pi^- + \tilde{\nu}^0 \\ \quad \to \pi^+ + \mu^0 + \tilde{\nu}^0 & \quad \to \pi^0 + \mu^0 + \tilde{\nu}^0 \\ K^- \to \mu^- + \pi^0 + \nu^0 & \tilde{K}^0 \to \mu^- + \pi^+ + \nu^0 \\ \quad \to \pi^- + \tilde{\mu}^0 + \nu^0, & \quad \to \pi^0 + \tilde{\mu}^0 + \nu^0, \end{array}\right\} \quad (2\cdot 34)$$

$$\left.\begin{array}{ll} K^+ \to e^+ + \pi^0 + \nu^0 & K^0 \to e^+ + \pi^- + \nu^0 \\ \quad \to \pi^+ + \tilde{e}^0 + \nu^0 & \quad \to \pi^0 + \tilde{e}^0 + \nu^0 \\ K^- \to e^- + \pi^0 + \tilde{\nu}^0 & \tilde{K}^0 \to e^- + \pi^+ + \tilde{\nu}^0 \\ \quad \to \pi^- + e^0 + \tilde{\nu}^0, & \quad \to \pi^0 + e^0 + \tilde{\nu}^0. \end{array}\right\} \quad (2\cdot 35)$$

From (2·34) and (2·35) we observe that a part of the anormalous three body decay of V^0-particles may be interpreted as the alternative decay modes of $K_{\mu 3}$ and K_{e3}. Experimental observation of the processes such as $K^0 \to \pi^0 + \mu^0 + \tilde{\nu}^0$, $\pi^0 + \tilde{e}^0 + \nu^0$; $\tilde{K}^0 \to \pi^0 + \tilde{\mu}^0 + \nu^0$, $\pi^0 + e^0 + \tilde{\nu}^0$ which are included in (2·34) or (2·35), will be very difficult. Moreover our scheme contains the following decay modes:

$$K^+ \to \pi^+ + \mu^0 + \tilde{\nu}^0 \quad \text{or} \quad \pi^+ + \tilde{e}^0 + \nu^0, \quad (2\cdot 36)$$

$$K^- \to \pi^- + \tilde{\mu}^0 + \nu^0 \quad \text{or} \quad \pi^- + e^0 + \tilde{\nu}^0. \quad (2\cdot 37)$$

As will be shown in section 6, relative to the normal $K_{\mu 3}$ and K_{e3} decay modes, the respective branching ratios of (2·36) and (2·37) are nearly equal to one, so in our scheme (2·36) and (2·37) will be permitted to be realized in nature with nearly the same frequency as the normal decay modes. However the decay mode $K^+ \to \pi^+ + \mu^0 + \tilde{\nu}^0$ or $\pi^+ + \tilde{e}^0 + \nu^0$ may not in practice be distinguished from $\tau'^+ \to \pi^+ + 2\pi^0$, since even one case of τ'^+ has not been identified directly as yet. We note here that there is a method of clarifying the difference between $\tau'^+ \to \pi^+ + 2\pi^0$ and $K^+ \to \pi^+ + \mu^0 + \tilde{\nu}^0$ or $\pi^+ + \tilde{e}^0 + \nu^0$. The method mentioned above is to investigate the energy spectrum of the secondary charged pion π in each of the decay modes $\tau' \to \pi + 2\pi^0$ and $K \to \pi + 2\nu$. Since Q value of the decay mode $K^+ \to \pi^+ + 2\nu$ is greater than that of $\tau'^+ \to \pi^+ + 2\pi^0$, the kinetic energy of the emitted positively charged pion π in the former case may, in some case, be greater than the maximum one of π in the latter, though such affirmation has not yet been attained.

In concluding this section we will add a few remarks on the allowed lepton reactions not mentioned in the above discussions where we have made restriction that the decays occur only into two and three particles. If we remove this restriction, we obtain the following reactions which are allowed by the selection rule (α);

$$K^+ \to \mu^+ + \tilde{\nu}^0 + \pi^+ + \pi^-, \quad (2\cdot 38)$$

$$K^+ \to e^+ + \nu^0 + \pi^+ + \pi^-, \quad (2\cdot 39)$$

$$\Xi^- \to \Lambda^0 + e^- + \tilde{\nu}^0 + \pi^0, \quad (2\cdot 40)$$

and their counter parts. Of course we do not here consider the cases where the real

emission of photons is concerned. If we consider (2·38) and (2·39) as the two π-mesonic decays of (2·10) and (2·10′) respectively, and (2·40) as the one π-mesonic decay of (2·17), the available phase space volumes for (2·38)$\sim$(2·40) become smaller than the respective ones of (2·11), (2·12) and (2·17). So the existence of the allowed processes (2·38)$\sim$(2·40) leads to no conflict with the fact that only the normal K- and Ξ-decays have been observed.

§ 3. Forbidden processes for lepton reactions

In this section we will shortly describe the main reactions which are not allowed by the selection rule (α). In particular we investigate the first forbidden processes which are forbidden by (α) but are allowed by the selection rule:

$$\begin{cases} \Delta N(\text{baryon}) = 0, \quad \Delta N(\text{lepton}) = 0, \\ |\Delta I| = \frac{1}{2}, \ \Delta Q = 0. \end{cases} \qquad (\beta)$$

Here we make assumption that the coupling strength in the first forbidden reactions is about 10^{-14} (dimensionless) times as small as that in the allowed lepton processes. Therefore in the lepton reactions the first forbidden processes are in general far beyond the present experimental affirmation. The typical processes in this category are given as follows:

$$\mu^- + P \to N + \tilde{\nu}^0, \qquad (3\cdot1)$$

$$\mu^+ \to e^+ + \mu^0 + \nu^0 \quad \text{or} \quad e^+ + e^0 + \nu^0, \qquad (3\cdot2)$$

$$N \to P + e^- + \tilde{\nu}^0, \qquad (3\cdot3)$$

$$\pi^+ \to \mu^+ + \tilde{\nu}^0 \quad \text{or} \quad e^+ + \nu^0, \qquad (3\cdot4)$$

$$K^+ \to \mu^+ + \tilde{\mu}^0, \ \mu^+ + \tilde{e}^0, \ e^+ + \mu^0 \quad \text{or} \quad e^+ + e^0, \qquad (3\cdot5)$$

$$K^+ \to \mu^+ + \tilde{\mu}^0 + \pi^0 \quad \text{or} \quad \mu^+ + \tilde{e}^0 + \pi^0, \qquad (3\cdot6)$$

$$K^+ \to e^+ + \mu^0 + \pi^0 \quad \text{or} \quad e^+ + e^0 + \pi^0, \qquad (3\cdot7)$$

$$K^+ \to \mu^+ + \tilde{\mu}^0 + \pi^+ + \pi^-, \qquad (3\cdot8)$$

$$K^+ \to e^+ + e^0 + \pi^+ + \pi^-. \qquad (3\cdot9)$$

Here we have neglected all the lepton reactions in which hyperons are included. Finally we add a few important forbidden reactions which do not belong to the first forbidden ones:

$$\begin{cases} \mu^- + P \to P + e^- \\ \mu^\pm \to e^\pm + e^+ + e^-, \end{cases} \quad |\Delta N(\text{lepton})| = 2. \qquad (3\cdot10)^{10)}$$

§ 4. Comparison with other models[11)]

(1) *Case of* $I(\mu\text{-meson}) = 1$

We consider here the case where the muon is assumed to have iso-spin one. This model

has recently been proposed by K. Senba et al.[11]

Table 2. Iso-spin assignment of model (1)

Particle	I	I_3	$Q=I_3+\frac{1}{2}(N+S)$	S	$U=N+S$	Transformation properties
μ^+, μ^0, μ^-	1	1, 0, −1	1, 0, −1	I_3	−1	0	iso-pseudovector
e^+, e^0	$\frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$	1, 0	$I_3+\frac{1}{2}$	0	1	1st kind iso-spinor
ν^0	0	0	0	I_3	−1	0	iso-scalar

Their model contains the several defects which arise from the assumption of the iso-scalar interaction Hamiltonians describing the decay processes: in the first place, as is seen from the iso-scalar interaction Hamiltonian describing the $K_{\mu 2}$-decay,

$$H(K\mu e)=g\{K^{+*}e^0\mu^+ + K^{0*}e^+\mu^- + \frac{1}{\sqrt{2}}(K^{+*}e^+ - K^{0*}e^0)\mu^0\} + h.c. \qquad (4\cdot 1)$$

$K^0\to\tilde{\mu}^- + e^+$ must be allowed with the same rate as the normal mode $K^+\to\tilde{\mu}^+ + e^0$. The former mode has not yet been detected although it should be easily identified, while the latter is the main mode of K-decays. Next, several unwanted processes become to be allowed when we consider the three body decays of K-meson. According to the iso-scalar interaction Hamiltonians corresponding to the $K_{\mu 3}$ and K_{e3} decays, the following modes of decay are contained in them:

$$\begin{cases} K^+\to\tilde{\mu}^- + e^+ + \pi^+ & c_0 \quad \text{no} \\ \to\tilde{\mu}^0 + e^+ + \pi^0 & c_0 \\ \to\tilde{\mu}^+ + e^+ + \pi^- & c_0 \quad \text{no} \end{cases} \qquad (4\cdot 2a)$$

$$\begin{cases} K^0\to\tilde{\mu}^- + e^0 + \pi^+ & c_0 \quad \text{yes} \\ \to\tilde{\mu}^0 + e^0 + \pi^0 & c_0 \\ \to\tilde{\mu}^+ + e^0 + \pi^- & c_0 \quad \text{yes} \end{cases} \qquad (4\cdot 2b)$$

$$\begin{cases} K^+\to\tilde{\mu}^0 + e^0 + \pi^+ & c_1 \\ \to\tilde{\mu}^+ + e^0 + \pi^0 & c_1 \quad \text{yes} \\ \to\tilde{\mu}^- + e^+ + \pi^+ & c_1/\sqrt{2} \quad \text{no} \\ \to\tilde{\mu}^+ + e^+ + \pi^- & c_1/\sqrt{2} \quad \text{no} \end{cases} \qquad (4\cdot 3a)$$

$$\begin{cases} K^0\to\tilde{\mu}^- + e^+ + \pi^0 & c_1 \quad \text{no} \\ \to\tilde{\mu}^0 + e^+ + \pi^- & c_1 \quad \text{yes} \\ \to\tilde{\mu}^- + e^0 + \pi^+ & c_1/\sqrt{2} \quad \text{yes} \\ \to\tilde{\mu}^+ + e^0 + \pi^- & c_1/\sqrt{2} \quad \text{yes} \end{cases} \qquad (4\cdot 3b)$$

where c_0, c_1, $\cdots$ are constant and mean the coefficient ratios between the reaction modes included in the iso-scalar Hamiltonian. We express by "yes" or by "no" whether the reaction mode under consideration is consistent with experiment or not. The reaction modes with no mark should be understood to be obscure whether they belong to "yes" or "no". There is no remarkable difference in the phase space volume, between the decay processes $K^+\to\tilde{\mu}^+ + e^0 + \pi^0$ and $K^+\to\tilde{\mu}^- + e^+ + \pi^+$, so that the two processes will be expected to occur with probabilities differing at most by a factor of order of unity. However in experiments no cases for the latter has been found while for the former about 30 cases have been reported till now. Thus we can conclude that, even if we consider only three body decay of K-mesons, the model used here is inconsistent with

experiments.

(2) *Case where the iso-spins are not assigned for any one of leptons.*

In order to allow all the well established lepton reactions, we must assume the following selection rule :

$$\begin{cases} \Delta N(\text{baryon}) = 0, \quad \Delta N(\text{lepton}) = 0, \\ |\Delta I| = |\Delta I_3| = 0, \frac{1}{2}, 1 ; \quad \Delta Q = 0. \end{cases} \tag{γ}$$

As is seen from (γ), the allowed lepton interaction Hamiltonians become in some case iso-scalar, and in other case iso-spinor or iso-vector. Therefore we cannot unify the interactions of the elementary particles by using the transformation properties of the interaction Hamiltonians in the iso-space. Moreover in this case the unwanted processes such as

$$\begin{cases} K \to \pi + e + e^-, \quad K \to \pi + \mu' + \mu'^-, \\ K^0 \to \mu^+ + \mu^-, \cdots \end{cases} \tag{4·4}$$

still remain not to be forbidden by the selection rule (γ). It will be very difficult to forbid all the prosesses (4·4) by some appropriate devices.

§ 5. A possible example of giving the mass difference between the charged and neutral leptons

In the investigation of the lepton processes we have made the following assumptions besides the assumption (I):

(II) (μ^+, μ^0), (e^0, e^-) and ν^0 are all taken as the normal Fermi particles, and have the iso-spins $\frac{1}{2}$, $\frac{1}{2}$ and 0 respectively. Moreover the rest masses of μ^0 and e^0, which are respectively the neutral counter parts of μ^+ and e^-, are vanishingly small in the practically observed lepton reactions. Therefore μ^0, e^0 and ν^0 are a kind of neutrino though the transformation properties in the iso-space are different from each other.

(III) For the lepton processes we set the following selection rule :

$$\begin{cases} \Delta N(\text{baryon}) = 0, \quad \Delta N(\text{lepton}) = 0, \\ \Delta I = 0, \quad \Delta S = 0. \end{cases} \tag{α}$$

By using the above three assumptions we have been able to give a unified interpretation for the lepton reactions. In this section we will try to analyze the assumptions we have made, especially the assumption (I). Analysis of the assumptions (I) and (III) will be made in the next section by connecting them to the universality of the weak interactions.

Speaking in a word, the assumptions (II) and (III) are another expression of the fact that at least one neutrino is involved in the lepton reactions. At first sight the assumptions (II) and (III) may seem to be contradictory to each other since the assumption (III) implies the charge independence, while the assumption (II) results in

the charge dependence. We show here a possible way of reconciling (III) with (II) by rough calculation. We introduce a neutral bose field which has nonzero rest mass and interacts only with the neutral leptons. We will call this field as the L-field. To make the discussion simple, we assume the L-field to be scalar and take the Hamiltonian describing the interaction between neutral lepton and the L-field as follows:

$$H=\frac{il}{M}\bar{\psi}\gamma_\lambda\psi\partial_\lambda L\,, \tag{5·1}$$

where M means the rest mass of the L-field, and l is the coupling constant. We will take $l^2/\hbar c\sim 1/137$. The L-mesonic self energy of the neutral lepton is given by the perturbational calculation as follows ($\hbar=c=1$):

$$\begin{aligned}\delta m=&-\frac{l^2}{8\pi^2M^2}\int_0^P\frac{d\vec{p}}{\sqrt{p^2+m^2}\,\sqrt{p^2+M^2}}\\&\times\Big\{\Big(\frac{\sqrt{p^2+m^2}+m}{\sqrt{p^2+m^2}+\sqrt{p^2+M^2}-m}-\frac{\sqrt{p^2+m^2}-m}{\sqrt{p^2+m^2}+\sqrt{p^2+M^2}+m}\Big)(p^2+M^2)+\\&+\Big(\frac{1}{\sqrt{p^2+m^2}+\sqrt{p^2+M^2}-m}-\frac{1}{\sqrt{p^2+m^2}+\sqrt{p^2+M^2}+m}\Big)p^2\sqrt{p^2+M^2}\Big\},\end{aligned} \tag{5·2}$$

where m means the mechanical mass of the neutral lepton, and P a cut off momentum. Since δm is negative, the rest masses of neutral leptons (μ^0 and e^0) may be reduced to zero, the mass of neutrino, if an appropriate cut off momentum and a sufficiently large mass of L-particle are assumed in (5·2).

In concluding this section we give a short remark on the observability of the L-particle. According to our assumption the L-particle interacts only with neutrinos, so that creation of L-particle occurs only through the neutrino-neutrino collision. But the detection of L-particle will be far beyond possibility.

§ 6. Interpretation of the universality of the weak interactions[12]

It has been remarked in the previous section that the neutrino is considered to be the source of the weak interactions in the lepton processes. In this section we will try to unify both of the weak interactions realized in the baryon-boson system and in the lepton reactions by using the first forbidden processes defined in the section 3.

In the first place we will consider the K-meson decays which consist of the $K-\pi$ decays ($\tau\ \to 3\pi$, $\theta\ \to 2\pi$) and $K-\nu$ decays (K_{μ^3}, $K_{\mu 2}$ and K_{e3} decays). G. Wentzel[13] has recently noted that the selection rule

$$|\Delta I|=\tfrac{1}{2} \tag{6·1}$$

accounts for showing the long life of θ^+ relative to that of θ^0 and the frequency ratio $4\cdot1\sim4\cdot6$ for τ^+ to τ'^+. In accordance with G. Wentzel we assume the selection rule (6·1) for the $K-\pi$ decays. Thus we get, on the one hand,

$$\tau^+ \to \pi^+ + \pi^+ + \pi^- , \tag{6·2}$$

and, on the other, have

$$\pi^+ \to \mu^+ + \tilde{\mu}^0, \tag{6·3}$$

$$\Delta I = 0.$$

$$\pi^+ \to e^+ + e^0, \tag{6·4}$$

We will notice here that the reaction (6·2) can be derived formally from the first forbidden lepton process (3·8) ((3·9)) by putting π^+ instead of $\mu^+ + \tilde{\mu}^0 (e^+ + e^0)$ in the right-hand side of (3·8) ((3·9)) by using the relation (6·3) ((6·4)). Therefore we can consider that the first forbidden process (6·2) in the baryon-boson system corresponds to the first forbidden process (3·8) ((3·9)) in the lepton reactions. Moreover owing to the above substitution the lepton disappears from the first forbidden lepton process (3·8) ((3·9)), and at the same time (3·8) ((3·9)) is transformed to the process (6·2) without neutrino. As mentioned in the previous section, the neutrino is considered to be the source of the weak lepton interaction, so the disappearance of the neutrino changes the coupling strength ($\sim 10^{-28}$) of the first forbidden lepton process (3·8) ((3·9)) to the coupling strength ($\sim 10^{-14}$) of the process (6·2) which is 10^{14} times larger than 10^{-28} and is the same strength of the allowed lepton processes ($\sim 10^{-14}$).

Similar discussion can be made for the process

$$\theta^+ \to \pi^+ + \pi^0, \qquad |\Delta I| = \tfrac{1}{2}, \tag{6·5}$$

and for the first forbidden lepton reactions (3·6) and (3·7). Analogous discussion is also true for the hyperon decays, for instance,

$$\Lambda^0 \to P + \pi^-, \qquad |\Delta I| = \tfrac{1}{2}, \tag{6·6}$$

corresponds to the first forbidden lepton reactions such as

$$\Lambda^0 \to P + \mu^- + \nu^0, \tag{6·7}$$

$$|\Delta I| = \tfrac{1}{2}.$$

$$\Lambda^0 \to P + e^- + \tilde{e}^0, \tag{6·8}$$

In short we may say that the source of the universality of the weak interactions realized in nature is on the one hand neutrino ($\Delta I = 0$) and on the other the selection rule $|\Delta I| = \frac{1}{2}$ when neutrino is not concerned.

§ 7. Application to the *K*-meson decays*

In the following discussion we assume that *K*-meson has zero spin and odd or even

* After this calculation has been finished, we knew about the similar investigation performed by Huruiti et al. from the lecture made in the annual meeting of the Physical Society of Japan, held at Sendai, in July, 1956.

parity according as it shows τ- or θ-mode of decay.[14] For the sake of simplicity we will call the respective K-meson with even or odd parity as K_+ or K_-. As was already mentioned in section 1, in the case of $K_{\mu 2}(K_{e2})$ decay the difference of the parity of parent has no effect to the decay life observed, so that it will be natural to assume that both K_+ and K_- have the same transition rate for the $K_{\mu 2}(K_{e2})$ decay.[15]

Now as is well known, the τ-decay is the important phenomena which give information about the spin-parity of the K-meson, but also the $K_{\mu 3}$- and K_{e3}-decays supply the knowledge on the spin of K-meson though nothing is said about the parity of K-meson. Hereafer the $K_{\mu 3}$- or K_{e3}-decay will be considered respectively to be the π-mesonic decay of the $K_{\mu 2}$-or K_{e2}-decays. Since we have used the gradient coupling for the $K_{\mu 2}$- and K_{e2}-decays, we take for the $K_{\mu 3}$- and K_{e3}-decays the following interaction Hamiltonian:

$$H=g'(\bar{\Psi} I'\gamma_u \psi)(\partial_u \Phi)\phi^*+h.c.\,, \tag{7·1}$$

where Ψ, ψ, Φ and ϕ denote respectively the muon or electron, neutrino, K-meson and π-meson. I' takes 1 or γ_5 according as Φ is scalar or pseudoscalar, if we assume Ψ and ψ have same parity. Since g' has dimension, it will be transformed to a dimensionless strength.

According to (3·1) the transition probabilities in the rest system of K-meson for the decays $K^+\to\mu^+ +\tilde{\nu}^0+\pi^0$ and $K^+\to e^+ +\nu^0+\pi^0$ per unit time are given by the perturbational calculation as follows:

$$w(K^+\to\mu^+ +\tilde{\nu}^0+\pi^0)=\frac{g'^2}{2\pi}m_K{}^5$$

$$\times\int_{\alpha_\mu}^{\alpha_\mu{}^0}(1+\alpha_\mu^2-\alpha_\pi^2-2\varepsilon)^2(\alpha_\mu^2+\varepsilon-2\varepsilon^2)(\varepsilon^2-\alpha_\mu^2)^{1/2}(1+\alpha_\mu^2-2\varepsilon)^{-2}d\varepsilon, \tag{7·2}$$

$$w(K^+\to e^+ +\nu^0+\pi^0)=\frac{g'^2}{2\pi}m_K{}^5$$

$$\times\int_{\alpha_e}^{\alpha_e{}^0}(1+\alpha_e{}^2-\alpha_\pi^2-2\varepsilon)^2(\alpha_e{}^2+\varepsilon-2\varepsilon^2)(\varepsilon^2-\alpha_e{}^2)^{1/2}(1+\alpha_e{}^2-2\varepsilon)^{-2}d\varepsilon \tag{7·3}$$

with abbreviation $\alpha_\mu=m_\mu/m_K$, $\alpha_e=m_e/m_K$, $\alpha_\pi=m_\pi/m_K$. $\alpha_\mu{}^0$ and $\alpha_e{}^2$ are defined by

$$\alpha_\mu{}^0=\tfrac{1}{2}+\tfrac{1}{2}\alpha_\mu^2-\tfrac{1}{2}\alpha_\pi^2\,, \tag{7·4}$$

$$\alpha_e{}^0=\tfrac{1}{2}+\tfrac{1}{2}\alpha_e{}^2-\tfrac{1}{2}\alpha_\pi^2. \tag{7·5}$$

$m_K\varepsilon$ means the energy (including the rest energy) of the charged secondary μ (e), so $m_K\alpha_\mu{}^0(m_K\alpha_e{}^0)$ is its maximum value. The integrand of (7·2) ((7·3)) gives the energy spectrum of the emitted charged particle μ (e) in the $K_{\mu 3}(K_{e3})$ decay. These results are shown in Fig. 2 and Fig. 3.

Remembering $\alpha_e\fallingdotseq 0.001$, we get from (7·3)

$$w(K^+\to e^+ +\nu^0+\pi^0)=\frac{g'^2}{2\pi}m_K{}^5$$

$$\times\left[-\frac{1}{2}\alpha^4+\frac{1}{3}(1-2\alpha_\pi^2)\alpha^3-\frac{1}{4}\alpha_\pi^2(1+\alpha^2)-\frac{1}{8}\alpha_\pi^4\log(1-2\alpha)\right], \quad (7\cdot6)$$

with
$$\alpha=\tfrac{1}{2}(1-\alpha_\pi^2). \quad (7\cdot7)$$

For the $K_{\mu 2}$-decay with gradient coupling we have

$$w(K^+\to\mu^++\tilde{\nu}^0)=\tfrac{1}{2}g^2m_K\alpha_\mu^2(1-\alpha_\mu^2)^2. \quad (7\cdot8)$$

Therefore we obtain from (7·6) and (7·8)

$$w(K^+\to e^++\nu^0+\pi^0)/w(K^+\to\mu^++\tilde{\nu}^0)\simeq 5\cdot g'^2m_\pi^4/g^2. \quad (7\cdot9)$$

Taking $\tau(K^+\to\mu^++\tilde{\nu}^0)=1.2\times10^{-8}$ sec, $m_K=966\,m_e$, $m_\pi=273\,m_e$, $m_\mu=207m_e$, we get $g^2=5.3\times10^{-14}$ and so obtain the following relation,

$$w(K^+\to e^++\nu^0+\pi^0)/w(K^+\to\mu^++\tilde{\nu}^0)\simeq 0.1, \quad (7\cdot10)$$

provided that the coupling constants g' and g are connected by

$$50g'^2m_\pi^4=g^2=5.3\times10^{-14}. \quad (7\cdot11)$$

On the other hand

$$w(K^+\to\mu^++\tilde{\nu}^0+\pi^0)/w(K^+\to e^++\nu^0+\pi^0)\simeq 0.8 \quad (7\cdot12)$$

is true if the same coupling constant is taken for (7·2) and (7·3). Combining (7·10) and (7·12),

$$w(K^+\to\mu^++\tilde{\nu}^0+\pi^0)/w(K^+\to\mu^++\tilde{\nu}^0)\simeq 0.08 \quad (7\cdot13)$$

is derived at once. The results of experiments (Table 3) are not inconsistent with (7·10), (7·12) and (7·13).

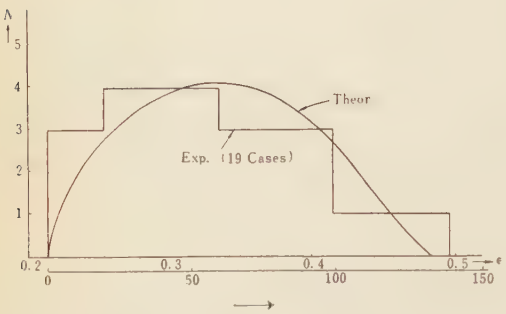
Table 3. Observed branching ratios for decay of K mesons produced and observed under different conditions[17]

Observer	Rochester (Hoang et al.)	Berkeley (Ritson et al.)	Paris (Crussard et al.)
mean modulation time	2.1×10^{-8} sec	1.5×10^{-8} sec	$\sim5\times10^{-8}$ sec
production mode	2.9 Bev protons	6.5 Bev protons	6.2 Bev protons
$K_{\mu 2}$	59%	57%	54%
$K_{\pi 2}$	21%	29%	26%
τ	5.2%	7.5%	7%
$K_{\pi 3}$	3.5%	2%	1%
$K_{\mu 3}$	6%	2%	4%
K_{e3}	5%	2%	8%

Also the following relations are easily derived :

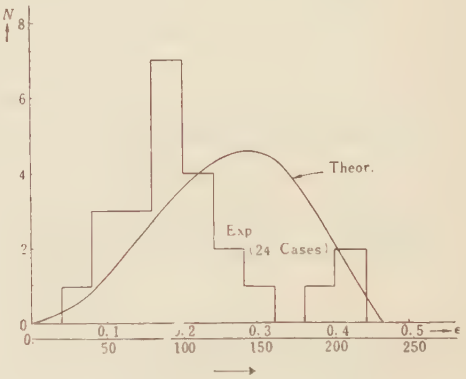
$$w(K^+\to\pi^+ +\mu^0+\tilde{\nu}^0) = w(K^+\to\pi^+ +\tilde{e}^0+\nu^0) \fallingdotseq 2w(K^+\to e^+ +\nu^0+\pi^0), \qquad (7\cdot 14)$$

where factor 2 in the right-hand side of (7·14) comes from the iso-spin consideration.



kinetic energy of μ^+ (Mev)

Fig. 2. The energy spectrum of μ^+ in the decay $K^+\to\mu^+ +\tilde{\nu}^0+\pi^0$



kinetic energy of e^+ (Mev)

Fig. 3. The energy spectrum of e^+ in the decary $K^+\to e^+ +\nu^0+\pi^0$

Now we consider the results of Fig. 2 and Fig. 3. The energy spectrum of the secondary μ^+ in our calculation agrees pretty well with that of experiments. In the case of K_{e3}-decay agreement is very bad, especially in the energy interval 80∼200 Mev, but this will not be vital if we consider that the data are very poor. Theoretically it is difficult to expect that the energy spectrum of the charged secondary in the K_{e3}-decay has a very different nature from that in the $K_{\mu 3}$ decay, for the effect of the secondary π^0 covers that of μ^+ or e^+, so the characteristic nature of μ^+ or e^+ does not contribute very much to the decay nature of $K_{\mu 3}$ or K_{e3}. At any rate it may be too early to derive any definite conclusion on the spin and type of coupling of the K-particle from the K_{e3}- and $K_{\mu 3}$-decay events. However in the decays $K_{\mu 2}$, $K_{\mu 3}$, K_{e3}, K-meson with zero spin is not inconsistent with experiments, and its coupling type is favourable to the gradient one.

§ 8. Conclusion

We have shown in this paper that the lepton reactions (which have been excluded in the frame of Nishijima-Gell-Mann's theory) can be consistently understood in a unified way under the assumptions (I), (II) and (III). In that case the assumption (II), which introduced the three kinds of neutrinos, played the most important role. In fact it is this assumption which makes possible to distinguish the lepton interactions sharply from the strong baryon-boson interactions and to unify the weak interactions.

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Letters to the Editor

Elastic Scattering of 24.6-Mev Alpha Particles by Au

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The differential elastic scattering cross sections of medium energy alpha particles from heavy elements have recently been calculated by several authors.[1-5] Wall, Rees and Ford[1] have calculated the cross sections of 22 Mev alpha particles, using the Blair model[6] as well as the fuzzy Blair model.[1] In the Blair model, which was proposed by Blair in order to successfully explain the energy dependence of the cross sections of alpha particles, it is assumed that η_l, the amplitude of the outgoing wave with orbital angular momentum l, is $e^{2i\sigma_l}$ for $l>l'$, where σ_l is the Coulomb phase shift, and is zero for $l\leq l'$. The critical value l' is assumed to be given by the relation

$$E=(\hbar^2/2M)\,l'(l'+1)/R^2+2Ze^2/R.$$

Here E is the kinetic energy of the incident alpha particle relative to the nucleus, M is the reduced mass of the relative motion, R is the interaction radius set equal to the sum of the radii of the nucleus and the alpha particle, Ze is the charge of the target nucleus. In the fuzzy Blair model proposed by Wall, Rees and Ford, the amplitude η_l with $l=l'$ is assumed to be $e^{2i\sigma_l}/2$ rather than to be zero, and the other η_l's with $l\neq l'$ are assumed to be the same as in the Blair model. The results of the calculations with these models showed the diffraction oscillations in backward directions which have not appeared in the experimental data. Other calculations[3-4] using the same model also showed similar features, revealing that these models are not suitable to explain the experimental data.[1),3-4]

N. Oda and K. Harada[5] have recently calculated the angular distributions of alpha particles, using the continuum theory and the Coulomb wave functions calculated by the W. K. B. approximation.[7] In this calculation, they also used the Blair like cut-off procedure to the summation over l in the partial wave analysis. However, they have mistaken the sign of A^l_{res} in eq. (2·6) in their paper, which is misprinted in Blatt and Weisskopf's textbook.[7] Not only N. Oda and K. Harada, but also the other authors have not indicated how the penetration factors reduce with increasing l values, so that it is not clear whether their cut-off at l' for the summation over l are reasonable or not.

In order to see how various partial waves contribute to the cross section, the differential elastic scattering cross section of 24.6 Mev alpha particles from Au is calculated by the exact partial wave method. The calculation has been carried out by using the continuum theory[8] for the interaction between the incident alpha

particles and the target nucleus, because the compound nucleus formed by Au after the capture of the alpha particle is supposed to have many open channels. The values of the Coulomb wave functions and their derivatives necessary for in the calculation have been obtained by using the recurrence relation of the Coulomb wave functions.[9] The values of the s-wave functions are tabulated in reference 10. This table can only be used in the case of $kR=2\gamma$, where k is the wave number, and γ is the Coulomb parameter.[7] If the interaction radius R is taken to be 9.36×10^{-13} cm, the above cited table of s-wave functions can be used under the condition $kR=\gamma=20$.

In the Table 1, the values of the penetration factors v_l[7] of the l-th partial wave are tabulated up to $l=14$. As one sees, the penetration factor does not fall off rapidly, so that one could not neglect the contributions from the partial waves up to $l=13$.

Table 1. The value of the penetration factor v_l

l	0	1	2	3	4
v_l	0.2321	0.2263	0.2149	0.1979	0.1758
l	5	6	7	8	9
v_l	0.1496	0.1206	0.0910	0.0633	0.0401
l	10	11	12	13	14
v_l	0.0229	0.0117	0.0053	0.0022	0.0008

In Figure 1, the differential elastic scattering cross section of 24.6 Mev alpha particles from Au is plotted in the ratio to the Coulomb cross section. The solid curve represents the cross section calculated by the exact partial wave analysis with the continuum theory. For the purpose of the comparison with Oda and Harada's calculation at 22 Mev incident energy, the cross section obtained by taking into account only the contributions from the partial waves with $l\leq4$ is also plotted by the dotted curve.

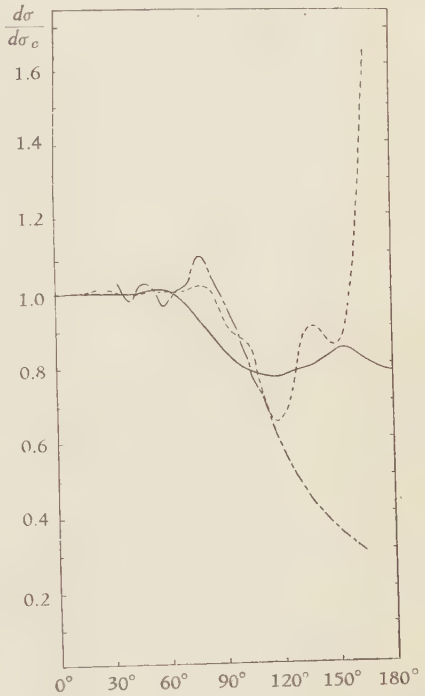

Figure 1.

The differential elastic scattering cross sections of 24.6 Mev alpha particles from Au are plotted in the ratio to the Coulomb cross section. Solid line represents the cross section calculated by the exact partial wave analysis with the continuum theory. The dotted line shows the cross section calculated by the Blair-like cut-off procedure at $l=4$. These calculations are done by using the square well potential with depth of 20 Mev and the interaction radius of 9.36×10^{-13} cm.

The dot-and-dashed curve represents the experimental result of 22 Mev alpha particles.

Although, in the present result, the diffraction oscillation does not appear in contrast to the calculation with the Blair or fuzzy Blair model, the ratio does not fall off rapidly at backward angles as compared with an expected experimental

data at 24.6 Mev, which would be similar to the data at 22 Mev[1] represented in Figure 1 by the dot-and-dashed line. Agreement with experimental data might probably be improved by adopting a complex potential with tail. Then, because of the reduction of the potential barrier at the smoothed-out nuclear boundary, we may expect the smaller elastic scattering cross section, and hence the smaller ratio to the Coulomb cross section which is very small at large angles. In fact, by using a diffused potential, Woods and Saxon[11] have succeeded in reducing the ratio of the elastic scattering cross section of 22 Mev protons to the Coulomb cross section at large angles, which was too high in the calculation with a square well optical potential[12] as compared with experimental data. The detailed study of the effect of this diffuseness is now under way and will shortly be submitted to this journal.

The authors would like to express their sincere thanks to Professor T. Miyazima for his kind interest and encouragement. Thanks are also due to M. Nagasaki for valuable discussions and suggestions.

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On the Effective Normal Fluid Density of Solution of He^3 in He^4

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November 8, 1956

Recently, Pellam[1] has reported measurements of the effective "normal fluid" density in a 3.3% (by mass) solution of He^3 in He^4 over a fairly wide range of temperatures. In the present note we have made use of a model recently proposed by us[2),3),4)], to explain the observed data reported by Pellam.

Assuming that He^3 atoms partake in the motion of the normal fluid, the density ρ_n of the normal fluid can be written as

$$\rho_n=\rho_{4n}+\rho_{3n}=\rho_{4n}+\rho_3(\mu/m_3), \qquad (1)$$

where ρ_{4n} is the density of the normal part of the He^4 atoms in solution while ρ_{3n} is the contribution to the normal density due to the presence of He^3. Following Pomeranchuk[5], we write

$$\rho_{3n}=\rho_3(\mu/m_3),$$

where μ is the effective mass of the He^3 atoms in solution while m_3 is the actual

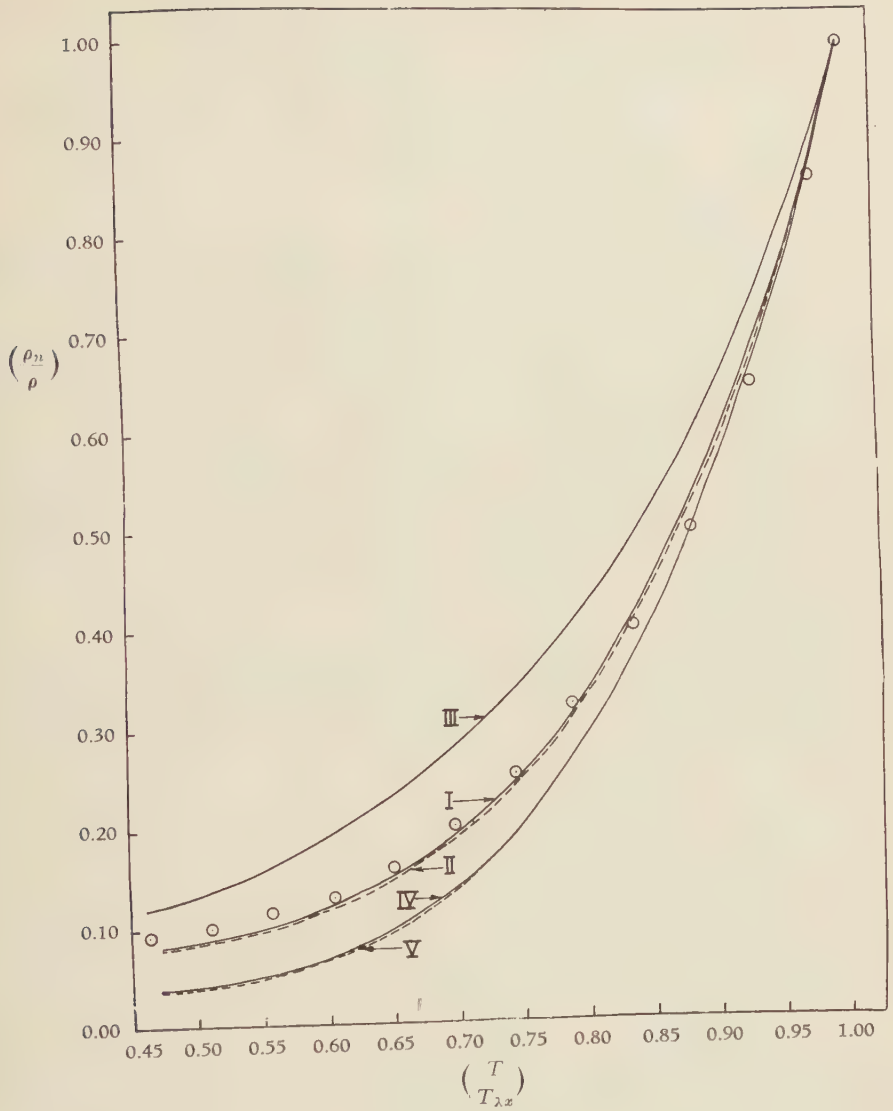

Fig. 1. The effective "normal fluid" concentration (ρ_n/ρ) for 3.3% (by mass) solution of He^3 in He^4, as a function of $(T/T_{\lambda x})$, $T_{\lambda x}$ being the λ-temperature of the solution.

Curve I: Present Model, Linear Expression for G_4.
Curve II: Present Model, Quadratic Expression for G_4.
Curve III: de Boer and Gorter's theory[7].
Curve IV: Mikura's "Modified B. E. Liquid" Model[8].
Curve V: Mikura's "Revised B. E. Liquid" Model[9].
⊙ : Smoothed experimental results of Pellam[1] (corrected for the skin depth effects).

mass. Thus the expression for the effective "normal fluid" concentration in a solution of He^3 in He^4 can be written as

$$\rho_n/\rho = x(1-x_3) + x_3(\mu/m_3), \qquad (2)$$

where x is the fraction of the total number of atoms of He^4 which constitute the normal fluid, while $x_3 (=\rho_3/\rho)$ is the concentration (by mass) of He^3 in the solution.

We have calculated the effective "normal fluid" concentration in a 3.3 percent (by mass) solution of He^3 in He^4 on the basis of our model by making use of the linear and quadratic expressions[6] for G_4 and using $\mu/m_3 = 2.17$, as required to explain the second sound velocity measurements in solutions.[3] The results thus obtained are compared with various theories,[7),8),9] using eq. (2) with $\mu = m_3$, and the experimental results.[1] It is evident from Fig. 1 that the present model describes the observed behaviour better than all the other theories.

Our grateful thanks are due to Professor D. S. Kothari for his interest in this investigation.

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ERRATA

Energy Loss of a Charged Particle Traversing Superconductors

S. Hayakawa and K. Kitao

Prog. Theor. Phys. **16** (1956), 131

		originally	should be read
p. 132	(2·2)	$\rho^0_\omega(Z_1e/2\pi c_l\beta)\dots$	$\rho^0_\omega=(Z_1e/2\pi c_l\beta)\dots$
p. 133	(2·6b)	$A_\omega=(Z_1e_l\beta/\pi c_l\beta)\dots$	$\boldsymbol{A}_\omega=(Z_1e\beta/\pi c_l\beta)\dots$
p. 133	l. 6	ϕ_ω and and $\boldsymbol{A}_\omega$	ϕ_ω and $\boldsymbol{A}_\omega$
p. 133	(2·9)	$-\left.\frac{dW}{dz_0}\right\|_{>r}=-\frac{r}{c_l\beta}\int_{-\infty}^{\infty}E_zH_\theta dz$	$-\left.\frac{dW}{dz_0}\right\|_{>r}=-\frac{r}{2_l\beta}\int_{-\infty}^{\infty}E_zH_\theta dz$
p. 135	(2·18)	$-\left.\frac{dW}{dz_0}\right\|_{<r,B}=\dots$	$-\left.\frac{dW}{dz_0}\right\|_{>r,B}=\dots$

Energy Loss of a Charged Particle Traversing Ionized Gas and Injection Energies of Cosmic Rays

S. Hayakawa and K. Kitao

Prog. Theor. Phys. **16** (1956), 139

		originally	should be read
p. 140	l. 15	auroral particles,	coronal particles
p. 142	l. 7	toi ncrease	to increase
p. 143	Fig. 1a (upper figure) must be turned upside down.		
p. 148	l. 19	in the corona[6)10)]	in the corona[8)10)]
p. 148	reference 1)	Phys. Rev. **94** (1954), 191	Phys. Rev. **94** (1954), 440

A New Formulation on the Electromagnetic Field

T. Ohmura

Prog. Theor. Phys. **16** (1956), 684 (L)

The title in CONTENTS on back cover, Vol. 16, No. 6, is to be replaced by the correct one, mentioned above.

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Volume 11, Number 12, December 1956

CONTENTS

Kiichi KIMURA, Isao KUMABE, Kozo MIYAKE, Hiroshi OGATA: The Reaction $O^{16}(n,\alpha)C^{13}$..1211
Jun-ichi HORI: On the Equivalence of Fluctuation-Dissipation Theorem and Irreversible Thermodynamics ..1220
Sukeaki HOSOYA, Seiya SATAKE and Satio TAKAGI: On the Polygonization of Rock Salt Crystals ..1228
Sukeaki HOSOYA: On the Polygonized Domain Pattern in Bent and Annealed NaCl Crystals Cisualized by Colour Centre Colloids ..1231
Shozo SAWADA: Thermal and Electrical Properties of Tungsten Oxide (WO_3) ..1237
Shozo SAWADA: Microscopic and X-ray Studies on Tungsten Oxide (WO_3) ..1246
Shuichiro TAKAHASHI: Internal Friction and Critical Stress of Copper Alloys ..1253
Eiko AIHARA and Kazuo NIIRA: The Bond Orbital Approximation for π-electron System in Benzene ..1261
Toshizo NAKAYA: Influence of Illumination from Outside on an As-Hg Discharge Tube ..1264
Itiro TANI and Hiroshi SATO: Boundary-Layer Transition by Roughness Element ..1284
Yoshio YASUOKA: On the Space Charge affected by the Magnetic Field Part II ..1292

SHORT NOTES

Syohei MIYAHARA and Haruyuki OHNISHI: Cation Arrangement and Magnetic Properties of Copper Ferrite-Chromite Series ..1295
Sumiaki IBUKI and Kenzō AWAZU: A Method of Growing CdS Single Crystals by Sublimation ..1297
Etsuro SAWAGUCHI, Takeshi MITSUMA and Zensho ISHII: Double Hysteresis Loop of $(Pb_\lambda Ca_{1-\lambda})TiO_3$ Ceramics ..1298
Yoshio INUISHI: On the Pulse Breakdown of Plastic Films at Higher Temperatures ..1299
Shūichi IIDA: Note on Superexchange Interactions and Magnetic Lattices of the Rhombohedral Sesquioxides ..1300
Eiichi FUKADA: On the Piezoelectric Effect of Silk Fibers ..1301
Kiyokata MATSUURA, Yoshikazu SUGIURA and G. M. HATOYAMA: Frequency Shift in Ammonia Absorption ..1301

Progress of Theoretical Physics, Vol. 17, No. 2, February 1957

On a Solution of Field Equations in Einstein's Unified Field Theory, II

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This paper contains some observations regarding a type of non-static solution of Einstein's field equations in "strong" form. Denoting the coordinates by x_k, x_m, x_n, x_l a static field $g_{\mu\nu}$ with components involving the coordinate x_k is made non-static by changing x_k into $x_k+\varepsilon x_l$ where ε is a constant. For the spherically symmetric case we find that such a solution of field equations is generally transformable into Bonnor's solution for the static field. When, however, the field components are all functions of $(x_k+\varepsilon x_l)$ only and the field is not spherically symmetric we obtain a non-trivial solution.

§ 1. Introduction

In this part of the paper we propose to investigate into a new type of solution of Einstein's field equations in the "strong" form in continuation of the simpler types considered in the first part.[1] Adopting the notations and abbreviations introduced therein and using general indices k, m, n, l as before we start with the fundamental non-symmetric field-tensor $g_{\mu\nu}$ having the following structure:

The components s_{kk}, s_{ll}, $a_{kl}(=-a_{lk})$ are functions of (x_k, x_l) only, while each of the components s_{mm}, s_{nn}, $a_{mn}(=-a_{nm})$ is the product of a function of (x_k, x_l) and a function of x_m of the type

$$\begin{aligned} s_{mm} &= \phi_{mm}(x_k, x_l)\,\psi_{mm}(x_m), \\ s_{nn} &= \phi_{mm}(x_k, x_l)\,\psi_{nn}(x_m), \\ a_{mn} &= \phi_{mn}(x_k, x_l)\,\psi_{mn}(x_m). \end{aligned} \tag{1·1}$$

On introducing polar coordinates $x_k=r$, $x_m=\theta$, $x_n=\phi$, $x_l=t$ this will yield solutions of field equations for a non-static tensor field.

For the calculation of the Γ's and the construction of the field equations corresponding to the tensor field (1·1) we make use of the general formulae given in one of my earlier papers.[2] For a spherically symmetric field, which is included in (1·1), these have already been obtained by Takeno, Ikeda and Abe,[3] Mavrides,[4] Rao[5] and others. As regards the solution of this complicated set of equations very little progress has been made so far. With a view to introduce simplicity we assume that all the functions of (x_k, x_l) which occur as components or as factors in the components of the tensor $g_{\mu\nu}$

in $(1\cdot1)$ are functions of the argument $(x_k+\varepsilon x_l)$, where ε is a constant real or imaginary. We characterize such a field as having a 'wave' form.[6] The resulting set of field equations has then been tackled by a new substitution method which finally yields the result that for a spherically symmetric field such a solution is transformable into Bonnor's solution discussed in my previous paper.[1] A new solution is, however, obtained having this "wave" character for the tensor field $g_{\mu\nu}$ in $(1\cdot1)$, all of whose components are functions of (x_k, x_l) only and of the typical form $f(x_k+\varepsilon x_l)$.

§ 2. Expression for the Γ and the general field equations

Referring to my earlier paper[2] we note that out of the 12 quantities A's, B's, H's only 7 survive with respect to the tensor field $(1\cdot1)$. We write them in the following from

$$\left.\begin{aligned} A_\sigma &= (\gamma_{mm\sigma}-c^n_m c^m_n \gamma_{mn\sigma})/(1-c^n_m c^m_n),\\ B_\sigma &= (\gamma_{mn\sigma}-\gamma_{mn\sigma})/(1-c^n_m c^m_n),\\ H_\sigma &= (\gamma_{kk\sigma}+\gamma_{ll\sigma}-2\gamma_{kl\sigma})/(1-c^l_k c^k_l),\\ H_m &= (\gamma_{mmm}+\gamma_{nnm}-2\gamma_{mnm})/(1-c^n_m c^m_n). \end{aligned}\right\} \qquad \sigma=(k,\,l) \qquad (2\cdot1)$$

In the above, symbols like γ_{mmk}, γ_{kll} denote respectively $\frac{s_{mm,k}}{s_{mm}}$, $\frac{a_{kl,l}}{a_{kl}}$ while symbols like c^k_l, c^m_n stand for a_{kl}/s_{kk}, a_{mn}/s_{mm}.

The non-vanishing Γ's can now be written in several groups as follows:

(i) 5 groups of 4 each*

$$\underset{\vee}{\Gamma'^m_{kn}} = -\tfrac{1}{2}c^m_n B_k \quad \Gamma'^m_{\underline{kn}} = \tfrac{1}{2}c^l_k c^m_n B_l,$$

$$\underset{\vee}{\Gamma'^m_{km}} = -\tfrac{1}{2}c^l_k A_l \quad \Gamma'^m_{km} = \tfrac{1}{2}A_k,$$

$$\Gamma'^k_{mm} = -\tfrac{1}{2}\frac{s_{mm}}{s_{kk}}(A_k + c^n_m c^m_n B_k),$$

(ii) 4 groups of 3 each

$$\underset{\vee}{\Gamma'^k_{kl}} = -\tfrac{1}{2}c^k_l H_k \quad \Gamma'^k_{\underline{kl}} = \tfrac{1}{2}\gamma_{kkl} + \tfrac{1}{2}c^l_k c^k_l H_l, \qquad (2\cdot2)$$

$$\Gamma'^k_{kk} = \tfrac{1}{2}\gamma_{kkk} \quad \Gamma'^k_{ll} = -\frac{s_{ll}}{s_{kk}}(\tfrac{1}{2}\gamma_{llk} + c^l_k c^k_l H_k),$$

(iii) one group of 2 only

$$\underset{\vee}{\Gamma'^k_{mn}} = -\tfrac{1}{2}\frac{a_{mn}}{s_{kk}}(A_k + B_k).$$

* The unwritten members are to be obtained from a particular Γ by suitable interchange of indices such as

$$\left.\begin{aligned} k &\longleftrightarrow l,\ m \longleftrightarrow n,\ k \longleftrightarrow m\\ & \qquad\qquad\quad l \longleftrightarrow n \end{aligned}\right\}.$$

For the existence of the above solutions we assume further

$$\begin{aligned} &1-c^l_k c^k_l \neq 0, \qquad 1-c^n_m c^m_n \neq 0, \\ &(1-c^l_k c^k_l)^2+4c^l_k c^k_l (1-c^n_m c^m_n) \neq 0, \\ &(1-c^n_m c^m_n)^2+4c^n_m c^m_n (1-c^l_k c^k_l) \neq 0. \end{aligned} \tag{2·3}$$

As for the field equations $R_{\mu\nu}=0$, $R_{(\mu\nu)}=0$ only 6 survive in the present case. These are

$$R_{kk}=0,\ R_{ll}=0,\ R_{mm}=0,\ R_{\underline{kl}}=0,\ R_{(kl)}=0,\ R_{(mn)}=0, \tag{2·4}$$

the explicit forms of which in terms of Γ's follow from the general formulae.[2]

Further, the equations $\Gamma'_\mu=0$ contribute

$$\Gamma'_k=0,\ \Gamma'_l=0,\ \Gamma'_n=0$$

yielding the relations

$$H_\sigma=2A_\sigma,\ (\sigma=k,\ l) \tag{2·5}$$

$$H_m=0. \tag{2·6}$$

Referring to (2·1) we notice that the condition (2·6) is satisfied by the field components in (1·1) if

$$\psi_{mn}=\surd\psi_{mm}\psi_{nn}\,. \tag{2·7}$$

It is to be observed that

$$A_\sigma=\frac{\partial}{\partial x_\sigma}\log(\phi^2_{mn}+\phi^2_{mn})^{1/2},$$

$$H_\sigma=\frac{\partial}{\partial x_\sigma}\log\left(1+\frac{s_{kk}s_{ll}}{a^2_{kl}}\right)$$

and (2·5) has the integrated form

$$s_{kk}s_{ll}=-a^2_{kl}\left[1-c(\phi^2_{mm}+\phi^2_{mn})\right], \tag{2·8}$$

where c is an arbitrary constant.

§ 3. System of modified field equations in terms of field components

Let us first consider the equations $R_{kk}=0$, $R_{ll}=0$. Combining them in the equation

$$R_{kk}-(s_{kk}/s_{ll})\,R_{ll}=0$$

and making proper substitutions from (2·2), (2·5), (2·6) we obtain after a straightforward calculation

$$\begin{aligned} (1-c^l_k c^k_l)\Big[&-A_{k,k}+\tfrac{1}{2}A^2_k+\tfrac{1}{2}c^n_m c^m_n B^2_k+A_k\gamma_{klk} \\ &-\frac{s_{kk}}{s_{ll}}\{-A_{l,l}+\tfrac{1}{2}A^2_l+\tfrac{1}{2}c^n_m c^m_n B^2_l+A_l\gamma_{lkl}\}\Big]=0. \end{aligned} \tag{3·1}$$

Next we take the equation $R_{kl}=0$. This readily reduces to the form

$$A_{k,k}+A_k(\gamma_{klk}-\gamma_{kkk})+\tfrac{1}{2}(3A_k^2-c_m^n c_n^m B_k^2)$$

$$+\frac{s_{kk}}{s_{ll}}[A_{l,l}+A_l(\gamma_{lkl}-\gamma_{lll})+\tfrac{1}{2}(3A_l^2-c_m^n c_n^m B_l^2)]=0. \qquad (3\cdot 2)$$

From $(3\cdot 1)$ and $(3\cdot 2)$ we derive the equation

$$A_k(\tfrac{1}{2}\gamma_{llk}+c_k^l c_l^k A_k)=\frac{s_{kk}}{s_{ll}}(-A_{l,l}+\tfrac{1}{2}A_l\gamma_{lll}-\tfrac{1}{2}A_l^2+\tfrac{1}{2}c_m^n c_n^m B_l^2). \qquad (3\cdot 3)$$

After some easy reduction using $(3\cdot 3)$, $(3\cdot 1)$, $(3\cdot 2)$ in the process the equation $R_{ll}=0$ becomes

$$(\tfrac{1}{2}c_m^n c_n^m B_k^2-\tfrac{5}{2}A_k^2+c_k^l c_l^k A_k^2)+\frac{s_{kk}}{s_{ll}}(\tfrac{1}{2}c_m^n c_n^m B_l^2-\tfrac{5}{2}A_l^2+c_k^l c_l^k A_l^2)$$

$$=\tfrac{1}{2}\frac{s_{kk}s_{ll}}{a_{kl}^2}\left[\gamma_{llk,k}+\tfrac{1}{2}\gamma_{llk}(\gamma_{llk}-\gamma_{kkk})+\frac{s_{kk}}{s_{ll}}\{\gamma_{kkl,l}+\tfrac{1}{2}\gamma_{kkl}(\gamma_{kkl}-\gamma_{lll})\}\right]. \qquad (3\cdot 4)$$

The equation $R_{kl}=0$ gives

$$(1-c_k^l c_l^k)(-A_{k,l}+c_k^l c_l^k A_k A_l+\tfrac{1}{2}A_k\gamma_{kkl}+\tfrac{1}{2}A_l\gamma_{llk}-\tfrac{1}{2}A_k A_l+\tfrac{1}{2}c_m^n c_n^m B_k B_l)=0. \qquad (3\cdot 5)$$

Eliminating $A_{k,k}+\frac{s_{kk}}{s_{ll}}A_{l,l}$ and $B_{k,k}+\frac{s_{kk}}{s_{ll}}B_{l,l}$ in succession between the equations $R_{mmm}=0$, $R_{mmn}=0$ we obtain

$$(1-c_m^n c_n^m)\left[B_{k,k}-B_k^2(1-c_m^n c_n^m)+B_k(A_k-\tfrac{1}{2}\gamma_{kkk}+\tfrac{1}{2}\gamma_{llk}+c_k^l c_l^k A_k)\right.$$

$$\left.+\frac{s_{kk}}{s_{ll}}\{B_{l,l}-B_l^2(1-c_m^n c_n^m)+B_l(A_l-\tfrac{1}{2}\gamma_{lll}+\tfrac{1}{2}\gamma_{kkl}+c_k^l c_l^k A_l)\}\right]=\frac{2s_{kk}}{s_{mmm}}f(x_m), \qquad (3\cdot 6)$$

$$(1-c_m^n c_n^m)\left[A_{k,k}+A_k(A_k-\tfrac{1}{2}\gamma_{kkk}+\tfrac{1}{2}\gamma_{llk}+c_k^l c_l^k A_k)\right.$$

$$\left.+\frac{s_{kk}}{s_{ll}}\{A_{l,l}+A_l(A_l-\tfrac{1}{2}\gamma_{lll}+\tfrac{1}{2}\gamma_{kkl}+c_k^l c_l^k A_l)\}\right]=-\frac{2s_{kk}}{s_{mmm}}f(x_m), \qquad (3\cdot 7)$$

where $f(x_m)$ stands for the expression

$$\tfrac{1}{2}[\gamma_{nnm,m}-\tfrac{1}{2}\gamma_{nnm}(\gamma_{mmm}-\gamma_{nnm})]. \qquad (3\cdot 8)$$

Using $(3\cdot 2)$ in $(3\cdot 7)$ we finally arrive at the equation

$$(A_k^2+c_m^n c_n^m B_k^2)+\frac{s_{kk}}{s_{ll}}(A_l^2+c_m^n c_n^m B_l^2)=-\frac{4s_{kk}f(x_m)}{(1-c_m^n c_n^m)s_{mmm}}. \qquad (3\cdot 9)$$

The system of 6 equations $(3\cdot 1)$, $(3\cdot 3)$, $(3\cdot 4)$, $(3\cdot 5)$, $(3\cdot 6)$, $(3\cdot 9)$ thus replaces the original system $(2\cdot 4)$. In equations (3.6) and $(3\cdot 9)$ there is a part $f(x_m)/\psi_{mmm}$ which is a function of x_m only while the rest are functions of x_k, x_l and therefore, for

consistency, we must have the auxiliary equation

$$4f(x_m)=\lambda\psi_{mn}\,, \tag{3·10}$$

where λ is a parameter which can take negative, positive or zero values. For $\lambda\neq 0$ it has been shown[1] that in the expressions for s_{mm}, s_{nn} in (1·1) we must have $\psi_{mm}=1$, $\psi_{nn}=\sin^2 x_m$ or $\sin h^2 x_m$ according as λ is negative or positive. When λ is negative the representation of $g_{\mu\nu}$ in (1·1) is spherically symmetric. Later on in our final solution we shall discuss two cases $\lambda\neq 0$ and $\lambda=0$, in the latter the field components being functions of x_k, x_l only.

§ 4. Simplified set of field equations and the solution

In accordance with our simplifying assumptions let us now impose the conditions

$$\gamma_{mml}/\gamma_{mmk}=\gamma_{mml}/\gamma_{mnk}=\gamma_{kkl}/\gamma_{kkk}=\gamma_{lll}/\gamma_{llk}=\gamma_{kll}/\gamma_{klk}=\varepsilon \tag{4·1}$$

in the field equations written above, remembering also

$$A_l/A_k=B_l/B_k=A_{k,l}/A_{k,k}=\varepsilon,\quad A_{l,l}/A_{k,k}=B_{l,l}/B_{k,k}=\varepsilon^2 \tag{4·2}$$

which follow from (4·1). We indicate the equations thus treated by putting a dash over their respective numbers. The equations (3·1′) and (3·5′) are now equivalent by virtue of the relation (2·5), that is

$$\gamma_{klk}-\tfrac{1}{2}\gamma_{kkk}-\tfrac{1}{2}\gamma_{llk}=-A_k(1-c_k^l c_l^k)\,. \tag{4·3}$$

We have then a new set of 5 equations to consider and for their solution we adopt the following substitution method:

Introduce two new variables P, Q defined by the equations

$$s_{kk}=\frac{A_k^2(\phi_{mn}^2+\phi_{mn}^2)^{1/2}}{P(1-c_k^l c_l^k)}\,,\quad s_{ll}=\frac{\varepsilon^2 A_k^2(\phi_{mm}^2+\phi_{mn}^2)^{1/2}}{Q(1-c_k^l c_l^k)} \tag{4·4}$$

so that

$$\varepsilon^2(s_{kk}/s_{ll})=Q/P. \tag{4·5}$$

By logarithmic differentiation of (4·4) we obtain

$$\begin{aligned}\gamma_{kkk}&=\frac{2A_{k,k}}{A_k}+A_k-\frac{P_k}{P}-2c_k^l c_l^k A_k\,,\\ \gamma_{llk}&=\frac{2A_{k,k}}{A_k}+A_k-\frac{Q_k}{Q}-2c_k^l c_l^k A_k\,,\end{aligned} \tag{4·6}$$

where we have omitted the commas in $P_{,k}$, $Q_{,k}$ denoting differentiation. Making use of (4·3) we get also

$$\gamma_{klk}=\frac{2A_{k,k}}{A_k}-\tfrac{1}{2}\left(\frac{P_k}{P}+\frac{Q_k}{Q}\right)-c_k^l c_l^k A_k\,. \tag{4·7}$$

Performing the above substitutions in the equations (3·1′), (3·3′) we find that the equations may be replaced by

$$A_{k,k}-c_k^l c_l^k A_k^2+\tfrac{1}{2}A_k^2-\tfrac{1}{2}A_k\left(\frac{P_k}{P}+\frac{Q_k}{Q}\right)+\tfrac{1}{2}c_m^n c_n^m B_k^2=0, \tag{4·8}$$

$$(P+Q)\,(c_k^l c_l^k A_k^2-\tfrac{1}{2}c_m^n c_n^m B_k^2)+\tfrac{1}{2}A_k(P_k+Q_k)=0, \tag{4·9}$$

which in conjunction with (4·7) yield

$$2\gamma_{klk}-2\,\frac{A_{k,k}}{A_k}+A_k+\frac{P_k+Q_k}{P+Q}+2c_k^l c_l^k A_k=0. \tag{4·10}$$

We also note that (4·8) can be written as

$$-A_{k,k}+A_k\gamma_{klk}+\tfrac{1}{2}\,(A_k^2+c_m^n c_n^m B_k^2)=0. \tag{4·11}$$

In view of (4·8) we rewrite (4·6) in the simpler form

$$\begin{aligned}\gamma_{kkk}&=Q_k/Q-c_m^n c_n^m\cdot B_k^2/A_k,\\ \gamma_{llk}&=P_k/P-c_m^n c_n^m\cdot B_k^2/A_k.\end{aligned} \tag{4·12}$$

We shall next prove that the equation (3·4′) is identically satisfied by the above substitutions and in consequence of the equations (4·8), (4·9).

Substituting for γ_{kkk}, γ_{llk} from (4·12) and simplifying, the righthand side of (3·4′) can be written in the form

$$\tfrac{1}{2}\,\frac{s_{kk}s_{ll}}{a_{kl}^2}\left[\left(\frac{\partial}{\partial x_k}+\tfrac{1}{2}\,\frac{P_k}{P}-\tfrac{1}{2}\,\frac{Q_k}{Q}\right)\left\{\frac{P_k+Q_k}{P}-c_m^n c_n^m\,\frac{B_k^2}{A_k}\left(1+\frac{Q}{P}\right)\right\}\right]. \tag{4·13}$$

Using (4·9) and carrying out the differentiation the above takes the form

$$\left(1+\frac{Q}{P}\right)[A_{k,k}-2\,(1-c_k^l c_l^k)\,A_k^2]+\tfrac{1}{2}A_k\left(1-\frac{Q}{P}\right)\left(\frac{P_k}{P}-\frac{Q_k}{Q}\right).$$

Using again (4·8), (4·9) we reduce the above to

$$\left(1+\frac{Q}{P}\right)(\tfrac{1}{2}c_m^n c_n^m B_k^2-\tfrac{3}{2}A_k^2+c_k^l c_l^k A_k^2),$$

which is equal to the left-hand side of (3·4′).

Consider now the equation (3·9′). Making use of (3·10), (4·4), we write it in the form

$$(P+Q)\,(A_k^2+c_m^n c_n^m B_k^2)=-\frac{A_k^2\phi_{mm}\lambda}{(1-c_k^l c_l^k)\,(\phi_{mn}^2+\phi_{mn}^2)^{1/2}}. \tag{4·14}$$

Integrating (4·10) we have

$$(P+Q)\,(1-c_k^l c_l^k)=\frac{A_k^2}{4c_1\,a_{kl}^2\,(\phi_{mm}^2+\phi_{mn}^2)^{1/2}}, \tag{4·15}$$

where c_1 is an arbitrary constant. Using (4·14) with $\lambda \neq 0$, we get from the above

$$-4\alpha a_{kl}^2 \phi_{mm} = A_k^2 + c_m^n c_n^m B_k^2 , \tag{4·16}$$

where α is put for $c_1\lambda$. Referring to (4·11) we therefore write (4·16) in the form

$$A_{k,k} - A_k \gamma_{klk} = -2\alpha \phi_{mm} a_{kl}^2 . \tag{4·17}$$

Again, using (4·9), (4·11), the equation (3·6′) becomes

$$2(P+Q)[B_{k,k} - B_k^2(1 - c_m^n c_n^m) - B_k \gamma_{klk}] = \frac{A_k^2 \phi_{mm} \lambda}{(1 - c_k^l c_l^k)(\phi_{mm}^2 + \phi_{mn}^2)^{1/2}} .$$

On putting $\bar{B}_k = (\phi_{mn}/\phi_{mm}) B_k$ and making use of (4·14) and (4·16) we get

$$\bar{B}_{k,k} - \bar{B}_k \gamma_{klk} = 2\alpha \phi_{mm} a_{kl}^2 . \tag{4·18}$$

For the final solution with $\lambda \neq 0$, besides the equations (4·16), (4·17) and (4·18) which involve ϕ_{mm}, ϕ_{mn} and a_{kl} we shall require two more involving s_{kk}, s_{ll} also. One of these is evidently the integrated form of (2·5), viz.,

$$s_{kk} s_{ll} = -a_{kl}^2 \{1 - c(\phi_{mm}^2 + \phi_{mn}^2)\} . \tag{4·19}$$

The other one is obtained as follows :

From (4·4) we have

$$\frac{1}{s_{kk}} + \frac{\varepsilon^2}{s_{ll}} = \frac{(P+Q)(1 - c_k^l c_l^k)}{A_k^2 (\phi_{mm}^2 + \phi_{mn}^2)^{1/2}} .$$

Using (4·15) we express the above as

$$\frac{1}{s_{kk}} + \frac{\varepsilon^2}{s_{ll}} = \frac{1}{4c_1 a_{kl}^2 (\phi_{mm}^2 + \phi_{mn}^2)} . \tag{4·20}$$

The 5 equations written above give the solution of the field components apparently as functions of $x_k + \varepsilon x_l$.

Consider now the transformation

$$x_k' = x_k + \varepsilon x_l , \quad x_l' = \phi(x_k, x_l) , \tag{4·21}$$

where

$$\partial\phi/\partial x_k = -\varepsilon s_{kk}/(\varepsilon^2 s_{kk} + s_{ll}) , \quad \partial\phi/\partial x_l = s_{ll}/(\varepsilon^2 s_{kk} + s_{ll}) ;$$

then the above solutions may be presented as involving only one variable x_k' and are of the same form discussed in the first paper.[1)]

When $\lambda = 0$ the field components do not involve x_m. From the equation (4·14) we find that either

$$\text{(i)} \quad A_k^2 - \bar{B}_k^2 = 0 \quad \text{or} \quad \text{(ii)} \quad P + Q = 0.$$

The relevant equations of the first case are then given by

$$A_k^2 - \bar{B}_k^2 = 0 ,$$

$$A_{k,k}-A_k\gamma_{klk}=0, \quad \bar{B}_{k,k}-\bar{B}_k\gamma_{klk}=0,$$

with (4·22)

$$s_{kk}s_{ll}=-a_{kl}^{2}\{1-c(\phi_{mm}^{2}+\phi_{mn}^{2})\},$$

$$\frac{1}{s_{kk}}+\frac{\varepsilon^2}{s_{ll}}=\frac{1}{4c_1a_{kl}^{2}(\phi_{mm}^{2}+\phi_{mn}^{2})}.$$

The transformation (4·21) applies to this case also and the solutions involve one variable x_k' and are of the same form as has been discussed before.[1]

When $P+Q=0$ we have also $P_k+Q_k=0$, hence

$$1/s_{kk}+\varepsilon^2/s_{ll}=0, \quad \gamma_{llk}=\gamma_{kkk}. \tag{4·23}$$

Out of the equations (3·1′), (3·3′), (3·4′), (3·6′), (3·9′), only (3·1′) survives, others being automatically satisfied since $P+Q=0$. Further we assume $\phi_{mm}=\phi_{mn}$ so that $B_k=0$, then equation (3·1′), i.e., (4·11), is integrable and we get

$$a_{kl}^{2}=(A_k^{2}/\phi_{mm})\cdot\text{const}, \tag{4·24}$$

where A_k is given by γ_{mmk}.

The equation (4·19) gives now

$$\varepsilon^2 s_{kk}^{2}=a_{kl}^{2}(1-2c\phi_{mn}^{2}). \tag{4·25}$$

The final solution may be presented as follows:

Let u^2 denote an arbitrary function of $(x_k+\varepsilon x_l)$. Then setting $\phi_{mm}=u^2$ we have $\phi_{mn}=u^2$, $\phi_{nn}=u^2$.

Equation (4·24) then gives

$$a_{kl}=\frac{c'}{u^2}\frac{\partial u}{\partial x_k},$$

where c' is an arbitrary constant.

Also (4·25) gives

$$s_{kk}=\frac{1}{\varepsilon}\frac{c'}{u^2}\frac{\partial u}{\partial x_k}(1-2cu^{4})^{1/2},$$

and consequently

$$s_{ll}=-\frac{\varepsilon c'}{u^2}\frac{\partial u}{\partial x_k}(1-2cu^{4})^{1/2}.$$

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Note on the Isotopic Spin of the Antiparticle

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Charge conjugation is studied in a different form from the usual one, and the assignments of the eigenvalues of the isotopic spin operator τ_3 for antiprotons and antineutrons are discussed. As an illustrative application, the annihilation process of a nucleon and an antinucleon is dealt with.

§ 1. Introduction

The eigenvalues of τ_3, the isotopic spin operator describing protons and neutrons, are usually taken as 1, -1, respectively. There seems to be two possibilities in the corresponding assignments for antiprotons and antineutrons; that is, $+1$, -1 or -1, $+1$. This arbitrariness has recently been argued by Lepore.[1] He has shown that one of the two alternatives is ruled out if one demands that the theory be invariant under charge conjugation. It is to be remarked here that Lepore's approach does not seem quite satisfactory since it rests upon Dirac's one-particle theory.* We shall therefore reformulate these considerations, basing ourselves on field theory. In § 3, as an illustrative example of the present formalism, we deal with the annihilation process of a nucleon and an antinucleon.[3] There are few new results, but the point of view of representation is new.

§ 2. Formulation

The operator describing the nucleon and the antinucleon is

$$\psi(x)=\begin{pmatrix}\psi_P(x)\\ \psi_N(x)\\ \psi_{\bar{P}}(x)\\ \psi_{\bar{N}}(x)\end{pmatrix}, \tag{2·1}$$

$$\psi_P(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\sqrt{m/|E_p|}\sum_{r=1}^{2}\{b_r^{(P)}(\boldsymbol{p})u^r(\boldsymbol{p})e^{ipx}+d_r^{(\bar{P})+}(\boldsymbol{p})v^r(\boldsymbol{p})e^{-ipx}\},$$

$$\psi_N(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\sqrt{m/|E_p|}\sum_{r}\{b_r^{(N)}(\boldsymbol{p})u^r(\boldsymbol{p})e^{ipx}+d_r^{(\bar{N})+}(\boldsymbol{p})v^r(\boldsymbol{p})e^{-ipx}\},$$

* This defect has been pointed out by Sugawara, too.[2]

$$\psi_{\bar{r}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\sqrt{m/|E_p|}\sum_r\{d_r^{(\bar{r})}(\boldsymbol{p})u_c^r(\boldsymbol{p})e^{ipx}+b_r^{(r)+}(\boldsymbol{p})v_c^r(\boldsymbol{p})e^{-ipx}\},$$

$$\psi_{\bar{N}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\sqrt{m/|E_p|}\sum_r\{d_r^{(\bar{N})}(\boldsymbol{p})u_c^r(\boldsymbol{p})e^{ipx}+b_r^{(N)+}(\boldsymbol{p})v_c^r(\boldsymbol{p})e^{-ipx}\},$$

$$u_c(\boldsymbol{p})=c[v(\boldsymbol{p})]^T \quad \text{or} \quad u(\boldsymbol{p})=c[\bar{v}_c(\boldsymbol{p})]^T, \quad v=u^+\gamma_4,$$

$$v_c(\boldsymbol{p})=c[\bar{u}(\boldsymbol{p})]^T \quad \text{or} \quad v(\boldsymbol{p})=c[\bar{u}_c(\boldsymbol{p})]^T, \quad \bar{u}=u^+\gamma_4.$$

Here c is the usual charge conjugation operator. $u(\boldsymbol{p})$ is a positive energy spinor of momentum $+\boldsymbol{p}$ and $v(\boldsymbol{p})$ is a negative energy spinor of momentum $-\boldsymbol{p}$. $b_r(\boldsymbol{p})$, $d_r(\boldsymbol{p})$ are destruction operators and $b_r^+(\boldsymbol{p})$, $d_r^+(\boldsymbol{p})$ are creation operators, and they satisfy the usual anticommutation relations. Each row (or column) of $\psi(x)$ (or $\bar{\psi}=\psi^+\gamma_4$) is characterised by operators :

$$\tau_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\qquad \tau_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},\qquad \tau_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},$$

$$\lambda_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\qquad \lambda_2=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix},\qquad \lambda_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{2·2}$$

The following assignment is adopted :

$$p\to(\tau_3=1,\ \lambda_3=1),\qquad n\to(\tau_3=-1,\ \lambda_3=1)$$

$$\bar{p}\to(\tau_3=1,\ \lambda_3=-1),\quad \bar{n}\to(\tau_3=-1,\ \lambda_3=-1). \tag{2·3}$$

Then the wave equation in an external electromagnetic field A_μ is*

$$[\gamma_\mu(\partial_\mu+ieQA_\mu)+M]\psi(x)=0. \tag{2·4}$$

The charge operator Q is given by

$$Q=[(1+\tau_3)/2]\lambda_3,\qquad 1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{2·5}$$

From (2·4), one can construct the wave function of its charge conjugate as

* The notation for the Dirac matrices is that of Schwinger.[4]

$$\psi_c(x)=\begin{pmatrix} c\bar{\psi}_P(x)^T \\ c\bar{\psi}_N(x)^T \\ c\bar{\psi}_{\bar{P}}(x)^T \\ c\bar{\psi}_{\bar{N}}(x)^T \end{pmatrix}=\begin{pmatrix} \psi_{\bar{P}}(x) \\ \psi_{\bar{N}}(x) \\ \psi_P(x) \\ \psi_N(x) \end{pmatrix}. \tag{2·6}$$

The equation for $\psi_c(x)$ is

$$[\gamma_\mu(\partial_\mu - ieQA_\mu)+M]\psi_c(x)=0,$$

where the relation $c^{-1}\gamma_\mu c=-\gamma_\mu^T$ is used.

If we define the matrix C in terms of

$$\psi_c(x)=C\psi(x),$$

we get

$$C=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}=\lambda_1. \tag{2·7}$$

Next we deal with pseudoscalar mesons. Assuming the pseudoscalar interaction and the charge independence, the wave equation is

$$[\gamma_\mu(\partial_\mu+ieQA_\mu)+if\gamma_5 I_\alpha\lambda_3\phi_\alpha+M]\psi(x)=0. \tag{2·8}$$

The I_α matrices are the isotopic spin matrices and given by

$$I_1=\tau_1\lambda_3, \qquad I_2=\tau_2, \qquad I_3=\tau_3\lambda_3. \tag{2·9}$$

From (2·7), the equation for ψ_c becomes

$$[\gamma_\mu(\partial_\mu-ieQA_\mu)+if\gamma_5 I_\alpha^*\lambda_3\phi_\alpha+M]\psi_c(x)=0, \tag{2·10}$$

where I_α^* is the complex conjugate operator of I_α.

If one now demands that the theory be invariant under charge conjugation, then the nucleon field equation remains unchanged provided that

$$\phi_1\to\phi_1, \qquad \phi_2\to-\phi_2, \qquad \phi_3\to\phi_3. \tag{2·11}$$

Further consider the interchange of proton and neutron. The equation for ψ_T becomes

$$[\gamma_\mu(\partial_\mu+ie\bar{Q}A_\mu)-if\gamma_5 I_\alpha^*\lambda_3\phi_\alpha+M]\psi_T(x)=0, \tag{2·12}$$

$$\psi_T=T\psi=\begin{pmatrix} \psi_N \\ -\psi_P \\ \psi_{\bar{N}} \\ -\psi_{\bar{P}} \end{pmatrix}, \quad T=i\tau_2, \quad \bar{Q}=[(1-\tau_3)/2]\lambda_3. \tag{2·13}$$

The charge symmetry requirement of meson-nucleon interaction is followed if

$$\phi_1 \to -\phi_1, \qquad \phi_2 \to \phi_2, \qquad \phi_3 \to -\phi_3 \tag{2·14}$$

under T-transformation. Summing up the results so far, we have

$$G = CT = i\tau_2\lambda_1 \tag{2·15}$$

for nucleon field and

$$\phi_c = C\phi, \qquad \phi_T = T\phi, \qquad \phi = \begin{pmatrix} (\phi_1 + i\phi_2)/\sqrt{2} \\ \phi_3 \\ (\phi_1 - i\phi_2)/\sqrt{2} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad G = CT = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2·16}$$

for meson field. Lee and Yang used this operator "G" and derived interesting selection rules concerning a nucleon-antinucleon system.[3]

The rule for composition of isotopic spin is

$$T_3 = \sum_i T_3^{(i)}. \tag{2·17}$$

This is assured if one observes that under a rotation through angle ϵ about the 3-axis in isotopic spin space ψ transforms as

$$\psi \to e^{(i/2) I_3 \epsilon}.$$

Thus one sees that if a state involving particles and antiparticles is constructed by operating on the vacuum state with appropriate creation operators, such a state transforms as

$$\Psi \to e^{-i/2 \cdot \sum_i I_3^{(i)} \epsilon} \Psi$$

under a rotation about the 3-axis in isotopic spin space. Similarly we have

$$T_1 = \sum_i T_1^{(i)}, \qquad T_2 = \sum_i T_2^{(i)}. \tag{2·18}$$

Finally, let us consider the following unitary transformation :

$$\psi' = U^{-1}\psi = \begin{pmatrix} \psi_P \\ \psi_N \\ \psi_{\bar{N}} \\ -\psi_{\bar{P}} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2·19}$$

Then we get*

$$I'_\alpha = U^{-1} I_\alpha U = \tau_\alpha, \qquad U^{-1}\lambda_3 U = \lambda_3. \tag{2·20}$$

* $U^{-1}\tau_1 U = \tau_1\lambda_3$, $U^{-1}\tau_2 U = \tau_2$, $U^{-1}\tau_3 U = \tau_3\lambda_3$.

The wave equation turns into

$$[\gamma_\mu(\partial_\mu+ieQ'A_\mu)+if\gamma_5\tau_\alpha\lambda_3\phi_\alpha+M]\psi'(x)=0 \qquad (2\cdot8)'$$

$$Q'=U^{-1}QU=(1/2)\cdot(\lambda_3+\tau_3). \qquad (2\cdot5)'$$

Further, we easily obtain

$$C'=U^{-1}CU=\tau_2\lambda_2,$$

$$T'=U^{-1}TU=i\tau_2,$$

$$G'=U^{-1}GU=i\lambda_2.$$

The equation $(2\cdot5)'$ is the formula for the charge of a nucleon proposed by Gell-Mann and Nishijima.[5]

§ 3. Application

As an illustrative example of the present formalism, we deal with the annihilation process of a particle and an antiparticle.[6] We have no new results other than those given by Lee and Yang,[3] but in this formulation the operator "G" is easily understood in terms of the product operators of charge conjugation and the interchange of proton and neutron.

The total isotopic spin $T=0$ state of a nucleon and an antinucleon $(p+\bar{p},\ n+\bar{n},\ p+\bar{n},\ n+\bar{p})$ is given by

$$T_3=0 \quad : \Psi^{\pm}_{0,0}=(1/2)\{(p_1\bar{p}_2+n_1\bar{n}_2)\pm(\bar{p}_1p_2+\bar{n}_1n_2)\} \qquad (3\cdot1)$$

and the $T=1$ states are given by

$$\begin{aligned} T_3=1 \quad &: \Psi^{\pm}_{1,1}=(1/\sqrt{2})(p_1\bar{n}_2\pm\bar{n}_1p_2), \\ T_3=0 \quad &: \Psi^{\pm}_{1,0}=(1/2)\{(p_1\bar{p}_2-n_1\bar{n}_2)\pm(\bar{p}_1p_2-\bar{n}_1n_2)\}, \\ T_3=-1 \quad &: \Psi^{\pm}_{1,-1}=(1/\sqrt{2})(p_1n_2\pm n_1\bar{p}_2). \end{aligned} \qquad (3\cdot2)$$

Here the suffices 1 and 2 distinguish the coordinates of two particles. One of the two alternatives in $(3\cdot1)$ and $(3\cdot2)$ is ruled out by the Pauli principle. The upper sign is appropriate to the states ${}^1S_0$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$,...etc., and the lower to the states ${}^3S_1$, ${}^1P_1$,...etc.

From $(2\cdot15)$, $(2\cdot16)$ and $(2\cdot19)$, we note that

$$G\phi=-\phi, \qquad Gp=-\bar{n}, \qquad Gn=\bar{p},$$

$$G\bar{n}=p, \qquad G\bar{p}=-n.$$

$$G_1G_2\Psi^{\pm}_{0,0}=\pm\Psi^{\pm}_{0,0}, \qquad G_1G_2\Psi^{\pm}_{1,1}=\mp\Psi^{\pm}_{1,1},$$

$$G_1G_2\Psi^{\pm}_{1,0}=\mp\Psi^{\pm}_{1,0}, \qquad G_1G_2\Psi^{\pm}_{1,1}=\mp\Psi^{\pm}_{1,-1}.$$

Thus we can easily write down the selection rules given by Lee and Yang.[3]

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A Method of Calculation of Electrical Conductivity

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A new method of calculating electrical conductivity is presented. The method is based upon the theory of relaxation function or after-effect function, from which the general formula for the electrical conductivity is obtained. Taking into account a time required for an observation, we introduce the irreversibility into the theory and investigate the relaxation behaviour of the process, by which the electrical conductivity is calculated and the practical expression is obtained. In the perturbation approximation the result is the same as the expression obtained on the basis of the Bloch theory.

§ 1. Introduction

The object of this article is to present a method of calculating the electrical conductivity, an outline of which has already been given in this journal.[1] Also the more details have been reported in Japanese,[2] and studies were made by a few authors on some problems concerning the present method.[3] It is a quantum-statistical method based upon the linear theory of irreversible process. This method is possible to be applied to any system without requiring any special consideration, if only a few generalizations are made in some details. To these generalizations we refer only a little in the appendix and the detailed discussions will be continued at the earliest opportunity hereafter.

A little after H. Takahasi made investigations on the theory of thermal fluctuations in classical system,[4] R. Kubo and K. Tomita generalized his theory to quantum-statistical systems for the purpose of discussing the problem of magnetic resonance absorption.[5] We make use of the same procedure in order to investigate the phenomena of electrical conduction.

Based upon the equation of motion of the density matrix we can find the relaxation function of the electric current, that is, the function of time which gives the density of electric current of a given system in the presence of an impulse of external electric field. The function is expressed in terms of the Hamiltonian and the operator of the current density and represents essentially the whole properties of response of the system to the electric field. By superposing the relaxation function appropriately we obtain the density of electric current placed in any time-dependent electric field, from which the formula for electrical conductivity is obtained. This is the formal aspect of the present method and is explained in § 2. Since this formula is not practical, we introduce, in § 3, the irreversibility into the theory by taking into account a time required for an observation and obtain the practical expression of the electrical conductivity and its perturbation

approximation. Based upon this formula we calculate in § 4 the conductivity of metal in which electrons make collisions with the thermal vibrations and other irregularities of the lattice of ions, and obtain in the perturbation approximation the same result as that based upon the Bloch theory.[5] The general formula obtained in § 2 is used again in § 4 for the system of free electrons in the presence of uniform magnetic field. Finally some remarks are presented in § 5 concerning the generalization of our method to other irreversible processes and the investigations of detailed mechanism of conduction phenomena.

§ 2. The formal theory, relaxation function and general formula for conductivity

In the first place we investigate the response of the system to the applied electric field in the scope of the linear theory of quantum-statistical mechanics. It is sufficient to find the response to an impulse field because the response to any applied field is obtained by superposing appropriately the responses to successive impulses. We denote by $\mathfrak{H}$ the Hamiltonian of the system in the absence of external electric field. As far as we do not give any notice, $\mathfrak{H}$ does not involve any external magnetic field. The motion of the system in the presence of an impulse field is described by the Hamiltonian $\mathfrak{H}-c^{-1}\boldsymbol{j}\cdot\boldsymbol{A}(t)$, where $\boldsymbol{j}$ is the operator of the current density and the vector potential $\boldsymbol{A}(t)$ is the step-function of time

$$\boldsymbol{A}(t)=\begin{cases} c\boldsymbol{F}(t_0)\,dt_0 & \text{for } t\leq t_0\,, \\ 0 & \text{for } t>t_0\,, \end{cases}$$

corresponding to the impulse field by the familiar relation $\boldsymbol{F}(t)=-c^{-1}\partial\boldsymbol{A}(t)/\partial t$. The Hamiltonian of the system as a function of time is therefore

$$\begin{cases} \mathfrak{H}-\boldsymbol{j}\cdot\boldsymbol{F}(t_0)\,dt_0 & \text{for } t\leq t_0\,, \\ \mathfrak{H} & \text{for } t>t_0\,. \end{cases} \tag{1}$$

The density matrix $\rho(t)$ of the system for $t\geq t_0$ can be written as

$$\rho(t)=\exp[-(i/\hbar)\,(t-t_0)\,\mathfrak{H}]\,\rho(t_0)\exp[(i/\hbar)\,(t-t_0)\,\mathfrak{H}]\,, \tag{2}$$

or its equation of motion is given by

$$i\hbar\,(\partial\rho(t)/\partial t)=[\mathfrak{H},\ \rho(t)]\,,$$

where $\rho(t_0)$ may be assumed as the canonical density matrix of the system with the Hamiltonian $\mathfrak{H}-\boldsymbol{j}\cdot\boldsymbol{F}(t_0)\,dt_0$, because this has been the Hamiltonian of the system all the time from $t=-\infty$ to t_0. Then we have

$$\rho(t_0)=\exp[-\beta\{\mathfrak{H}-\boldsymbol{j}\cdot\boldsymbol{F}(t_0)\,dt_0\}]\cdot\{T_r\exp[-\beta\{\mathfrak{H}-\boldsymbol{j}\cdot\boldsymbol{F}(t_0)\,dt_0\}]\}^{-1}. \tag{3}$$

With use of the expressions (2) and (3), the value of the electric current density at $t\geq t_0$ is written as

$$\boldsymbol{J}(t,\,t_0)=T_r[\rho(t)\boldsymbol{j}]=T_r[\rho(t_0)\boldsymbol{j}(t-t_0)]\,, \tag{4}$$

where $\boldsymbol{j}(t)$ is the operator of the current density in the Heisenberg picture

$$\boldsymbol{j}(t) = \exp(i/\hbar \cdot t\mathfrak{H})\, \boldsymbol{j} \exp(-i/\hbar \cdot t\mathfrak{H}), \tag{5}$$

which follows the equation of motion

$$(\hbar/i)\,\partial \boldsymbol{j}(t)/\partial t = [\mathfrak{H},\ \boldsymbol{j}(t)]\,.$$

Expanding (3) into the power series of $\boldsymbol{F}(t_0)$ and leaving the zeroth and the first order terms, we get

$$\rho(t_0) = \exp(-\beta\mathfrak{H}) \cdot \Big[1 + \int_0^\beta d\lambda\, e^{\lambda\mathfrak{H}} (\boldsymbol{j} - \boldsymbol{J}_C) \cdot \boldsymbol{F}(t_0)\, dt_0 e^{-\lambda\mathfrak{H}}\Big] \cdot [T_r \exp(-\beta\mathfrak{H})]^{-1}$$

$$= \rho_C \Big\{1 + dt_0 \int_0^\beta d\lambda\, e^{\lambda\mathfrak{H}} (\boldsymbol{j} - \boldsymbol{J}_C) \cdot \boldsymbol{F}(t_0)\, e^{-\lambda\mathfrak{H}}\Big\}, \tag{6}$$

where

$$\rho_C = \exp(-\beta\mathfrak{H}) / T_r \exp(-\beta\mathfrak{H}) \tag{6a}$$

is the density matrix for the canonical ensemble of systems with the Hamiltonian $\mathfrak{H}$, and

$$\boldsymbol{J}_C = T_r(\rho_C \boldsymbol{j}) \tag{6b}$$

is the average of the density of electric current over this ensemble or the value in thermal equilibrium. Substituting (6) in (4) we have

$$\boldsymbol{J}(t,\ t_0) = \boldsymbol{J}_C + dt_0 \int_0^\beta d\lambda\, T_r\{\rho_C e^{\lambda\mathfrak{H}} (\boldsymbol{j} - \boldsymbol{J}_C) \cdot \boldsymbol{F}(t_0)\, e^{-\lambda\mathfrak{H}} \boldsymbol{j}(t - t_0)\}\,. \tag{7}$$

Since any spontaneous current does not exist in almost all systems, we neglect $\boldsymbol{J}_C$. Then Eq. (7) is written in terms of the orthogonal components:

$$J_\mu(t,\ t_0) = dt_0 \sum_\nu F_\nu(t_0)\, \varphi_{\nu\mu}(t - t_0),\quad (\mu \text{ or } \nu = x,\ y,\ z), \tag{8}$$

where $\varphi_{\nu\mu}(t)$ is the relaxation or after-effect function

$$\varphi_{\nu\mu}(t) = \int_0^\beta d\lambda\, T_r\{\rho_C e^{\lambda\mathfrak{H}} j_\nu e^{-\lambda\mathfrak{H}} j_\mu(t)\}\,. \tag{9}$$

On the basis of the superposition principle in the linear theory we can obtain the expression of the density of electric current $\boldsymbol{J}(t)$ at time t of the system in the presence of an external electric field $\boldsymbol{F}(t)$ as

$$J_\mu(t) = \int_{-\infty}^t dt_0 J_\mu(t,\ t_0) = \sum_\nu \int_0^\infty d\tau\, F_\nu(t - \tau)\, \varphi_{\nu\mu}(\tau)\,. \tag{10}$$

The frequency-dependent complex conductivity $\sigma_{\nu\mu}(\omega)$ is therefore given by the Fourier transform of the relaxation function $\varphi_{\nu\mu}(t)$,

$$\sigma_{\nu\mu}(\omega)=\int_0^\infty dt\,\varphi_{\nu\mu}(t)\,e^{-i\omega t}=\int_0^\infty dt\,e^{-i\omega t}\int_0^\beta d\lambda\,T_r\{\rho_C e^{\lambda\mathfrak{H}}j_\nu e^{-\lambda\mathfrak{H}}j_\mu(t)\}\,. \tag{11}$$

If we denote the eigenstates of $\mathfrak{H}$ by m, n and so on and the corresponding eigenvalues by E_m, E_n and so on, we can rewrite the relaxation function as

$$\varphi_{\nu\mu}(t)=\sum_m{}'\sum_n{}'\frac{e^{-\beta E_n}-e^{-\beta E_m}}{E_m-E_n}\,\Re_e[(m|j_\nu|n)\,(n|j_\mu|m)\cdot$$
$$\cdot\exp\{i/\hbar\cdot t(E_n-E_m)\}]\cdot(\sum_n{}'e^{-\beta E_n})^{-1}, \tag{12}$$

where $\Re_e[\cdots]$ means the real part of the expression in the bracket.

The properties of electrical conduction of the system are essentially condensed in the relaxation function. As shown in Appendix 1 the relaxation function satisfies the equations which relate to the microscopic reversibility of dynamics

$$\varphi_{\mu\nu}(t)=\varphi_{\nu\mu}(-t)=\varphi_{\nu\mu}(t)\,. \tag{13}$$

By substituting them in (11) we have the Onsager reciprocal relation

$$\sigma_{\mu\nu}(\omega)=\sigma_{\nu\mu}(\omega)\,. \tag{14}$$

The static conductivity σ is

$$\sigma=\int_0^\infty dt\,\varphi_{\mu\mu}(t)=\int_0^\infty dt\int_0^\beta d\lambda\,T_r\{\rho_C e^{\lambda\mathfrak{H}}j_\mu e^{-\lambda\mathfrak{H}}j_\mu(t)\}\,. \tag{15}$$

If we make use of the expression (12), we can rewrite (15) as[7]

$$\sigma=(\pi\hbar/kT)\cdot\sum_m{}'\sum_n{}'(E_m=E_n)\,e^{-\beta E_n}|(m|j_\mu|n)|^2\cdot(\sum_n{}'e^{-\beta E_n})^{-1}. \tag{15a}$$

§ 3. Irreversibility and the approximate procedure of calculating conductivity

As the general formula obtained in the preceding section is not practical, we reformulate it in the following, introducing the irreversibility by taking into account the fact that a time is required for a measurement. We can then obtain the tractable expression of electrical conductivity.

The density matrix at time $t\geq t_0$ of the system to which the impulse of electric field is applied at t_0 is now assumed, in a certain stage of approximation as shown in Appendix 2, to be written down in terms of an "effective vector potential" $\boldsymbol{A}(t,\,t_0)$ as follows

$$\rho'(t)=\exp[-\beta\{\mathfrak{H}-c^{-1}\boldsymbol{j}\cdot\boldsymbol{A}(t,\,t_0)\}]/T_r\exp[-\beta\{\mathfrak{H}-c^{-1}\boldsymbol{j}\cdot\boldsymbol{A}(t,\,t_0)\}]\,,$$

where

$$c^{-1}\boldsymbol{A}(t_0,\,t_0)=\boldsymbol{F}(t_0)\,dt_0\,. \tag{16a}$$

We can trace the change of $\boldsymbol{A}(t,\,t_0)$ with time $t>t_0$ by making use of the equation

of motion of $\rho(t)$ in the form (2), by means of which we have for $t+\Delta t > t > t_0$

$$\exp[-\beta\{\mathfrak{H}-c^{-1}\boldsymbol{j}\cdot\boldsymbol{A}(t+\Delta t,\, t_0)\}]$$
$$=\exp(-i/\hbar\cdot\Delta t\mathfrak{H})\exp[-\beta\{\mathfrak{H}-c^{-1}\boldsymbol{j}\cdot\boldsymbol{A}(t,\, t_0)\}]\exp(i/\hbar\cdot\Delta t\mathfrak{H}).$$

Now we expand both hands of this equation with respect to the effective vector potential, multiply them with j_μ and take their traces. We get to the first order in the vector potential

$$\sum_\nu \frac{A_\nu(t+\Delta t,\, t_0)-A_\nu(t,\, t_0)}{\Delta t}\, a_{\nu\mu} = -\sum_\nu A_\nu(t,\, t_0)\, b_{\nu\mu}\,, \tag{16b}$$

where

$$a_{\nu\mu}=\int_0^\beta d\lambda\, T_r(\rho_C e^{\lambda\mathfrak{H}} j_\nu e^{-\lambda\mathfrak{H}} j_\mu),\quad b_{\nu\mu}=\int_0^\beta d\lambda\, T_r[\rho_C e^{\lambda\mathfrak{H}} j_\nu e^{-\lambda\mathfrak{H}}\{j_\mu - j_\mu(\Delta t)\}/\Delta t)]\,. \tag{16c}$$

If the time interval Δt is of the same order of magnitude as that required for a measurement, the difference quotient may be regarded as the differential quotient in the left-hand side. Furthermore we can rewrite $b_{\nu\mu}$ in a more practical form, as is shown in Appendix 3, as follows,

$$b_{\nu\mu}=\hbar^{-2}\int_0^\infty dt\, T_r\{\rho_C[j_\nu,[\mathfrak{H},\, j_\mu(t)]]\}+(i\hbar)^{-1}T_r\{\rho_C[j_\nu,\, j_\mu]\}\,. \tag{16d}$$

Then the equation (16b) can be written as

$$dA_\mu(t)/dt=-\sum_\nu A_\nu(t,\, t_0)\, c_{\nu\mu}\,, \tag{16}$$

where

$$c_{\nu\mu}=\sum_\rho b_{\nu\rho}\, a^{-1}_{\rho\mu} \tag{16e}$$

with the use of the inverse matrix $a^{-1}_{\nu\mu}$ of $a_{\nu\mu}$ given by (16c). We can find the expression of $A_\mu(t,\, t_0)$ by solving eq. (16) in the initial condition (16a).

In the case in which the applied field is not an impulse but is expressed as $\boldsymbol{F}(t)$ as a function of time,

$$\boldsymbol{A}(t)=\int_{-\infty}^t dt_0 \boldsymbol{A}(t,\, t_0) \tag{17}$$

appears instead of $\boldsymbol{A}(t,\, t_0)$, and it is clear that the same equation as for $A_\mu(t,\, t_0)$

$$dA_\mu(t)/dt=-\sum_\nu A_\nu(t)\, c_{\nu\mu} \tag{18}$$

is fulfilled for $A_\mu(t)$. With the use of $A_\mu(t)$, we get the average value $J_\mu(t)$ of the electric current density at t, as follows

$$J_\mu(t)=c^{-1}\sum_\nu A_\nu(t)\, a_{\nu\mu}\,. \tag{19}$$

By making use of the transformation matrix $s_{\nu\mu}$ which transforms $c_{\nu\mu}$ to the diagonal form as

$$\sum_{\rho}{}' \sum_{\sigma}{}' s_{\nu\rho} c_{\rho\sigma} s^{-1}_{\sigma\mu} = c_{\nu} \delta_{\nu\mu} ,$$

we can get the solution of Eq. (16) in the initial condition (16a) as follows

$$A_{\mu}(t, t_0) = c dt_0 \sum_{\nu}{}' F_{\nu}(t_0) \sum_{\rho}{}' s^{-1}_{\nu\rho} s_{\rho\mu} e^{-c_{\rho}(t-t_0)} . \tag{20}$$

Then we get by Eq. (17)

$$A_{\mu}(t) = c \sum_{\nu}{}' F_{\nu} e^{i\omega t} \sum_{\rho}{}' s^{-1}_{\nu\rho} s_{\rho\mu} (c_{\rho} + i\omega)^{-1} = c \sum_{\nu}{}' F_{\nu} e^{i\omega t} (c + i\omega)^{-1}_{\nu\mu} , \tag{21}$$

in the case in which the applied field is the trigonometric function $F_{\nu}(t) = F_{\nu} \cdot e^{i\omega t}$. We then obtain the density of electric current $J_{\mu}(t)$ at time t

$$J_{\mu}(t) = \sum_{\nu}{}' F_{\nu} e^{i\omega t} \{ (ab^{-1} + i\omega)^{-1} a \}_{\nu\mu} . \tag{22}$$

Accordingly the frequency-dependent and complex conductivity tensor $\sigma_{\nu\mu}(\omega)$ is given by

$$\sigma_{\nu\mu}(\omega) = [(ba^{-1} + i\omega)^{-1} a]_{\nu\mu} \tag{23}$$

with the use of (16c) or (16d). The static conductivity tensor is

$$\sigma_{\nu\mu} = (ab^{-1} a)_{\nu\mu} . \tag{23a}$$

We now investigate an approximate procedure of obtaining the concrete expression for conductivity tensor from the formula (23) or (23a) on the assumption that the Hamiltonian $\mathfrak{H}$ is the sum of the two terms, the unperturbed $\mathfrak{H}_0$ and the perturbation or the collision term $\mathfrak{H}'$ and that the density of electric current $\boldsymbol{j}$ is commutative with $\mathfrak{H}_0$. The commutativity is satisfied in all cases except in the presence of external magnetic field. We further assume that $\mathfrak{H}'$ has no diagonal element in the scheme in which $\mathfrak{H}_0$ is diagonal. As we shall see in Appendix 3, we can get approximate but more concrete expressions of matrix elements of a and b by making use of the perturbation calculation as follows,

$$a_{\nu\mu} = i\beta \langle j_{\nu} j_{\mu} \rangle , \quad b_{\nu\mu} = i\beta\pi / \hbar^3 \cdot \langle | j_{\nu}, \mathfrak{H}'_{\omega} | [\mathfrak{H}'_{-\omega}, j_{\mu}] \rangle |_{\omega=0}$$
$$+ i/\hbar^3 \int_{-\infty}^{\infty} d\omega / \omega^2 \cdot \langle [[j_{\nu}, \mathfrak{H}'_{\omega}], [\mathfrak{H}'_{-\omega}, j_{\mu}]] \rangle . \tag{16f}$$

The conductivity tensor obtained by substituting (16f) in (23) or (23a) is a little complicated and we do not write them down. In the system of spherical symmetry the conductivity which is equal to $\sigma_{\mu\mu}$ is expressed as

$$\sigma = \hbar^2 / \pi k T \cdot \langle j_{\mu}{}^2 \rangle^2 \cdot \langle [j_{\mu}, \mathfrak{H}'_{\omega}] [\mathfrak{H}'_{-\omega}, j_{\mu}] \rangle^{-1} |_{\omega=0} , \tag{23b}$$

and the non-diagonal components of the tensor all vanish. In (23b)

$$\langle q \rangle = T_r (e^{-\beta \mathfrak{H}_0} q) / T_r (e^{-\beta \mathfrak{H}_0}) \tag{24}$$

is the average value of q at thermal equilibrium of the unperturbed system, and

$$\mathfrak{H}_\omega' = 1/2\pi \cdot \int_{-\infty}^{\infty} dt\, \mathfrak{H}'(t)\, e^{-i\omega t} \tag{25}$$

is the Fourier component of $\mathfrak{H}'(t) = e^{it\mathfrak{H}_0/\hbar}\mathfrak{H}' e^{-it\mathfrak{H}_0/\hbar}$.

§ 4. Some concrete calculations

As one approximation we shall calculate on the basis of our formula the conductivity of the degenerate gas of electrons swimming in the field of crystal lattice waves (a typical model of metals).

We make use of the method of second quantization. The Hamiltonian of the system is expressed as

$$\mathfrak{H} = \mathfrak{H}_0 + \mathfrak{H}', \tag{26}$$

$$\mathfrak{H}_0 = \sum_{\boldsymbol{K}} \varepsilon(\boldsymbol{K})\, a_{\boldsymbol{K}}^* a_{\boldsymbol{K}} + \sum_{\boldsymbol{k}} \hbar\omega(\boldsymbol{k})\, b_{\boldsymbol{k}}^* b_{\boldsymbol{k}}\,, \tag{26a}$$

$$\mathfrak{H}' = \gamma \sum_{\boldsymbol{K}} \sum_{\boldsymbol{k}} k\omega(\boldsymbol{k})^{-1/2} (a_{\boldsymbol{K}+\boldsymbol{k}}^* a_{\boldsymbol{K}} b_{\boldsymbol{k}} + a_{\boldsymbol{K}}^* a_{\boldsymbol{K}+\boldsymbol{k}} b_{\boldsymbol{k}}^*), \tag{26b}$$

where $a_{\boldsymbol{K}}^*$ and $a_{\boldsymbol{K}}$ are the creation and annihilation operator of electron in the state of wave number $\boldsymbol{K}$, and $b_{\boldsymbol{k}}^*$ and $b_{\boldsymbol{k}}$ are those of the phonon corresponding to the normal mode of wave number $\boldsymbol{k}$. Further $\varepsilon(\boldsymbol{K})$ is the energy of the electron with wave number $\boldsymbol{K}$, $\omega(\boldsymbol{k})$ is the frequency of the normal mode of wave number* $\boldsymbol{k}$ and γ is a parameter which represents the strength of the electron-phonon coupling. The operator of the density of electric current is

$$j_\mu = -\frac{e}{\hbar} \sum_{\boldsymbol{K}} \frac{\partial \varepsilon(\boldsymbol{K})}{\partial K_\mu}\, a_{\boldsymbol{K}}^* a_{\boldsymbol{K}}\,. \tag{27}$$

As we shall see in Appendix 4, we have for such system

$$a_{\nu\mu} = a_{\mu\nu} = (e/\hbar)^2 (4\pi^3)^{-1} \iint_{\Omega_F} dS\, |\operatorname{grad}_{\boldsymbol{K}} \varepsilon|^{-1} \frac{\partial \varepsilon}{\partial K_\nu} \frac{\partial \varepsilon}{\partial K_\mu}\,, \tag{28a}$$

$$b_{\nu\mu} = b_{\mu\nu} = (e\gamma)^2 (16\hbar^3\pi^3 kT)^{-1} \iint_{\Omega_F} dS \iiint_V d\boldsymbol{k}\, |\operatorname{grad}_{\boldsymbol{K}} \varepsilon|^{-1} k^2 \frac{\partial}{\partial K_\nu} \{\varepsilon(\boldsymbol{K}+\boldsymbol{k}) - \varepsilon(\boldsymbol{K})\} \cdot$$

$$\cdot \frac{\partial}{\partial K_\mu} \{\varepsilon(\boldsymbol{K}+\boldsymbol{k}) - \varepsilon(\boldsymbol{K})\} \{e^{\hbar\omega(\boldsymbol{k})/kT} - 1\}^{-1} \{1 - e^{-\hbar\omega(\boldsymbol{k})/kT}\}^{-1} \cdot$$

$$\cdot \delta\left(\frac{\varepsilon(\boldsymbol{K}+\boldsymbol{k}) - \varepsilon(\boldsymbol{K})}{\hbar} - \omega(\boldsymbol{k})\right). \tag{28b}$$

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* We call 2π times the true wave number and frequency by the names of wave number and frequency. The use of the same letter for the wave number of normal mode as for the Boltzmann constant will not cause any confusion.

where dS is the element of Fermi surface Ω_F in the wave number space of electron over which the integration covers and $d\boldsymbol{k}$ is the volume element in a simple unit cell V of the reciprocal lattice over which the integration covers. By substituting (28a) and (28b) in (23) or (23a) we get the conductivity tensor.

We shall further develop the calculation in the case of spherical symmetry that $\mathcal{E}(\boldsymbol{K})$ and $\omega(\boldsymbol{k})$ do not depend on the directions of $\boldsymbol{K}$ and $\boldsymbol{k}$ respectively but only on the magnitudes $\boldsymbol{K}$ and $\boldsymbol{k}$. In this case, as we shall show in Appendix 4, we have $a_{\nu\mu}=b_{\nu\mu}=0$ for $\nu\neq\mu$ and

$$a_{\mu\mu}=[e/\pi\hbar\cdot(\partial\mathcal{E}/\partial K_0)K_0]^2/3\,, \tag{28c}$$

$$b_{\mu\mu}=(6\pi/k\Theta)(\gamma eN/\hbar)^2(T/\Theta)^5 J_5(\Theta/T)\,, \tag{28d}$$

where K_0 and $(\partial\mathcal{E}/\partial K)_0$ are the values of K and $\partial\mathcal{E}/\partial K$ at the Fermi surface of electron, Θ is the Debye temperature, and further

$$J_5(x_0)=\int_0^{x_0} dx\,\frac{x^5}{(e^x-1)(1-e^{-x})}\,.$$

Substituting these in (23b), we get

$$\sigma=\frac{e^2k\Theta n(\partial\mathcal{E}/\partial K)_0^2K_0}{18\pi^3\hbar^2N^2\gamma^2(T/\Theta)^5J_5(\Theta/T)}\,,$$

where n and N are the numbers of electrons and lattice ions per unit volume respectively. If we use a concrete expression of γ

$$\gamma=(2\hbar C^2/9NM)^{1/2},$$

where C is Sommerfeld-Bethe's coupling constant,[6] we get

$$\sigma=\frac{e^2nMsk_0K_0(\partial\mathcal{E}/\partial K)_0^2}{4\pi^3\hbar^2C^2(T/\Theta)^5J_5(\Theta/T)}\,, \tag{29}$$

where s is the sound velocity in the crystal and k_0 is the wave number of phonon corresponding to the cut-off frequency. The asymptotic forms of this expression for low and high temperatures are

$$\sigma=\begin{cases}\dfrac{e^2nMsk_0K_0(\partial\mathcal{E}/\partial K)_0^2}{480\left(\sum_{m=1}^{\infty}m^{-5}\right)\pi^3\hbar^3C^2}\left(\dfrac{\Theta}{T}\right)^5 & \text{for } T\ll\Theta\,, \qquad (29a)\\[2ex] \dfrac{e^2nMsk_0K_0(\partial\mathcal{E}/\partial K)_0^2}{\pi^3\hbar^2C^2}\,\dfrac{\Theta}{T} & \text{for } T\gg\Theta\,. \qquad (29b)\end{cases}$$

The expressions (29) and (29a, b) just coincide with those obtained on the basis of the Bloch theory.[6]

If the Hamiltonian has two different kinds of collision terms, $\mathfrak{H}'$ and $\mathfrak{H}''$, the resistivity of the system is equal to the sum of two terms which represent the resistivities in the cases that the collision term is respectively $\mathfrak{H}'$ and $\mathfrak{H}''$ only, as far as $\mathfrak{H}'$ and

$\mathfrak{H}''$ are independent of each other. The independence of $\mathfrak{H}'$ and $\mathfrak{H}''$ means that any eigenstate of $\mathfrak{H}_0$ cannot return to itself by any successive transitions due to $\mathfrak{H}'$ and $\mathfrak{H}''$ or the product $\mathfrak{H}'\,\mathfrak{H}''$ has no diagonal element in the scheme that $\mathfrak{H}_0$ is diagonal. This fact is usually believed and it is true in the case that the system is of spherical symmetry. It is, however, not true if the system is not of spherical symmetry, as is seen from (23a) and (16f). With this reason we should like to stress that the sum rule of resistivity is not satisfied in the system of spherical asymmetry even if the independence of collision terms is fulfilled.

Now in the following we investigate the system of spherical symmetry. The resistivity of the system can be written down as

$$\sigma^{-1}=\sigma'^{-1}+\sigma''^{-1}+\Delta\,, \tag{30}$$

where σ', σ'' and σ are conductivities respectively in the cases that the collision term is $\mathfrak{H}'$, $\mathfrak{H}''$ and both of these. The interference term Δ is given by

$$\Delta=\{\hbar\langle j_\mu{}^2\rangle\}^{-2}\cdot 2\pi kT\langle[\mathfrak{H}'_\omega,\ j_\mu][j_\mu,\ \mathfrak{H}''_{-\omega}]\rangle|_{\omega=0}\,. \tag{30a}$$

If $\mathfrak{H}'$ and $\mathfrak{H}''$ are independent of each other the interference term Δ vanishes. These discussions can be straightforwardly generalized to the cases that three or more kinds of collision terms are present.

In the case in which $\mathfrak{H}'$ is the term corresponding to the collisions of electrons with the phonons of lattice vibrations and $\mathfrak{H}''$ is the term corresponding to the collisions of electrons with the lattice irregularities, for example defects, impurities, disordering in alloys and so on, the sum rule that the interference vanishes is known as the Mathiesen rule, where σ' is given by (29) and σ'' will be given in the following. With the use of the Fourier component $U(\boldsymbol{K})$ of the irregularity potential for an electron we have

$$\mathfrak{H}''=\sum_{\boldsymbol{K}}\sum_{\boldsymbol{K}'}U(\boldsymbol{K}'-\boldsymbol{K})\,a^*_{\boldsymbol{K}'}a_{\boldsymbol{K}}\,. \tag{26c}$$

The condition of independence is fulfilled in the present case. The conductivity σ'' due to $\mathfrak{H}''$ is found in Appendix 4 to be

$$\sigma''=(2e^2/3\pi\hbar)\,(\partial\mathcal{E}/\partial K)_0{}^2\cdot\Big[\int_0^\pi d\theta\,\sin\theta\,(1-\cos\theta)\,U(\theta)\Big]^{-1}, \tag{29c}$$

where $U(\theta)$ is equal to $U(\boldsymbol{K}'-\boldsymbol{K})$ in which the magnitudes of $\boldsymbol{K}'$ and $\boldsymbol{K}$ are both equal to K_0 and the angle between them is θ. If the system is not of spherical symmetry, the interference term appears and the Mathiesen rule is no longer satisfied.

We investigate lastly, as an example of application of the original formula (11), the free electron assembly in the presence of uniform magnetic field of magnitude H. If we take the direction of the magnetic field as the z-axis, the eigenstate of one electron is indicated with the set of three quantum numbers, n, γ_0 and λ, which we represent by a single letter, q. The Hamiltonian of the system is expressed as[10]

$$\mathfrak{H}=\sum_q(n\hbar\omega_0+\lambda^2/2m)\,a_q{}^*a_q\,, \tag{31}$$

where

$$\omega_0 = eH/mc \tag{31a}$$

is twice the Larmor frequency, and a_η^* and a_η are the creation and annihilation operators of the state q. The operators of the components of electric current density are

$$\left.\begin{aligned} j_x &= -(e/m)\sum_q{}'\sum_{q'}{}'(q\,|\,p_x + m\omega_0 y\,|\,q')\,a_q^*\,a_{q'}\,, \\ j_y &= -(e/m)\sum_q{}'\sum_{q'}{}'(q\,|\,p_y\,|\,q')\,a_q^*\,a_{q'}\,,\quad j_z = -(e/m)\sum_q{}'\sum_{q'}{}'(q\,|\,p_z\,|\,q')\,a_q^*\,a_{q'} \end{aligned}\right\} \tag{32}$$

in terms of the matrix elements of the one electron problem, which are null except in the following

$$\left.\begin{aligned} &(q\,|\,p_x\,|\,q) = -m\omega_0\gamma_0,\quad (q\,|\,y\,|\,q) = \gamma_0,\quad (q\,|\,p_z\,|\,q) = \lambda, \\ &(q\,|\,p_y\,|\,q') = \begin{cases} i\sqrt{m\hbar\omega_0/2}\ \sqrt{n+1} & \text{for } q' = (n+1,\ \gamma_0,\ \lambda), \\ -i\sqrt{m\hbar\omega_0/2}\ \sqrt{n} & \text{for } q' = (n-1,\ \gamma_0,\ \lambda), \end{cases} \\ &(q\,|\,y\,|\,q') = \begin{cases} \sqrt{\hbar/(2m\omega_0)}\ \sqrt{n+1} & \text{for } q' = (n+1,\ \gamma_0,\ \lambda), \\ \sqrt{\hbar/(2m\omega_0)}\ \sqrt{n} & \text{for } q' = (n-1,\ \gamma_0,\ \lambda), \end{cases} \end{aligned}\right\} \quad q = (n,\ \gamma_0,\ \lambda). \tag{32a}$$

With the use of (32) and (32a) we get from (9) after a short calculation

$$\left.\begin{aligned} &\varphi_{xy}(t) = -\varphi_{yx}(t) = (ne^2/m)\sin\omega_0 t,\quad \varphi_{xx}(t) = \varphi_{yy}(t) = (ne^2/m)\cos\omega_0 t, \\ &\varphi_{yz}(t) = \varphi_{zy}(t) = \varphi_{xz}(t) = \varphi_{zx}(t) = 0,\quad \varphi_{zz}(t) = \text{independent of time.} \end{aligned}\right\} \tag{33}$$

From these we obtain the static conductivity tensor, which components are

$$\sigma_{yx} = -\sigma_{xy} = nec/H,\quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{zx} = \sigma_{xz} = \sigma_{zy} = \sigma_{yz} = 0\,. \tag{34}$$

On the other hand σ_{zz} is not finite, and it is necessary to take into account the collision terms to get a finite result.

§ 5. Concluding remarks

We have investigated a method of calculating electrical conductivity based upon the quantum-statistical theory of irreversible process. It has a feature fairly different from usual Bloch's theory which is based upon the Boltzmann equation of transport phenomena, and the applicability of the former is not so limited as the latter. The present method is useful indeed in discussing different kinds of relaxation phenomena as we shall see later at the earliest opportunity.

The method is formally developed on the basis of the relaxation function, but the formula for the relaxation function is hardly of practical use in order to investigate the conduction phenomenon in actual systems. We have to introduce the irreversibility of the phenomena into the theory and then we can get the practical expression of the conductivity. In the preceding papers[1),2)] we made an approximation of the relaxation function of the actual system according to Kubo-Tomita's method[3)] on the assumption

that the relaxation function is asymptotically equal to an exponentially fading function determined by the perturbation, in which it was not so clear how the irreversibility was introduced and accordingly the generalization to a wider application was difficult. In the present procedure we have discussed in § 3 that the meaning of irreversibility is fairly clear and the generalization is quite easy to include the effect of the displacement of the phonon distribution which we shall show quite soon. We shall further be able to develop a general theory of irreversible process, on which some authors have made investigations[9),11)] and we also shall investigate according to our procedure.

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Appendices

1. The symmetry properties of the relaxation function.

The equation (13) can be proved on the basis of the properties of the current density operator and the Hamiltonian. With use of the variable $\lambda'=\beta-\lambda$, we have

$$T_r\{e^{-\beta\mathfrak{H}}e^{\lambda\mathfrak{H}}j_\nu e^{-\lambda\mathfrak{H}}j_\mu(t)\}=T_r\{e^{-\beta\mathfrak{H}}e^{\lambda'\mathfrak{H}}j_\mu(t)\,e^{-\lambda'\mathfrak{H}}j_\nu\}$$
$$=T_r\{e^{-\beta\mathfrak{H}}e^{\lambda'\mathfrak{H}}j_\mu e^{-\lambda'\mathfrak{H}}j_\nu(-t)\}.$$

Substituting it in (9) and changing the integration variable λ for λ', we get the equation

$$\varphi_{\nu\mu}(t)=\varphi_{\mu\nu}(-t).$$

In a certain representation operators are real, purely imaginary or complex, though they are all hermitian of course. In the representation that the coordinate x is diagonalized, the coordinate itself is real and the momentum $p=\hbar/i\cdot\partial/\partial x$ is purely imaginary. The Hamiltonian $\mathfrak{H}$ is real in the absence of the magnetic field as it is a quadratic function of momenta of electrons and others. The current density proportional to the total momentum of electrons is purely imaginary. Representing the conjugate complex by the superscript bar, we have

$$\overline{j_\mu(t)}=\overline{\exp(i/\hbar\cdot t\mathfrak{H})\,j_\mu\exp(-i/\hbar\cdot t\mathfrak{H})}=-j_\mu(-t).$$

By substituting it in the conjugate complex of (9), we obtain

$$\overline{\varphi_{\nu\mu}(t)}=\varphi_{\nu\mu}(-t).$$

On the other hand $\varphi_{\nu\mu}(t)$ is real, of course, as we can in fact verify from the equations

$$\int_0^\beta d\lambda\,T_r\{e^{-\beta\mathfrak{H}}e^{\lambda\mathfrak{H}}j_\nu e^{-\lambda\mathfrak{H}}j_\mu(t)\}^*=\int_0^\beta d\lambda\,T_r\{j_\mu(t)\,e^{-\lambda\mathfrak{H}}j_\nu e^{\lambda\mathfrak{H}}e^{-\beta\mathfrak{H}}\}$$
$$=\int_0^\beta d\lambda\,T_r\{e^{-\beta\mathfrak{H}}e^{\lambda\mathfrak{H}}j_\nu e^{-\lambda\mathfrak{H}}j_\mu(t)\},$$

where the asterisk represents the Hermitian conjugate matrix. Then we get

$$\varphi_{\nu\mu}(t)=\varphi_{\nu\mu}(-t).$$

The equations (13) have been all proved.

In the presence of external magnetic field the Hamiltonian $\mathfrak{H}(\boldsymbol{H})$ is not real and the electric current density $\boldsymbol{j}(\boldsymbol{H})$ is not purely imaginary but both are complex, and

$$\boldsymbol{j}(\boldsymbol{H})=-\boldsymbol{j}(-\boldsymbol{H}),\ \mathfrak{H}(\boldsymbol{H})=\mathfrak{H}(-\boldsymbol{H}).$$

Accordingly we get, instead of (13),

$$\varphi_{\nu\mu}(t,\ \mathfrak{H})=\varphi_{\mu\nu}(t,\ -\mathfrak{H}). \tag{13a}$$

2. The expression for the density matrix after applying an impulse field.

It is possible to express any operator of a given system in terms of operators which, as a whole, form a complete set of operators of the system. Accordingly, writing operators in a complete set as $x_1, x_2, \cdots, x_i, \cdots$, we can express the density matrix $\rho(t)$ or its logarithm $\log\rho(t)$ at any time t in terms of these operators as

$$\log\rho(t)=\text{const}+\sum_i X_i(t)\cdot x_i.$$

We have therefore

$$\rho(t)=C\cdot\exp\left[\sum_i X_i(t)\cdot x_i\right],$$

where C is the normalization constant.

We must next take into account that the problems under investigation is not simple quantum-mechanical processes but the thermodynamical (irreversible) processes. In other words, we do concern with the thermodynamical states and not with the simple quantum-mechanical states. Therefore it is sufficient to make use of the set of gross or coarse-grained operators[9] instead of the complete set of operators in the quantum-mechanically exact sense. Thus we must regard $x_1, x_2, \cdots, x_i, \cdots$ as the set of gross operators instead of the complete set of operators. It must be determined from the physical viewpoint how many and which of gross operators are to be taken. In the present paper we have expressed the density matrix after applying an impulse field at t_0 as

$$\rho(t)=C\cdot\exp\left[-\beta\left\{\mathfrak{H}-\sum_\mu A_\mu(t,\ t_0)j_\mu\right\}\right].$$

If we take further $B_\mu(t,\ t_0)$ which is the intensive parameter conjugate to the total momentum of the whole system of lattice vibrations, we can take into account the state of affairs that the distribution of lattice phonons is displaced from that in thermal equilibrium. If we take, instead of a single effective potential $A_\mu(t,\ t_0)$, $A_{\mu\,\boldsymbol{K}}(t,\ t_0)$ of different $\boldsymbol{K}$ which is conjugate to the part of electric current contributed from the electrons with wave numbers in the vicinity of $\boldsymbol{K}$, we can make calculation which takes into account the dependence of the collision time upon the wave number of the electron.

3. The symmetric properties of the matrices $a_{\nu\mu}$ and $b_{\nu\mu}$, the calculation of them and the perturbation approximation.

We shall show in the first place that the expression of $b_{\nu\mu}$ in (16c) can be rewritten as in (16d). We have first

$$b_{\nu\mu} = -(\Delta t)^{-1}\int_0^{\beta} d\lambda \int_0^{\Delta t} dt\, T_r\{\rho_c e^{\lambda\mathfrak{H}} j_\nu e^{-\lambda\mathfrak{H}} dj_\mu(t)/dt\}$$

$$= -(i\hbar\Delta t)^{-1}\int_0^{\beta} d\lambda \int_0^{\Delta t} dt\, T_r\{\rho_c j_\nu \partial j_\mu(t+i\hbar\lambda)/\partial\lambda\}$$

$$= (i\hbar\Delta t)^{-1}\int_0^{\Delta t} dt\, T_r\{\rho_c[j_\nu,\ j_\mu(t)]\},$$

further this can be written as

$$b_{\nu\mu} = (i\hbar\Delta t)^{-1}\int_0^{\Delta t} dt \int_0^{t} d\tau\, T_r\{\rho_c[j_\nu,\ dj_\mu(\tau)/d\tau]\} + (i\hbar)^{-1} T_r\{\rho_c[j_\nu,\ j_\mu]\}$$

and by changing the integration variables τ, t for τ, $\tau' = t-\tau$ in the integral in the first term we get

$$b_{\nu\mu} = (i\hbar\Delta t)^{-1}\int_0^{\Delta t} d\tau \int_0^{\Delta t-\tau} d\tau'\, T_r\{\rho_c[j_\nu,\ dj_\mu(\tau)/d\tau]\} + (i\hbar)^{-1} T_r\{\rho_c[j_\nu,\ j_\mu]\}$$

$$= -\hbar^{-2}\int_0^{\Delta t} d\tau\,(\Delta t-\tau)\ T_r\{\rho_c[j_\nu,\ [j_\mu(\tau),\ \mathfrak{H}]]\} + (i\hbar)^{-1} T_r\{\rho_c[j_\nu,\ j_\mu]\}.$$

It will be possible to assume that the trace appearing in the integrand in the last expression is going to fade rapidly as τ goes beyond a certain value τ_c which is perhaps much smaller than Δt of the same order of magnitude as a time required for a measurement, and therefore we can rewrite as (16d).

Next we show that $a_{\nu\mu}$ and $b_{\nu\mu}$ are both the real and symmetric matrices. That they are real is trivial. The properties of symmetry are also immediately verified. For the matrix $a_{\nu\mu}$ the symmetry is a direct consequence of that of the relaxation function because $a_{\nu\mu} = \varphi_{\nu\mu}(0)$. We have for $b_{\nu\mu}$ from (16d)

$$b_{\nu\mu} = \overline{b_{\nu\mu}} = -\hbar^{-2}\int_0^{\infty} dt\, \overline{T_r\{\rho_c[j_\nu, [j_\mu(t), \mathfrak{H}]]\}} - (i\hbar)^{-1}\overline{T_r\{\rho_c[j_\nu,\ j_\mu]\}}$$

$$= -\hbar^{-2}\int_0^{\infty} dt\, T_r\{\rho_c[j_\nu,\ [j_\mu(-t),\ \mathfrak{H}]]\} - (i\hbar)^{-1} T_r\{\rho_c[j_\nu,\ j_\mu]\}$$

$$= -\hbar^{-2}\int_0^{\infty} dt\, T_r\{\rho_c[j_\nu(t),\ [j_\mu,\ \mathfrak{H}]]\} + (i\hbar)^{-1} T_r\{\rho_c[j_\mu,\ j_\nu]\}$$

$$= -\hbar^{-2}\int_0^{\infty} dt\, T_r\{\rho_c[j_\mu,\ [j_\nu(t),\ \mathfrak{H}]]\} + (i\hbar)^{-1} T_r\{\rho_c[j_\mu,\ j_\nu]\} = b_{\mu\nu}$$

by making use of the relation $j_\nu(t)=-j_\nu(-t)$, $\bar{\mathfrak{H}}=\mathfrak{H}$. In the case of the presence of magnetic field, we have

$$b_{\nu\mu}(\boldsymbol{H})=b_{\mu\nu}(-\boldsymbol{H})$$

because $j_\mu(t,\ \mathfrak{H})=-j_\mu(-t,\ -\mathfrak{H})$, $\mathfrak{H}(\boldsymbol{H})=\mathfrak{H}(-\boldsymbol{H})$. These symmetry properties are the direct consequences of the same properties of the relaxation function, if we rely on the original expression, (16c), which is rewritten as

$$b_{\nu\mu}=\frac{\varphi_{\nu\mu}(\Delta t)-\varphi_{\nu\mu}(0)}{\Delta t}$$

in terms of the relaxation function.

Now we shall further find the perturbation approximation in the case of absence of magnetic field. By substituting the perturbation developments,

$$j_\mu(t)=j_\mu+(i\hbar)^{-1}\int_0^t d\tau[j_\mu,\ \mathfrak{H}'(\tau)]+\cdots,$$

$$\exp[-\beta(\mathfrak{H}_0+\mathfrak{H}')]=\exp(-\beta\mathfrak{H}_0)\Big|1-\int_0^\beta d\lambda\, e^{\lambda\mathfrak{H}_0}\mathfrak{H}'e^{-\lambda\mathfrak{H}_0}+\cdots\Big|,$$

in (16d) and retaining the lowest order term only, we get

$$b_{\nu\mu}=-\hbar^{-2}\int_0^\beta d\lambda\int_0^\infty dt\langle e^{\lambda\mathfrak{H}_0}[\mathfrak{H}',\ j_\nu]e^{-\lambda\mathfrak{H}_0}[j_\mu,\ \mathfrak{H}'(t)]\rangle.$$

By substituting the Fourier decomposition of $\mathfrak{H}'(t)$ in this expression and making use of the integral formula

$$\int_0^\infty dt\exp(i\omega t)=\pi\delta(\omega)+i/\omega,$$

we get the expression (16f). We shall not need to explain about deriving the perturbational expression for $a_{\nu\mu}$ in (16f).

4. The conductivity of the system of free electrons and lattice vibrations or lattice irregularities of other kind.

Using (27), we have

$$\langle j_\nu j_\mu\rangle=\Big(\frac{e}{\hbar}\Big)^2\Big\langle\sum_{\boldsymbol{K}}\frac{\partial\varepsilon}{\partial K_\nu}a_{\boldsymbol{K}}^*a_{\boldsymbol{K}}\sum_{\boldsymbol{K}'}\frac{\partial\varepsilon'}{\partial K_{\mu'}}a_{\boldsymbol{K}'}^*a_{\boldsymbol{K}'}\Big\rangle$$

$$=\Big(\frac{e}{\hbar}\Big)^2\sum_{\boldsymbol{K}}\frac{\partial\varepsilon}{\partial K_\nu}\frac{\partial\varepsilon}{\partial K_\mu}(f_{\boldsymbol{K}}-f_{\boldsymbol{K}}^2)=\Big(\frac{e}{\hbar}\Big)^2kT\sum_{\boldsymbol{K}}\frac{\partial\varepsilon}{\partial K_\nu}\frac{\partial\varepsilon}{\partial K_\mu}\frac{\partial f}{\partial\varepsilon},$$

where

$$f_{\boldsymbol{K}}\equiv\langle a_{\boldsymbol{K}}^*a_{\boldsymbol{K}}\rangle=f(\varepsilon(\boldsymbol{K}))=[\exp\{(\varepsilon(\boldsymbol{K})-\zeta)/kT\}+1]^{-1}$$

is the function of the Fermi distribution in which ζ is the Fermi energy, and where the use is made of the abbreviations ε and ε' for $\varepsilon(\boldsymbol{K})$ and $\varepsilon(\boldsymbol{K}')$. The summation over $\boldsymbol{K}$ can be replaced by the integration over $\boldsymbol{K}$. Taking into account the spin degeneracy and taking the curvilinear coordinates on the surface $\varepsilon(\boldsymbol{K})=$const. and along its normal, the summation over $\boldsymbol{K}$ is replaced by the integration

$$2(2\pi)^{-3}\iiint dK_x\,dK_y\,dK_z = 2(2\pi)^{-3}\int d\varepsilon\,|\operatorname{grad}_{\boldsymbol{K}}\varepsilon|^{-1}\iint dS\,,$$

and we get (28a).

Secondly, by making use of the commutation relation

$$[a_{\boldsymbol{K}}^{*}a_{\boldsymbol{K}'},\ a_{\boldsymbol{K}''}^{*}a_{\boldsymbol{K}'''}] = a_{\boldsymbol{K}}^{*}a_{\boldsymbol{K}'''}\delta_{\boldsymbol{K}'\boldsymbol{K}''} - a_{\boldsymbol{K}''}^{*}a_{\boldsymbol{K}'}\delta_{\boldsymbol{K}\boldsymbol{K}'''}\,,$$

we have

$$[j_\nu,\ \mathfrak{H}_\omega'] = (e/\hbar)\gamma\sum_{\boldsymbol{K},\boldsymbol{k}}(\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})-\hbar\omega(\boldsymbol{k})=\pm\hbar\omega)\,k\omega(\boldsymbol{k})^{-1/2}\cdot$$
$$\cdot(\partial/\partial K_\nu)\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}(\partial/\partial K_\mu)\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{k})\}(a_{\boldsymbol{K}+\boldsymbol{k}}^{*}a_{\boldsymbol{K}}b_{\boldsymbol{k}}-a_{\boldsymbol{K}}^{*}a_{\boldsymbol{K}+\boldsymbol{k}}b_{\boldsymbol{k}}^{*})\,,$$

where the summation is taken over the values which satisfy the equation $\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})-\hbar\omega(\boldsymbol{k})=\hbar\omega$ and $-\hbar\omega$ respectively for the first term $a_{\boldsymbol{K}+\boldsymbol{k}}^{*}a_{\boldsymbol{K}}b_{\boldsymbol{k}}$ and the second term $a_{\boldsymbol{K}}^{*}a_{\boldsymbol{K}+\boldsymbol{k}}b_{\boldsymbol{k}}^{*}$. By substituting it in the second expression of (16f) we have

$$b_{\nu\mu}=\pi(e^2\gamma^2/\hbar^4)\sum k^2\omega(\boldsymbol{k})^{-1}(\partial/\partial K_\nu)\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}(\partial/\partial K_\mu)\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}\cdot$$
$$\cdot\{f_{\boldsymbol{K}+\boldsymbol{k}}(1-f_{\boldsymbol{K}})\cdot(n_{\boldsymbol{k}}+1)+f_{\boldsymbol{K}}(1-f_{\boldsymbol{K}+\boldsymbol{k}})n_{\boldsymbol{k}}\}\,|_{\hbar^{-1}\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}-\omega(\boldsymbol{k})=0}$$
$$=2\pi(e^2\gamma^2/\hbar^4)\sum k^2\omega(\boldsymbol{k})^{-1}\cdot\partial/\partial K_\nu\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}\cdot$$
$$\cdot\partial/\partial K_\mu\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}(f_{\boldsymbol{K}}-f_{\boldsymbol{K}+\boldsymbol{k}})n_{\boldsymbol{k}}(n_{\boldsymbol{k}}+1)\,|_{\hbar^{-1}\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}-\omega(\boldsymbol{k})=0}$$
$$\simeq 2\pi(e^2\gamma^2/\hbar^3)\sum k^2(\partial/\partial K_\nu)\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}(\partial/\partial K_\mu)\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}\cdot$$
$$\cdot(\partial f/\partial\varepsilon)n_{\boldsymbol{k}}\,|_{\hbar^{-1}\{\varepsilon(\boldsymbol{K}+\boldsymbol{k})-\varepsilon(\boldsymbol{K})\}-\omega(\boldsymbol{k})=0}\,,$$

where

$$n_{\boldsymbol{k}}\equiv\langle b_{\boldsymbol{k}}^{*}b_{\boldsymbol{k}}\rangle = n(\omega(\boldsymbol{k})) = [\exp\{\hbar\omega(\boldsymbol{k})/kT\}-1]^{-1}$$

is the function of the Planck distribution. If we replace the summations by the integrations, we get the expression (28b).

In the system of spherical symmetry $\varepsilon(\boldsymbol{K})=\varepsilon(K)$ and $\omega(\boldsymbol{k})=\omega(k)=sk$ with the use of the sound velocities independent of $\boldsymbol{k}$, we have

$$|\operatorname{grad}_{\boldsymbol{K}}\varepsilon| = |\partial\varepsilon/\partial K|\,,\quad \frac{\partial\varepsilon}{\partial K_\nu}=\frac{\partial\varepsilon}{\partial K}\frac{K_\nu}{K}\,,$$

and the integration in (28a) can be calculated as

$$\iint\limits_{\Omega_F} dS\,|\partial\varepsilon/\partial K|^{-1}(\partial\varepsilon/\partial K)^2\frac{K_\nu}{K}\frac{K_\mu}{K}=\frac{4\pi}{3}K_0^2\left(\frac{\partial\varepsilon}{\partial K}\right)_0\delta_{\nu\mu}\,,$$

which gives (28c). The integration in (28b) is calculated as follows

$$\iint_{\Omega_F} dS \int dk \int d(\cos\theta) \int d\varphi k^4 |\partial\mathcal{E}/\partial K|^{-1} (\partial\mathcal{E}/\partial K)^{-2} k_\nu k_\mu K^{-2} (e^{\hbar\beta k^3}-1)^{-1} \cdot$$

$$\cdot (1-e^{-\hbar\beta k^3})^{-1} \delta\left(\hbar^{-1} k\cos\theta \frac{\partial\mathcal{E}}{\partial K} - sk\right) = \frac{8}{3} \pi^2 \hbar \delta_{\nu\mu} \int_0^{k_0} dk k^5 (e^{\hbar\beta k^3}-1)^{-1} (1-e^{-\hbar\beta k^3})^{-1},$$

where θ is the angle between $\boldsymbol{K}$ and $\boldsymbol{k}$. Substituting it in (28b) we get (28d).

Taking the collision term (26e) we have

$$\langle [\mathfrak{H}_\omega'', j_\nu] [j_\mu, \mathfrak{H}_{-\omega}''] \rangle|_{\omega=0} = \left(\frac{e}{\hbar}\right)^2 \sum_{\boldsymbol{K},\boldsymbol{K}'} |U(\boldsymbol{K}-\boldsymbol{K}')|^2 \left(\frac{\partial\mathcal{E}'}{\partial K_\nu'} - \frac{\partial\mathcal{E}}{\partial K_\nu}\right)\left(\frac{\partial\mathcal{E}'}{\partial K_\mu'} - \frac{\partial\mathcal{E}}{\partial K_\mu}\right) \cdot$$

$$\cdot f_{\boldsymbol{K}'}(1-f_{\boldsymbol{K}}) |_{\hbar^{-1}\{\mathcal{E}(\boldsymbol{K}')-\mathcal{E}(\boldsymbol{K})\}=0},$$

which is, with the use of the integrations instead of the summations,

$$\frac{e^2 kT}{4\pi^3\hbar} \iint_{\Omega_F} dS \iint_{\Omega_F} dS' |\operatorname{grad}\mathcal{E}|^{-1} |\operatorname{grad}\mathcal{E}'|^{-1} \left(\frac{\partial\mathcal{E}'}{\partial K_\nu'} - \frac{\partial\mathcal{E}}{\partial K_\nu}\right) \cdot$$

$$\cdot \left(\frac{\partial\mathcal{E}'}{\partial K_\mu'} - \frac{\partial\mathcal{E}}{\partial K_\mu}\right) \cdot |U(\boldsymbol{K}-\boldsymbol{K}')|^2.$$

In the case in which $\mathcal{E}(\boldsymbol{K})$ is a function only of the magnitude of $\boldsymbol{K}$, we have further

$$\frac{e^2 kT}{32\pi^6\hbar} \iint_{\Omega_F} dS \iint_{\Omega_F} dS' (K_\mu - K_\mu')^2 K^{-2} |U(\boldsymbol{K}'-\boldsymbol{K})|^2 \cdot \delta_{\nu\mu}.$$

With the use of the angles θ between $\boldsymbol{K}$ and $\boldsymbol{K}'$, θ between $\boldsymbol{K}$ and the μ-axis, θ' between $\boldsymbol{K}'$ and μ-axis and φ between the plane containing $\boldsymbol{K}$ and μ-axis and that containing $\boldsymbol{K}$ and $\boldsymbol{K}'$, we have

$$\frac{e^2 kT}{6\pi^4\hbar} K_0^4 \int_0^\pi d\theta |U(\theta)|^2 \sin\theta (1-\cos\theta)$$

where use has been made of the relation $\cos\theta' = \cos\theta\cos\theta + \sin\theta\sin\theta\cos\varphi$. Substituting this and the expression for $\langle j_\mu^2 \rangle$ above obtained in (23b), we get (29c).

Note added in proof

We must correct here the wrong estimation of the time interval Δt in eq. (16b) we have made in this paper. To do it some investigation of the slowly varying property of the gross variable is required. The gross variable of the macroscopic system, such as the current density of the system of a great many electrons as in metals, is slowly varying in time. Although every piece of current density due to a single electron fluctuates very rapidly, the total current density of the system varies only very slowly because the sharp fluctuation of every piece of current is cancelled one another in total.

The condition of slowly-varying of the gross variable has been formulated by N. G. Van Kampen.[9] According to him this condition for a quantity x is given by

$$\Delta X_{\text{obs}} \gg \dot{X} \cdot \Delta t_{\text{obs}} , \tag{N1}$$

where ΔX_{obs} is the inaccuracy in the observed value of x and Δt_{obs} is a time required for an observation. On the other hand, the difference quotient appearing in the left hand side of (16b) can be regarded physically as the differential quotient, if the time interval Δt is of the order given by

$$[dA_\nu(t, t_0)/dt] \cdot \Delta t \gtrsim \Delta A_\nu(t, t_0)_{\text{obs}} . \tag{N2}$$

On account of the relation (19) the same condition as (N1) is fulfilled for $A_\nu(t, t_0)$ viz.

$$\Delta A_\nu(t, t_0)_{\text{obs}} \gg [dA_\nu(t, t_0)/dt] \cdot \Delta t_{\text{obs}} . \tag{N3}$$

Comparing (N2) and (N3), we get

$$\Delta t \gg \Delta t_{\text{obs}} . \tag{N4}$$

Instead we have written in the present article as

$$\Delta t \approx \Delta t_{\text{obs}} , \tag{N5}$$

which is a wrong estimate. It is indeed because Δt satisfies (N4) and not (N5) that we can replace in the expression for $b_{\nu\mu}$ the upper limit of integration Δt by ∞ and neglect $t/\Delta t$ compared to the unity when we calculate $b_{\nu\mu}$ in Appendix 3.

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One-Center Expansion of Molecular Wave Function, II

— Lower Excited States of H_2 —

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The method of one-center expansion of molecular wave function is applied to the calculation of electronic energies of lower excited levels, ${}^3\Sigma_g^+$, ${}^{3,1}\Sigma_u^+$ and ${}^{3,1}\Pi_u$ of H_2. As for the ground ${}^1\Sigma_g^+$ level the present author has already reported some calculations where he stated to have obtained the best value, 3.8 e.V., as the electronic dissociation energy. Unfortunately it was later found that there were mistakes in numerical calculations. An alternative 'best' value, 3.28 e.V., will be reported in the second section of the present paper. As a whole, the results of the present calculations may be said moderately satisfiable.

§ 1. Introduction

The problem of lower excited states of H_2 has been treated by many authors. Kemble and Zener[1] and Guillemin and Zener[2] employed the Heitler-London method. Hylleraas[3] constructed the wave functions of the excited states of H_2 from the wave functions of H_2^+. Present[4] and James, Coolidge and Present[5] used the variational method which proved to be very successful in the treatment of the ground state of H_2[6]. These authors treated only the so-called two-quantum states, namely in the current notations, $1s\sigma 2s\sigma\ {}^3\Sigma_u^+$, $1s\sigma 2p\sigma\ {}^3\Sigma_u^+$, ${}^1\Sigma_u^+$ and $1s\sigma 2p\pi\ {}^3\Pi_u$, ${}^1\Pi_u$. One of the reasons for this limitation is, undoubtedly, the well-known difficulties in calculation: when we attempt to include higher quantum states, using the wave function with two reference centers (two protons in this case), integrals of electron-electron interactions create troubles.

Besides the above calculations, the work of MacDonald[7] is noteworthy. In his treatment the wave functions of H_2 are built up from the functions associated with one electron in the normal state of H_2^+ and with the other in an n-quantum state of the hydrogen atom located at the middle point between two nuclei, together with interchange. This idea was suggested by the facts that many excited states have the equilibrium nuclear distance of about 1.0 Å which is the one of the normal hydrogen molecule ion H_2^+ and the levels of the outer electron of H_2 are spaced as is expressed by the ordinary Rydberg formula[8]. This appeals to our intuitive imagination and, for a molecule as closely bound as hydrogen, it is expected that the method of MacDonald will be more effective than the Heitler-London method, especially for higher excited states. In fact MacDonald treated three-quantum states of H_2 successfully by his method. Now let us consider the potential field for an electron due to two separate attractive centers. In Fig. 1 Z_A, Z_B

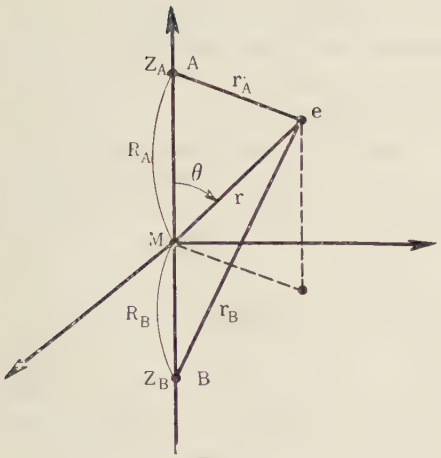

Fig. 1.

are two nuclear charges and R_A+R_B is the nuclear distance. M is the center of mass of two charges, namely $Z_A R_A=Z_B R_B$. The position of an electron, e, is expressed by the spherical coordinates around the origin M. The potential field $V(r, \theta, \varphi)$ is expressed as follows;

$$V(r, \theta, \varphi) = -(Z_A/r_A) - (Z_B/r_B),$$

$$1/r_A = \sum_{l=0}^{\infty} (r^l/R_A^{l+1}) P_l(\cos\theta), \qquad r < R_A,$$

$$= \sum_{l=0}^{\infty} (R_A^l/r^{l+1}) P_l(\cos\theta), \qquad r \geq R_A, \quad (1\cdot1)$$

$$1/r_B = \sum_{l=0}^{\infty} (r^l/R_B^{l+1}) P_l(\cos\theta)\,(-1)^l, \quad r < R_B,$$

$$= \sum_{l=0}^{\infty} (R_B^l/r^{l+1}) P_l(\cos\theta)\,(-1)^l, \quad r \geq R_B,$$

where $P_l(\cos\theta)$ is the Legendre Polynomial. Then for $r > R_A,\ R_B$,

$$V(r, \theta, \varphi) = -\sum_{l=0}^{\infty} \{Z_A(R_A^l/r^{l+1}) + (-1)^l Z_B(R_B^l/r^{l+1})\} P_l(\cos\theta) \qquad (1\cdot2)$$

$$= -(Z_A+Z_B)/r\left[1+1/(Z_A+Z_B)\sum_{l=1}^{\infty}\{Z_A(R_A/r)^l + (-1)^l Z_B(R_B/r)^l\} P_l(\cos\theta)\right]$$

and in the case of $Z_A=Z_B=Z$ and $R_A=R_B=R$,

$$V(r, \theta, \varphi) = -(2Z/r)\left[1+\sum_{s=1}^{\infty}(R/r)^{2s} P_{2s}(\cos\theta)\right]. \qquad (1\cdot3)$$

Here the second term shows the degree of deviation from the spherical potential field $(-2Z/r)$ due to the charge $2Z$ located at the middle point M and it is seen that the deviation falls off rapidly for the outer region, $r \gg R$. Then it is natural to expect that atomic wave functions around the middle point M offer reasonable approximations for the electron which moves in outer regions. This is especially the case when the bare nuclear field is screened by the inner electron and the outer electron moves in the screened field. However, for the inner region, $r \lesssim R$, a notion like this does not work at all. Thus, MacDonald adopted the following function designed by Guillemin and Zener[9] for the normal H_2^+ as the wave function of the inner electron of H_2:

$$M(\xi, \eta) = C\, e^{-\alpha R\xi} \cosh\beta R\eta, \qquad (1\cdot4)$$

where C, α and β are constants, $2R$ is the nuclear separation and ξ and η are spheroidal coordinates. However, MacDonald's manner of constructing wave functions of H_2 suffers a serious disadvantage: because his wave functions contain three reference centers (two nuclei and the middle point) the integrals of electron-electron interactions are extremely difficult and he had to resort to tedious graphical integrations. Nevertheless, this disad-

vantage is not inherent in MacDonald's original idea. If we have a good wave function having its coordinate center at M also for the inner electron, the difficulty of three-center integrals disappears at once and the whole calculation will become a very easy matter. The present author attempted an examination of this possibility in a preceding work[10] (hereafter called I). The conclusion was that the situation is not necessarily hopeless and with this in mind a calculation has been made on several of the two-quantum states of H_2 along the same line of reasoning as in the paper of MacDonald, as is reported in the following.

§ 2. Calculations on lower excited states of H_2

In this section an analytical version of MacDonald's calculation is described. The point is that we use, in place of the two-center function (1·4), the following one-center function for the inner electron which was obtained in the previous calculation;[10]

$$\phi_{in}=0.6614319[(1s\sigma;\ 1)+0.5126091(4s\sigma;\ 3)+0.2707095(4d\sigma;\ 3)], \qquad (2\cdot1)$$

$$(ns\sigma;\ \zeta)=N(n,\ \zeta)r_i^{n-1}\ e^{-\zeta r}\ Y_{00},$$

$$(nd\sigma;\ \zeta)=N(n,\ \zeta)r^{n-1}\ e^{-\zeta r}\ Y_{20},$$

$$N(n,\ \zeta)=(2\zeta)^{n+1/2}[(2n)!]^{-1/2},$$

where the electron coordinates are measured from the middle point between two nuclei. This function yields -1.074755 a.u. (atomic units) as the total electronic energy of the normal state of H_2^+, while the exact value is -1.102625 a.u.[11], the nuclear distance being 2.0 a.u. For the outer electron we adopt a single term of atomic wave functions, the center of which is located at the middle point between two nuclei. Thus our wave function has a form:

$$\Psi(1,\ 2)=(1/\sqrt{2})[\phi_{in}(1)\phi_{out}(2)\pm\phi_{out}(1)\phi_{in}(2)], \qquad (2\cdot2)$$

where ϕ_{in} is the approximate wave function for normal H_2^+ given by (2·1) and ϕ_{out} is an atomic function to be chosen properly for each case. The plus sign stands for the singlet state and the minus sign for the triplet state. Our problem is to calculate the expectation values of the following Hamiltonian (expressed in atomic units) and to minimize it with respect to an effective charge parameter η contained in ϕ_{out}:

$$H(1,\ 2)=-\frac{1}{2}(\Delta_1+\Delta_2)-\left(\frac{1}{r_{A1}}+\frac{1}{r_{B1}}+\frac{1}{r_{A2}}+\frac{1}{r_{B2}}\right)+\frac{1}{r_{12}}+\frac{1}{2R}, \qquad (2\cdot3)$$

$$E(\eta)=(\Psi,\ H\Psi)/(\Psi,\ \Psi). \qquad (2\cdot4)$$

In the following calculations the strict minimization of $E(\eta)$ with respect to η is not attempted and, instead, six discrete values of η, 0.25, 0.5, 0.75, 1.0, 1.25 and 1.5, are examined. Actual calculations are performed only for two-quantum excited states: $1s\sigma 2s\sigma$, ${}^3\Sigma_u^+$, $1s\sigma 2p\sigma$ ${}^3\Sigma_u^+$, ${}^1\Sigma_u^+$ and $1s\sigma 2p\pi$ ${}^3\Pi_u$, ${}^1\Pi_u$, but the advantage of the present method over others will be more prominent in the treatment of higher levels.

Strictly speaking, the notations such as $1s\sigma 2s\sigma$, etc., are not adequate in the present scheme but we shall use them for convenience.

(a) $1s\sigma 2s\sigma \quad {}^3\Sigma_g^+, \ ({}^1\Sigma_g^+)$.

For the first ${}^3\Sigma_g^+$ state, we assume for ϕ_{out} the following form,

$$\phi_{out} = (2s\sigma\,;\ \eta) = N(2,\ \eta) r e^{-\eta r}\ Y_{00}. \qquad (2\cdot 5)$$

The calculation is performed at a fixed nuclear distance $2R=2.0$ a.u. Because the ${}^3\Sigma_g^+$ function is automatically orthogonal to the ground ${}^1\Sigma_g^+$ function we can freely perform the variational calculation. But if we use the plus sign form of $\Psi(1, 2)$ in (2, 2) we merely obtain a crude approximation for the ground ${}^1\Sigma_g^+$. For the ground ${}^1\Sigma_g^+$ a separate discussion is found in sub-section (d). The results are shown in Table I.

Table I

state	η	E(a.u.)
${}^3\Sigma_g^+$	0.25	−0.679027
	0.5	−0.699134
	0.75	−0.629125
${}^1\Sigma_g^+$	1.0	−1.011309
	1.25	−1.029406
	1.5	−1.017308

Table II

state	η	E(a.u.)
${}^3\Sigma_u^+$	0.75	−0.804265
	1.0	−0.821220
	1.25	−0.769867
${}^1\Sigma_u^+$	0.25	−0.669988
	0.5	−0.705872
	0.75	−0.687671

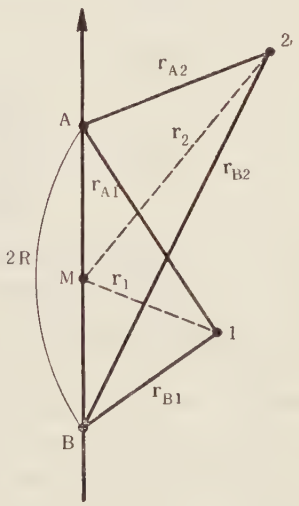

Fig. 2.

The electronic energy −1.029406 a.u. ($\eta=1.25$) for ${}^1\Sigma_g^+$ means the electronic dissociation energy $D_e=0.8$ e.V. at $2R=2$ a.u.

(b) $1s\sigma 2p\sigma \quad {}^3\Sigma_u^+, \ {}^1\Sigma_u^+$.

There is no particular reason to calculate the energies of these two states only at a fixed nuclear distance 2.0 a.u. The experimentally known equilibrium nuclear distance of the lowest ${}^1\Sigma_u^+$ state is not 2 a.u. but about 2.5 a.u. and the lowest ${}^3\Sigma_u^+$ state is unstable. Let us, however, calculate them and see the outcome. Our assumption is

$$\phi_{out} = (2p\sigma\,;\ \eta) = N(2,\ \eta) r e^{-\eta r}\ Y_{10} \qquad (2\cdot 6)$$

and the results are listed in Table II. Contrary to the case of Σ_g, the triplet state has

lower energy than that of the singlet state. This corresponds just to the situation in the helium atom where $(1s)^2$ ^1S is the ground state and 1s2s ^3S lies above, while 1s2p ^3P is lower than 1s2p ^1P.

(c) $1s\sigma 2p\pi$ $^3\Pi_u$, $^1\Pi_u$.

For these Π states there is a two-fold degeneracy if we neglect rotational motions of the molecule. The outer electron may be described by

$$\phi_{out} = (2p\pi;\ \eta) = N(2,\eta) r e^{-\eta r} (1/\sqrt{2}) [Y_{1,-1}(\theta,\varphi) + (-1) Y_{1,+1}(\theta,\varphi)], \quad (2\cdot 7)$$

$$Y_{1,\pm 1}(\theta,\varphi) = \mp \sqrt{3/4}\ \sin\theta \cdot e^{\pm i\varphi}/\sqrt{\pi}.$$

The results of calculation are displayed in Table III.

Table III

state	η	E(a.u.)
$^3\Pi_u$	0.25	−0.668623
	0.5	−0.706030
	0.75	−0.686257
$^1\Pi_u$	0.25	−0.665847
	0.5	−0.678971
	0.75	−0.662441

Again the triplet state lies lower than the singlet state, in agreement with the experiment.

(d) $1s\sigma 1s\sigma$ $^1\Sigma_g^+$.

This is the ground state of H_2. In the above we have discussed some excited states in which the outer electron has a wave function spread out into the outer space and then we have expected that a single atomic wave function offers a good approximation. We cannot, however, expect this for the ground state because here two electrons must be moving rather closely around two nuclear attractive centers. Nevertheless it will be interesting to see what happens if we use the same type of approximation also for calculating the ground state energy. Thus, our wave function is

$$\Psi(1,2) = (1/\sqrt{2})[\phi_{in}(1)\phi_{out}(2) + \phi_{out}(1)\phi_{in}(2)],$$

$$\phi_{in} = c[(1s\sigma;\ \zeta) + \kappa(4s\sigma;\ \zeta') + \lambda(4d\sigma;\ \zeta')], \quad (2\cdot 8)$$

$$\phi_{out} = (1s\sigma;\ \eta) = N(1,\eta) e^{-\eta r}\ Y_{00},$$

where ζ, ζ', η, c, κ and λ are variable parameters. The calculation is performed at a fixed nuclear distance $2R = 1.4$ a.u. which is approximately the observed equilibrium distance. ϕ_{in} is almost the same as the wave function for H_2^+ at the nuclear distance 1.4 a.u.[10] With parameter values $\zeta = 1.1$, $\zeta' = 4.3$, $\eta = 0.8$, $c = 0.489949475$, $\kappa = 0.524208$ and $\lambda = 0.273048$, the electronic energy is $E = -1.107194$ a.u. ($2R = 1.4$ a.u.) which means the electronic dissociation energy $D_e = 2.92$ e.V. This result is compared rather favourably with the values $D_e = 3.14$ e.V. ($2R = 1.645$ a.u.) of the original Heitler-London theory and $D_e = 2.68$ e.V. ($2R = 1.61$ a.u.) of the naive molecular oribital method, if we consider that the present method requires only little labor for calculation. The values of ζ and ζ' are the same as in the previous calculation on H_2^+ at the same nuclear distance and c, κ and λ are determined through the Ritz procedure. η was fixed through

several trial calculations. Now let us try to go one step further and introduce a wave function which is able to include the so-called left-right correlation of two electrons. The whole wave function is

$$\Psi(1,\ 2)=c_1(1/\sqrt{2})[\phi_{in}(1)\phi_{out}(2)+\phi_{out}(1)\phi_{in}(2)]+c_2(2p\sigma;\ \bar{\xi})\ (2p\sigma;\ \bar{\xi}),\quad (2\cdot9)$$

where the first term is just the same as given in (2·8) and the second term is newly added one. By varying the parameter $\bar{\xi}$ with an interval of 0.1, we find that the best energy value is obtained at $\bar{\xi}=1.6$ (see Table IV). Then we determine two coefficients c_1 and c_2 by Ritz procedure to be $+0.4877954$ and -0.09366858 respectively.

Table IV

$\bar{\xi}$	E(a.u.)	D_e(a.u.)	D_e(e.V.)
1.5	−1.1202602	0.1202602	3.2723
1.6	−1.1205660	0.1205660	3.2806
1.7	−1.1203778	0.1203778	3.2755

D_e : electronic dissociation energy.

In I the present author stated that by the method of one-center expansion of molecular wave function he could obtain the electronic dissociation energy of 3.8 e.V. Later it was found that some mistakes were made in the numerical calculation and the wave function given there could provide only a very poor value, $D_e=2.8$ e.V. at best. In this respect we should like to express our gratitude to Dr. Joy for his kind information about this point.

§ 3. Discussions

Table V

$\Delta_1=|E_{cal}-E_{cal}(X)|$, $\Delta_2=|E_{cal}-E_{obs}(X)|$, $\Delta_3=|E_{obs}-E_{obs}(X)|$

state	E_{cal}(a.u.)	Δ_1(cm^{-1})	Δ_2(cm^{-1})	Δ_3(cm^{-1})
$C\ {}^1\Pi_u$	−0.678971	96919	108695	100058
$a\ {}^3\Sigma_g^+$	−0.699134	92494	104270	95938
$c\ {}^3\Pi_u$	−0.706030	90980	102757	95744
$B\ {}^1\Sigma_u^+$	−0.705872	91015	102791	(91690)
$b\ {}^3\Sigma_u^+$	−0.821220	65699	77475	(66500)
$X\ {}^1\Sigma_g^+$	−1.1205660	0	11776	0

The main subject of this section is to discuss possible implications of the present method for calculations of excited states of molecules and the related problems. In Table V we display the experimental matters and the results of the present calculations. The observed data are taken from the appendix of the text-book of Herzberg[12]. Δ indicates energy differences between the higher energy states and the ground state in cm^{-1}. In

observing the figures in Table V we cannot say that the results of calculations are sufficiently satisfiable. However, our labor spent on calculation here is very small compared with other elaborate calculations and there is a good chance to improve the present results if we are not forced to confine ourselves within poor facilities of numerical calculations. The lowest ${}^3\Sigma_u^+$ is unstable and the value 66500 cm^{-1} in J_2 is obtained from a rough estimation from Fig. 1 of Hirschfelder and Linnett[13]. The calculated position of ${}^1\Sigma_u^+$ is too high but it might reflect the fact that the electronic energy curve of this state has its minimum point at the nuclear distance 2.5 a.u., not at 2.0 a.u., the adopted value in the present calculation. It is to be noted that the approximations for wave functions described in the previous section are in very preliminary stages and we can improve them in many respects. In the present calculations we are concerned with the states automatically orthogonalized to the ground ${}^1\Sigma_g^+$ state, which permits us to perform variational calculations very simply. But this does not mean that the present scheme of calculation is not suitable for the discussions of higher states, being not orthogonal to the ground state. If we use approximate wave functions in the form of linear combinations of several configurations for the outer electron and solve the secular equation for the coefficients, we can obtain a better result for the lowest state together with the ones for accompanied higher states of the same symmetry, orthogonalities between them being secured automatically. In the present method wave functions of all states are expressed in some series of one-center functions and this makes the whole calculations very easy. The method will be suitable also for calculations of various physical quantities other than electronic term values. In the near future we shall treat the problems of coupling between electronic and rotational motions and the fine structure of the spectra of H_2 molecule.

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One-Center Expansion of Molecular Wave Function, III

—— He-He Repulsive Potential ——

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The repulsive potential between two normal He atoms is calculated by the method of one-center expansion of molecular wave function. The calculation is performed at smaller internuclear distances where the discrepancies between the experiment and the existing theoretical estimations are very large. At an internuclear distance of 1.0 a.u. the present calculation gives a better result than any of the calculations hitherto performed. The implication of the result is discussed.

§ 1. Introduction

The problem of theoretical calculation of the repulsive potential between two normal He atoms has received considerable attention from many physicists. This is because the He-He repulsive interaction presents a simplest and typical case of the interaction between two closed shell systems. Slater[1] treated this problem by the valence bond method using a rather elaborate wave function but he neglected higher order permutations in the reduction of the energy integrals. Rosen[2] adopted a simpler atomic orbital and he also calculated the interaction by the valence bond method. He took into account all permutations. In the infinite separation Rosen's wave function for a He atom is $\exp(-Z_1r_1)\exp(-Z_2r_2)+\exp(-Z_2r_1)\exp(-Z_1r_2)$ with $Z_1=2.15$ and $Z_2=1.19$ (the normalization constant omitted). This function yields -2.8754 a.u. (atomic units) as the ground state energy of a He atom. Griffing and Wehner[3] (G-W) started with a still simpler atomic orbital,

$$\chi_a=(Z^3/\pi)^{1/2}\exp(-Zr_a),\quad Z=27/16, \tag{1}$$

and constructed two molecular orbitals,

$$\begin{aligned}\varphi_1&=N_1(\chi_a+\chi_b)\ ;\quad \sigma_g\,,\\ \varphi_2&=N_2(\chi_a-\chi_b)\ ;\quad \sigma_u\,,\end{aligned} \tag{2}$$

where N_1 and N_2 are normalization constants. Using these molecular orbitals they calculated the He-He interaction energy by the antisymmetrized molecular orbital method. The corresponding atomic wave function for an isolate He atom is $\exp[-Z(r_1+r_2)]$ with $Z=27/16$ and this gives -2.84766 a.u. as the ground state energy, a little bit inferior to that of Rosen. It is easily seen that, so long as we use an atomic orbital of the form (1), the antisymmetrized molecular orbital method is just equivalent to the simple

Heitler-London theory. Comparing the results of the above two calculations with each other, we notice a rather curious fact : Rosen started with a wave function for an isolate He atom better than the one of G-W and the result is that his energy curve lies considerably above the curve obtained by the simpler calculation of G-W. This is shown in columns (1) and (2) of Table I.

Recently Sakamoto and Ishiguro[1] (S-I) have performed an interesting calculation on the present subject. All of previous calculations before S-I did not explicitly take into account the deformation of atomic orbitals due to the approach of another atom. Sakamoto and Ishiguro have attempted this by making use of the modified atomic orbital of Inui[5] and Nordsieck[6], which was successful in the treatment of the ground state of H_2 molecule. The repulsive interaction energy of the He-He system obtained by them is lower than any of previous calculations at all values of the internuclear distance but it must be mentioned that the difference between the results of the simple calculation of G-W and of the elaborate calculation of S-I is rather small, namely within 1 e.V. over nearly the whole range of internuclear distance in discussion. Further, if we compare these theoretical results with the values experimentally obtained by Amdur and Harkness,[7] we find a very large discrepancy which amounts to about 30 e.V. at the internuclear distance of 0.5 Å (see Table I of the present paper and Table V and Fig. 1 of S-I). If the experimental result of Amdur and Harkness is reliable, the situation must be said to be very challenging. In this connection the present author has attempted a united atom treatment of the problem and the result obtained here seems to indicate the breaking down of the simple Heitler-London approach at small internuclear distance.

§ 2. A united atom treatment of the He-He interaction

The notion of the united atom is very useful for the qualitative understanding of electronic structures of molecules but it has generally been believed that the united atom treatment of molecules offers only poor approximations for quantitative calculations. Matsen[8] was the first who showed by actual calculations the usefulness of the method for the calculations of excited states of H_2^+ molecular ion. The present author has extended his method and applied it to the discussions of the ground state of H_2^+ and the ground and excited states of H_2.[9] Tien Chi Chen[10] is making calculations on polyelectronic molecules along the same line. Carter[11] has applied successfully the method of one-center expansion of molecular wave function to methane and silane.

One might say, however, that a united atom treatment of the He-He interaction would be nonsensical because two electrons on each atom form the so-called closed shell and the electron cloud acts as something like small rigid sphere. The following simple calculation will tell us whether it is true or not.

As the first step, we calculate the He-He repulsive interaction energy at smaller internuclear distances by using, so to speak, a single term apporoximation, the meaning of which will be self-explanatory in the following description. In the united atom point of view, two electrons of the He-He system occupy ($1s\sigma$) orbital and other two occupy

$(2p\sigma)$ orbital (see Fig. 1). If we denote

$$\varphi_1 = (1s\sigma\,;\,\zeta) = N(1,\zeta)\exp(-\zeta r)\,Y_{00}\,; \quad \sigma_g,$$
$$\varphi_2 = (2p\sigma\,;\,\eta) = N(2,\eta)\,r\exp(-\eta r)\,Y_{10}\,; \quad \sigma_u, \tag{3}$$

where $N(n,\zeta) = (2\zeta)^{n+(1/2)}/[(2n)!]^{1/2}$, then these just correspond to φ_1 and φ_2 of G-W in (2). The wave function of the He-He system is

$$\psi = (1/24)^{1/2}\sum_{\lambda}(-1)^{\lambda}P_{\lambda}[(\varphi_1\alpha)^1(\varphi_1\beta)^2(\varphi_2\alpha)^3(\varphi_2\beta)^4], \tag{4}$$

where P_λ is the operation permuting the superscripts which denote electrons, and α and β are the usual orthonormal spin eigenfunctions. The operator for the present problem is

$$H = \sum_{\mu=1}^{4} H^\mu + \sum_{\mu>\nu=1}^{4}(1/r^{\mu\nu}) + (4/2R),$$

where

$$H^\mu = -(1/2)\Delta_\mu - (2/r_{a\mu}) - (2/r_{b\mu}).$$

According to Roothaan[12], the total energy for the system is

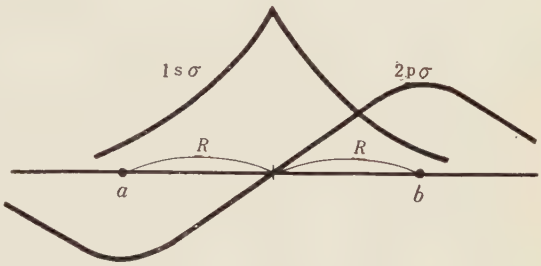

Fig. 1.

$$E = \int \psi H \psi d\tau = 2H_1 + 2H_2 + J_{11} + 4J_{12} + J_{22} - 2K_{12} + (4/2R), \tag{5}$$

where

$$H_i = \int \varphi_i^\mu H^\mu \varphi_i^\mu d\tau^\mu,$$

$$J_{ij} = \int \varphi_i^\mu \varphi_j^\nu (1/r^{\mu\nu}) \varphi_i^\mu \varphi_j^\nu d\tau^{\mu\nu},$$

$$K_{ij} = \int \varphi_i^\mu \varphi_j^\nu (1/r^{\mu\nu}) \varphi_j^\mu \varphi_i^\nu d\tau^{\mu\nu}.$$

The internuclear distance $2R$ and the energies are measured in Hartree's atomic units (a.u.). We are going to compare our results mainly with the ones of G-W and S-I in which $E_{R=\infty}$ is assumed to be -5.69531 a.u. Then it is natural to define the interaction potential energy in the present calculation as the calculated total energy E minus the approximate value of energy of two He atoms, -5.69531 a.u.

The use of φ_1 and φ_2 of (3) makes calculations very simple and straightforward, and the analytical expressions of formulas are

$$H_1 = (\zeta^2/2) - (2/R)[2 - (2+2R\zeta)\exp(-2R\zeta)],$$
$$H_2 = (\eta^2/2) - (2/R^3\eta^2)[(6+2R^2\eta^2)$$
$$- (6+12R\eta+14R^2\eta^2+11R^3\eta^3+6R^4\eta^4+2R^5\eta^5)\exp(-2R\eta)],$$

$$J_{11}=5\zeta/8,$$
$$J_{12}=(\eta/2)[1-\{\eta^4(3\zeta+\eta)/(\zeta+\eta)^5\}], \tag{6}$$
$$J_{22}=501\eta/1280,$$
$$K_{12}=28\zeta^3\eta^5/3(\zeta+\eta)^7.$$

If we assume $\zeta=\eta$ for simplicity, we have

$$E(R,\zeta)=2\zeta^2-(4/R)[2-(2+2R\zeta)\exp(-2R\zeta)]$$
$$-(4/R^3\zeta^2)[(16+2R^2\zeta^2)$$
$$-(6+12R\zeta+14R^2\zeta^2+11R^3\zeta^3+6R^4\zeta^4+2R^5\zeta^5)\exp(-2R\zeta)] \tag{7}$$
$$+2.620572917\zeta+(4/2R),$$

and $\partial E(r,\zeta)/\partial\zeta=0$ yields the formula

$$\zeta=(1/R^3\zeta^3)[(12+24R\zeta+24R^2\zeta^2+19R^3\zeta^3+14R^4\zeta^4+6R^5\zeta^5+4R^6\zeta^6)\times$$
$$\times\exp(-2R\zeta)-12]-0.6551432292. \tag{8}$$

Solving this equation numerically for a fixed value of R, we obtain the best value of variational parameter ζ for the value of internuclear distance. The calculations are performed at three fixed values of internuclear distance, $2R=0.6$, 0.8 and 1.0 a.u. It seems that the simple assumption $\zeta=\eta$ is good enough for the present purpose at $2R=0.8$ and 1.0 a.u. The results of calculations are listed in column (4) of Table I in which the set of parameter values $\zeta=2.0$ and $\eta=1.85$ at $2R=0.6$ a.u. are not exactly the best but are probably nearest to the best. In Table I we also display other data for comparison. The values of the column (3) is calculated by the formula,

$$V(R)=15.02222\exp[-2.36478(2R)]\text{ a.u.}\quad(R\text{ in a.u.}), \tag{9}$$

which is given in (11) of S-I's paper. In the present paper the internuclear distance is $2R$. It is remarkable that, already at $2R=0.8$ a.u., the simple and very crude united atom treatment gives a result better than that of the elaborate valence bond theory of S-I. Even at $2R=1.0$ a.u. our result is definitely better than that of Rosen. This is rather an unexpected situation and it may not be unreasonable to infer that at the internuclear distance of about 1 a.u. the Heitler-London method becomes quite inadequate. In order to confirm this view, let us proceed to the next stage of the united atom approximation. The refinement consists in taking

$$\varphi_1=c_1(1s\sigma\,;\zeta)+c_2(4s\sigma\,;\zeta')+c_3(4d\sigma\,;\zeta'), \tag{10}$$

where

$$(4s\sigma\,;\zeta')=N(4,\zeta')r^3\exp(-\zeta' r)\,Y_{00},$$

$$(4d\sigma\,;\zeta')=N(4,\zeta')r^3\exp(-\zeta' r)\,Y_{20}.$$

The first approximation described above is the case of $c_2=c_3=0$, which corresponds to Matsen's original treatment of ground state of H_2^+. The present author attempted to extend Matsen's method by taking three terms instead of one, just like in (10)[9]. Now we repeat the same for, say, He_2^{+++} problem. The calculation is performed at a fixed internuclear distance, 1.0 a.u. The values of variable parameters ζ and ζ' can be loosely assigned through guess-work and are fixed to be 1.75 and 6.25 respectively in order to use the numerical table of $A_n(\alpha)$ in KAS's table[13]. The ground state of He_2^{+++} is given by H_1 in (5). The single term $(1s\sigma\,;1.75)$ gives $H_1=-3.8621409$ a.u., while three terms approximation of (10) yields $H_1=-4.2659549$ a.u., a significant improvement which amounts to 0.404 a.u. or 11 e.V. The values of coefficients c_1, c_2 and c_3 are determined to be 0.68241138, 0.32284165 and 0.16923435 respectively by the Ritz method. Thus we obtain an approximate wave function of one electron in He_2^{+++} and we are going to use this function as the molecular orbital of two inner core electrons of the He-He system, the outer orbital being the same as in (3) ;

$$\varphi_1=0.68241138\,(1s\sigma\,;1.75)+0.32284165\,(4s\sigma\,;6.25)+0.16923435\,(4d\sigma\,;6.25),\tag{11}$$

$$\varphi_2=(2p\sigma\,;\eta).$$

Again we assign a fixed value 1.5 for η by guess-work. The calculation is simple and goes on just the same way as in the first single term approximation. Since the whole wave function is expressed about a single coordinate center, the energy integrals of the electron-electron interaction can be performed very easily. Useful formulas are given in Appendix. The result is $E=-4.42012$ a.u. This means that the He-He interaction energy at $2R=1.0$ a.u. is 34.698 e.V. which is 3.7 e.V. lower than that of S-I, 38.414 e.V. It is clear that the wave function used above is a very crude one-center expansion of the real wave function of the He-He system. The fact that the elaborate wave function of S-I and the present one-center wave function gives about the same result at the internuclear distance 1.0 a.u. tells us that both are equally far apart from the real wave function and further efforts are required for theoretical side in order to diminish the existing large discrepancy between theory and experiment.

§ 3. Discussions

Applying the method of one-center expansion of molecular wave function to the He-He system, we have obtained some interesting results which indicate that the present theoretical investigations of the interactions between two closed shell systems remain in a rather unsatisfactory stage. We have to make efforts to improve the current approximations. A further refinement of one-center expansion may be one way but it does not seem so suitable for covering the physically most important range of the internuclear distances. Here the current two-center description will be preferable. Before concluding

this article, let us try to look for possible improvements of calculations on the He-He repulsive potential.

First we examine the approximation used by Griffing and Wehner[3]. They used the molecular orbitals given in (2). The essential point is that the effective charge Z is the same for both $\chi_a+\chi_b$ and $\chi_a-\chi_b$ orbitals. In this connection it is instructive for us to remind that in the H_2^+ ion problem there exists a remarkable difference between the manners of variations of effective charges for $\chi_a+\chi_b(\sigma 1s)$ and $\chi_a-\chi_b(\sigma^* 1s)$: for shorter internuclear distances the value for $\sigma 1s$ rises up steeply and the one for $\sigma^* 1s$ falls down rapidly. This is shown, for example, in Fig. 4.6 on p. 85 of Coulson's text book.[14] In addition, for the present He-He system we must take account of the exclusion effect of the Pauli principle and this reinforces the requirement of separate variation of effective charges for $\chi_a+\chi_b$ and $\chi_a-\chi_b$. If we assume a common effective charge Z for both of them, the necessary deformation of the electron cloud is suppressed entirely. It is reported in G-W's paper that the variation of Z from the value 27/16 produces no significant improvement. This is probably due to the cancellation of the effects of opposite directions on σ_g and σ_u orbitals. On the other hand, the unexpected success of the method of one-center expansion may be due to its flexibility of separate variation of the effective charges of both orbitals. In the calculation of Sakamoto and Ishiguro[4] there are two variable parameters for the deformation of the electron cloud but unfortunately they cannot produce the deformation required in the above discussion. As for the work of Rosen[2] more careful examination is necessary but a conjecture is that, if we proceed with the molecular orbital method, instead of his valence bond method, in constructing σ_g orbital from two inner atomic orbitals and σ_u from two outer atomic orbitals, we shall get a better result. A calculation based on the molecular orbital method using different effective charges for $\chi_a+\chi_b(\sigma_g)$ and $\chi_a-\chi_b(\sigma_u)$ orbitals is now in progress.

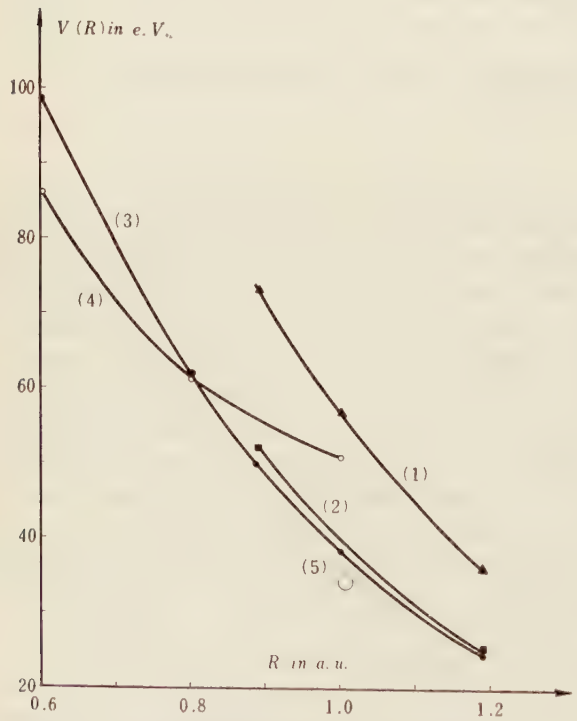

Fig. 2 He-He repulsive potential in small separation

Table I. Interatomic potential for He_2 (in e.V.)

$2R$(a.u.)	(1)	(2)	(3)	(4)	(5)	(6)
0.6			98.874	86.110*		
0.8			61.618	61.457**		
0.889	72.955	51.843	49.944			10.959
1.0	56.235		38.414	50.942***	34.698	9.011
1.186	36.530	24.862	24.783			6.647

(1) Rosen
(2) Griffing and Wehner } (according to Table V of S-I)
(3) Sakamoto and Ishiguro
(4) Single term approximation of the present paper
* $\zeta=2.0$, $\eta=1.85$ ** $\zeta=\eta=1.71$ *** $\zeta=\eta=1.57$
(5) Three term approximation of the present paper
$\zeta=1.75$, $\zeta'=6.25$, $\eta=1.5$
(6) Experimental result of Amdur and Harkness (according to Table V of S-I)

Appendix

In the method of one-center expansion of molecular wave function the molecular integrals offer no troubles. All of the integrals are familiar to us in atomic problems and it seems to be almost unnecessary to describe the details of the calculations. The following formulas are presented for those who may find them useful. The most tedious are, of course, the integrals of electron-electron interaction energy. The integrations over angular variables are straightforward, leaving the radial part integrals which reduce to the following type of integration;

$$I_p(M,\alpha|N,\beta)=\int_0^\infty\int_0^\infty (r_<^p/r_>^{p+1})r_1^M\exp(-\alpha r_1)r_2^N\exp(-\beta r_2)r_1^2dr_1r_2^2dr_2,$$

where $r_<$ means the smaller one of r_1 and r_2 and $r_>$ the larger. Using the following relation between binomial coefficients,

$$\sum_{k=0}^{m-l}\binom{n+k}{n}\binom{m-k}{l}=\binom{m+n+1}{l+n+1},$$

we can easily arrive at the next useful formula:

$$I_p(M,\alpha|N,\beta)=[(M-p+1)!(N+p+2)!/\alpha^{M-p+2}(\alpha+\beta)^{M+N+4}]\cdot\sum_{l=0}^{M-p+1}\binom{M+N+4}{l+N+p+3}\alpha^{M-p+1-l}\beta^l$$

$$+[(N-p+1)!(M+p+2)!/\beta^{N-p+2}(\alpha+\beta)^{M+N+4}]\cdot\sum_{l=0}^{N-p+1}\binom{M+N+4}{l+M+p+3}\beta^{N-p+1-l}\alpha^l.$$

Especially in the case of $M=N$ and $\alpha=\beta$,

$$I(M,\alpha|M,\alpha)=\lfloor(M-p+1)!(M+p+2)!/2^{2M+3}\alpha^{2M+5}\rfloor\sum_{l=0}^{M-p+1}\binom{2M+4}{l+M+p+3}.$$

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Note added in proof. The calculation mentioned in the last sentence of §3 has been completed. The result is summarized in the following approximate formula:

$$V(2R)=8.58817\ \exp[-2.16097(2R)]\ \text{a.u.}\quad (2R\ \text{in a.u.}),\ 1\lesssim 2R\lesssim 2.$$

At $2R=1.0$ this gives 26.92 e.V. as the repulsive He-He interaction energy, which is about 12 e.V. lower than the corresponding value of S-I.

Erratum: R in Fig. 2 should be read as $2R$.

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Theory of the Magnetic Properties of Ferrous and Cobaltous Oxides, I

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The magnetic properties of antiferromagnetics, FeO and CoO, are investigated from the standpoint of the one-ion approximation. In their crystalline field of cubic symmetry the orbital degeneracies of Fe^{++} and Co^{++} are not completely removed and the residual orbital angular momenta play an important role through the presence of the spin-orbit coupling and through the direct effect of the orbital magnetic moments. After deriving the effective Hamiltonian for these degenerate cases, which corresponds to the Hamiltonian derived by Pryce for the non-degenerate case, we discuss the paramagnetic susceptibility, the Néel temperature and the state of each ion at absolute zero. We can give a reasonable quantitative interpretation of the susceptibility of CoO in paramagnetic state and obtain a good agreement with experiment for the magnetic moment of CoO in antiferromagnetic state. For FeO, sufficiently reliable data to compare with theory are not available at present, but a preliminary comparison is made. Further, in connection with the investigation of the validity of our theory, the origins of the crystalline field are discussed and it is pointed out that the covalent effect may make an appreciable contribution to the crystalline field.

§ 1. Introduction

Iron-group monoxides are known to be antiferromagnetic substances[1]. Their powder susceptibilities[2] show typical behaviours of antiferromagnetism and their spin arrangements have been determined by Shull, Strauser and Wollan[3] by neutron diffraction. They all have the structure of NaCl type except for a slight deformation[4] which takes place below the Néel temperature. The purpose of this paper and the succeeding one is to make a theoretical investigation of ferrous and cobaltous oxides from the standpoint of the one-ion approximation which seems to be appropriate when applied to these ionic crystals.

Each metallic ion of these oxides can be regarded as being placed in a strong electric field arising from other ions. Above the Néel temperature this crystalline field has cubic symmetry. Below the Néel temperature there is a crystalline field of a lower symmetry produced by a deformation; however, as will be discussed fully in the next paper, this deformation is caused mainly by magnetostriction and has a smaller effect on the energy levels than the spin-orbit coupling energy. A particular situation with ferrous and cobaltous ions as compared with manganous and nickelous ions is that their orbital angular momenta are not completely quenched by the Stark effect due to the cubic field. As shown in Fig. 1, when a cubic field is present, the ground orbital states of the free ferrous and cobaltous ions are split into two and three sublevels[5], respectively, of which the lowest ones, Γ_5' and Γ_4', respectively, are triply degenerate. Therefore, the residual orbital angular momentum plays an important role for their magnetic properties through the spin-orbit

coupling and directly through the orbital magnetic moment.

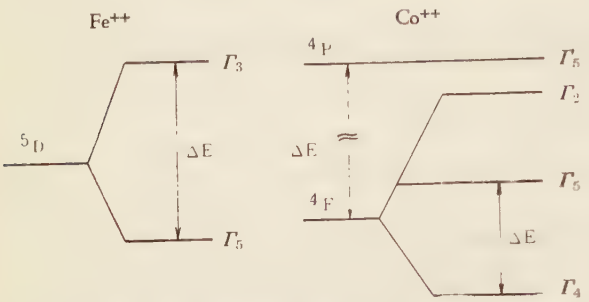

Fig. 1. Orbital level schemes of Fe^{++} and Co^{++} in the cubic electric field.

Néel[6] and Anderson[7] discussed the antiferromagnetic ordering of spins in these substances. Extending Anderson's calculation, Smart[8] has however, found a difficulty in interpreting the paramagnetic susceptibility of CoO, as will be referred to in § 4. These treatments do not take into account the presence of the residual orbital angular momentum.

Measurement[3] of the absolute intensity of the neutron diffraction in antiferromagnetic state shows that, in CoO, there is a significant contribution arising from the orbital angular momentum. However, the data could not be interpreted quantitatively[3] by the simple classical picture that the spin and the residual orbital angular momentum of each cation are parallel with each other. This difficulty can be removed by taking account of the spin-orbit coupling energy quantum-mechanically, as will be shown in § 6.

Recently, Li[9] has analysed the neutron diffraction data of iron group monoxides and discussed the relation between the magnetic anisotropy and the deformation below the Néel temperature, but his discussions indicate nothing of the origin of the magnetic anisotropy energy and of the mechanism of the deformation below the Néel temperature.

We shall give in this and the next papers a reasonable interpretation of the susceptibility, magnitude of the magnetic moment of each cation in antiferromagnetic state, magnetic anisotropy energy and deformation of these two oxides by considering the presence of the residual orbital angular momentum. In § 2 of this paper we discuss the form of the effective Hamiltonian of the spin-orbit coupling energy and the Zeeman energy for the degenerate orbital levels which corresponds to the Hamiltonian derived by Pryce[10] for the nondegenerate case. The paramagnetic susceptibility is discussed in § 3 and § 4 ; the effect of the orbital angular momentum on the Néel temperature in § 5 ; the state of each metallic ion at the absolute zero of temperature in § 6. Finally, in § 7, we discuss the origins of the crystalline field and thereby investigate the validity of our theory. In the next paper we shall discuss the magnetic anisotropy energy and deformation below the Néel temperature.

§ 2. Effective Hamiltonian for the lowest orbital state

Pryce[10] derived the effective Hamiltonian for the non-degenerate ground orbital state, assuming that the energy separations between the ground orbital state and upper ones are so large that one can treat the spin-orbit coupling energy as perturbation. We shall here derive the corresponding effective Hamiltonian for the degenerate orbital state

under a similar assumption that the energy separations caused by the cubic field are so large that other terms of the Hamiltonian can be treated as perturbations.

The Hamiltonian of each metallic ion consists of the following terms:

$$\mathcal{H} = (\text{potential energy due to the crystalline field}) + (\text{exchange energy}) + \lambda \boldsymbol{L}\cdot\boldsymbol{S} + \mu_B \boldsymbol{H}\cdot(2\boldsymbol{S}+\boldsymbol{L}). \tag{1}$$

Here we assume that the so-called crystalline field can be represented by the electrostatic potential which depends on one-electron coordinates. This assumption will be discussed in a later section (§ 7) in connection with the theory of the origin of the crystalline field. Further, we assume cubic symmetry for the potential energy. Leaving the discussion of the exchange energy to later sections, we calculate here the effective Hamiltonian of the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling and Zeeman energy for the degenerate ground state by using the perturbation theory.

In (1), we have neglected the internal spin-spin coupling because of its smallness and also the orbit-orbit interactions between the cations (which are not necessarily small), because they make no contribution in paramagnetic state where the orbital momenta of the cations have random orientations. The detailed discussion of these interactions will be given in the next paper in relation to the calculation of the anisotropy energy in antiferromagnetic state. We shall treat the ferrous ion and the cobaltous ion separately.

The ground state of a free ferrous ion is ${}^5D$. We denote the orbital magnetic quantum number by M and the corresponding orbital function by φ_M. The three orbital wave functions which are degenerate in the cubic field are given by

$$\psi_1 = -(1/\sqrt{2})(\varphi_1-\varphi_{-1}),\quad \psi_2 = (1/\sqrt{2}i)(\varphi_2-\varphi_{-2}) \quad\text{and}\quad \psi_3 = (i/\sqrt{2})(\varphi_1+\varphi_{-1}). \tag{2}$$

These functions give the representation l'_5 of the octahedral group. The other two wave functions, which belong to l'_3, are given by

$$\psi_4 = (1/\sqrt{2})(\varphi_2+\varphi_{-2}) \quad\text{and}\quad \psi_5 = \varphi_0. \tag{3}$$

The matrices of the orbital angular momentum with respect to these ψ_i are

$$L_z = \begin{bmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 2i & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad\text{and}\quad L_x+iL_y = L^+ = \begin{bmatrix} 0 & i & 0 & 1 & -\sqrt{3} \\ -i & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -i & -\sqrt{3}i \\ -1 & 0 & i & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{3}i & 0 & 0 \end{bmatrix}. \tag{4}$$

The submatrices of $\boldsymbol{L}$ in space of l'_5 are therefore connected with the matrices of the angular momentum operator $\boldsymbol{l}$ of magnitude 1 by the relation[11]:

$$\boldsymbol{L} = -\boldsymbol{l}. \tag{5}$$

Using (5), the first order terms of the spin-orbit coupling energy and Zeeman energy

for the ground orbital states are written as

$$\lambda' \boldsymbol{l}\cdot\boldsymbol{S}(\lambda'=-\lambda) \quad \text{and} \quad \mu_B \boldsymbol{H}(2\boldsymbol{S}-\boldsymbol{l}). \tag{6), (7}$$

The second order terms are produced by those matrix elements of $\boldsymbol{L}$ which connect l'_5 and l'_3. Using the perturbation theory, the second order term of $\lambda \boldsymbol{L}\cdot\boldsymbol{S}$ can be expressed by the following matrix of the l'_5 space:

$$\mathcal{H}_1 = 2(\lambda^2/\Delta E)\begin{vmatrix} -2S_\eta^2 & S_\eta S_\zeta & S_\eta S_\xi \\ S_\zeta S_\eta & -2S_\zeta^2 & S_\zeta S_\xi \\ S_\xi S_\eta & S_\xi S_\zeta & -2S_\xi^2 \end{vmatrix},$$

where ΔE is the energy separation between l'_5 and l'_3. In operator form this matrix can be written as

$$\mathcal{H}_1 = -(4\lambda^2/\Delta E)\cdot S(S+1) + (2\lambda^2/\Delta E) \times \{3(l_x^2 S_x^2 + l_y^2 S_y^2 + l_z^2 S_z^2) - (\boldsymbol{l}\cdot\boldsymbol{S})^2 - (\boldsymbol{l}\cdot\boldsymbol{S})\}. \tag{8}$$

Similarly, the cross term of $\lambda \boldsymbol{L}\cdot\boldsymbol{S}$ and $\mu_B \boldsymbol{H}\cdot\boldsymbol{L}$ gives rise to

$$\mathcal{H}_2 = (8|\lambda|/\Delta E)\cdot\mu_B(\boldsymbol{H}\cdot\boldsymbol{S}) - (2|\lambda|/\Delta E)\mu_B \times [6(H_x l_x^2 S_x + H_y l_y^2 S_y + H_z l_z^2 S_z) - \{(\boldsymbol{H}\cdot\boldsymbol{l})(\boldsymbol{l}\cdot\boldsymbol{S}) + (\boldsymbol{l}\cdot\boldsymbol{S})(\boldsymbol{H}\cdot\boldsymbol{l})\}]. \tag{9}$$

$\mathcal{H}_2$ corresponds to the $(g-2)$-tensor of the non-degenerate case. The second order perturbation $\mu_B \boldsymbol{H}\cdot\boldsymbol{L}$ is given by

$$\mathcal{H}_3 = -(4\mu_B^2 H^2/\Delta E) + (2\mu_B^2/\Delta E) \times \{3(H_x^2 l_x^2 + H_y^2 l_y^2 + H_z^2 l_z^2) - (\boldsymbol{H}\cdot\boldsymbol{l})^2\}. \tag{10}$$

This corresponds to that term of the non-degenerate case which gives the temperature-independent susceptibility.

For cobaltous ion, the situation is a little more complicated, because the ground state ^4F of a free cobaltous ion is perturbed by a ^4P state through the cubic crystalline field. For the time being, we neglect this effect; however, the effect can be taken into account by a certain multiplicative factor as will be shown a little later. The ground orbital state of ^4F in the cubic crystalline field is given by

$$\psi_1 = (1/4i)\{\sqrt{5}(\varphi_3+\varphi_{-3}) + \sqrt{3}(\varphi_1+\varphi_{-1})\},\ \psi_2 = \varphi_0,$$
$$\psi_3 = -(1/4)\{\sqrt{5}(\varphi_3-\varphi_{-3}) - \sqrt{3}(\varphi_1-\varphi_{-1})\}. \tag{11}$$

They give the representation l'_4. The triply degenerate state, which gives the representation l'_5 and is separated in energy by ΔE from the ground state, is given by

$$\psi_4 = (1/4)\{\sqrt{3}(\varphi_3-\varphi_{-3}) + \sqrt{5}(\varphi_1-\varphi_{-1})\},$$
$$\psi_5 = (1/\sqrt{2})(\varphi_2+\varphi_{-2}),$$
$$\psi_6 = (1/4i)\{-\sqrt{3}(\varphi_3+\varphi_{-3}) + \sqrt{5}(\varphi_1+\varphi_{-1})\}. \tag{12}$$

The highest non-degenerate state is given by

$$\psi_7=(1/\sqrt{2}i)\,(\varphi_2-\varphi_{-2}). \tag{13}$$

The matrices of $\boldsymbol{L}$ with respect to ψ_i are

$$L_z=\begin{vmatrix} 0 & 0 & -3i/2 & \sqrt{15}i/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3i/2 & 0 & 0 & 0 & 0 & -\sqrt{15}i/2 & 0 \\ -\sqrt{15}i/2 & 0 & 0 & 0 & 0 & i/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2i \\ 0 & 0 & \sqrt{15}i/2 & -i/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2i & 0 & 0 \end{vmatrix},$$

$$L^+=\begin{vmatrix} 0 & 3i/2 & 0 & 0 & \sqrt{15}i/2 & 0 & 0 \\ -3i/2 & 0 & -3/2 & -\sqrt{15}/2 & 0 & -\sqrt{15}i/2 & 0 \\ 0 & 3/2 & 0 & 0 & -\sqrt{15}/2 & 0 & 0 \\ 0 & \sqrt{15}/2 & 0 & 0 & -1/2 & 0 & -2i \\ -\sqrt{15}i/2 & 0 & \sqrt{15}/2 & 1/2 & 0 & -i/2 & 0 \\ 0 & \sqrt{15}\,i/2 & 0 & 0 & i/2 & 0 & -2 \\ 0 & 0 & 0 & 2i & 0 & 2 & 0 \end{vmatrix}. \tag{14}$$

The submatrices of L in the space of I'_4 can be expressed by $\boldsymbol{l}$ of magnitude 1 as

$$\boldsymbol{L}=-(3/2)\,\boldsymbol{l}. \tag{15}$$

Corresponding to (6), we obtain the first order energy as

$$\lambda'\boldsymbol{l}\cdot\boldsymbol{S}(\lambda'=-(3/2)\,\lambda) \quad \text{and} \quad \mu_B\boldsymbol{H}(2\boldsymbol{S}-(3/2)\,\boldsymbol{l}). \tag{16), (17}$$

Corresponding to (8), (9) and (10), the second order energies are given by

$$\mathcal{H}_1=-(15/4)\,(\lambda^2/\Delta E)\,\{2(l_x^2S_x^2+l_y^2S_y^2+l_z^2S_z^2)-(\boldsymbol{l}\cdot\boldsymbol{S})^2\}, \tag{18}$$

$$\mathcal{H}_2=(15/4)\,(|\lambda|\mu_B/\Delta E)\,\{4(H_xl_x^2S_x+H_yl_y^2S_y+H_zl_z^2S_z) - (\boldsymbol{H}\cdot\boldsymbol{l})\,(\boldsymbol{l}\cdot\boldsymbol{S})-(\boldsymbol{l}\cdot\boldsymbol{S})\,(\boldsymbol{H}\cdot\boldsymbol{l})\}, \tag{19}$$

$$\mathcal{H}_3=-(15/4)\,(\mu_B^2/\Delta E)\,\{2(H_x^2l_x^2+H_y^2l_y^2+H_z^2l_z^2)-(\boldsymbol{H}\cdot\boldsymbol{l})^2\}. \tag{20}$$

We now calculate the effect of the hybridization of ^4P into the ground orbital sublevels. Attaching a prime to the wave functions of ^4P, we define ψ_i' by the relation:

$$\psi_1'=(i/\sqrt{2})\,(\varphi_1'+\varphi_{-1}'),\ \psi_2'=\varphi_0' \quad \text{and} \quad \psi_3'=-(1/\sqrt{2})\,(\varphi_1'-\varphi_{-1}'). \tag{21}$$

The symmetry property of these wave functions is the same as that of ψ_1, ψ_2, ψ_3. Therefore, we obtain the wave functions of the ground orbital state in the form,

$$\Psi_i=\alpha\psi_i+\beta\psi_i' \qquad (\alpha^2+\beta^2=1,\ i=1,\ 2,\ 3). \tag{22}$$

Since the matrices of $\boldsymbol{L}$ in the space of ^4P are the same as those of $\boldsymbol{l}$, (15) is modified to

$$\boldsymbol{L}=-(3/2)\alpha^2\boldsymbol{l}+\beta^2\boldsymbol{l}=-(3/2)\boldsymbol{l}+(5/2)\beta^2\boldsymbol{l}, \tag{23}$$

and (16) and (17) now become

$$\lambda'=-(3/2)(1-(5/3)\beta^2)\lambda \quad \text{and} \quad \mu_B\boldsymbol{H}\{2\boldsymbol{S}-(3/2)(1-(5/3)\beta^2)\boldsymbol{l}\}. \tag{24), (25}$$

Furthermore, since (18), (19) and (20) operate only on $\zeta''_{\gamma\gamma}$, each of them must now acquire a factor $(1-\beta^2)$ each.

If we adopt the one-electron approximation, the energy matrix of the cubic crystalline field potential[11),12)] constructed between $\psi_i(i=1, 2, 3)$ and $\psi_i'(i=1, 2, 3)$ becomes

$$\begin{bmatrix} -(3/4)\Delta E' & -(1/2)\Delta E' \\ -(1/2)\Delta E' & \Delta E'' \end{bmatrix}, \tag{26}$$

where $\Delta E'$ is now the energy separation between Γ'_4 and Γ'_5 of ^4F and $\Delta E''$ the separation between ^4F and ^4P of free ion. $\Delta E''$ is about 14,800 cm^{-1} according to the spectroscopic data.[13)] From (26) β is given, in the first approximation, by

$$\beta=\frac{(1/2)\Delta E'}{\Delta E''+(3/4)\Delta E'} \tag{27}$$

and the true separation ΔE between Γ'_4 and Γ'_5 is given by

$$\Delta E=(1+\beta/2)\Delta E'. \tag{27'}$$

In actual calculations, β, though small, cannot be neglected.

§ 3. Theory of the paramagnetic susceptibility

In the paramagnetic state above the Néel temperature the energy levels of each ion are determined mainly by the first order spin-orbit coupling energy (6) or (16), and they are specified by the magnitude of the resultant angular momentum $\boldsymbol{j}=\boldsymbol{l}+\boldsymbol{S}$. In the case of CoO, there are three energy levels which correspond to $\boldsymbol{j}=1/2$, $3/2$ and $5/2$. Adopting the value -180 cm $^{-1}$ for the coefficient λ of the spin-orbit coupling, we obtain the separations of these levels as

$$E_{3/2}-E_{1/2}=405\ \mathrm{cm}^{-1} \quad \text{and} \quad E_{5/2}-E_{1/2}=1080\ \mathrm{cm}^{-1}.$$

Here we neglected the effect of ^4P. Since the Néel temperature of CoO is about 293°K or 204 cm $^{-1}/k$ (k: Boltzmann constant), there are appreciable contributions from the higher levels to the paramagnetic susceptibility, and this makes the calculation rather complicated. A similar situation occurs in the case of FeO.

For the exchange energy we adopt the molecular field approximation and write it as

$$\mathcal{H}_{ex}=2Jz\langle\boldsymbol{S}\rangle\cdot\boldsymbol{S}. \tag{28}$$

$\langle \boldsymbol{S} \rangle$ means the average vector of each spin. Since we have orbital degeneracy, it is in general open to question whether we can adopt the same form, (28), for the exchange energy as in the non-degenerate case. Above the Néel temperature, however, the exchange field has cubic symmetry on the average so that the assumption (28), which depends only on spins, is justified. This problem will be discussed further in the next paper. The value of $2Jz$ alone cannot determine the Néel temperature, because the exchange energies among geometrically nearest neighbours are ineffective in antiferromagnetic state[7], whereas they are effective in paramagnetic state and is included in $2Jz$. Thus we take $2Jz$ as an adjustable parameter and determine it by comparing the calculated paramagnetic susceptibility with the corresponding experimental data.

We shall here discuss the case of CoO. The second order terms of the Zeeman and spin-orbit coupling energies (18), (19) and (20) have anisotropic forms of cubic symmetry and they produce a cubic anisotropy of the paramagnetic susceptibility. Therefore, in order to compare our calculation with the measured powder susceptibility, which is the only available magnetic data of CoO at present, it is necessary to take the average of the susceptibility over the direction of the external field. For the sake of convenience, however, we adopt the following simplified procedure which is correct in the first order of $|\lambda|/\Delta E$ and $\mu_B H/\Delta E$. We divide each of the Hamiltonians (18), (19) and (20) into two parts : an isotropic part and an anisotropic part. For the latter, we choose a form such that transforms under coordinate transformation in the same manner as the spherical harmonics of the fourth degree ; then this anisotropic part makes no contribution to the powder susceptibility in the first order, since it vanishes when averaged, and we can confine ourselves to the isotropic part. Thus, we write (18) in the following form :

$$\begin{aligned}\mathcal{H}_1 = &-(15/4)(\lambda^2/\Delta E)[\{2(l_x^2 S_x^2 + l_y^2 S_y^2 + l_z^2 S_z^2) \\ &-(2/5)(S(S+1)l(l+1) + (\boldsymbol{l}\cdot\boldsymbol{S}) + 2(\boldsymbol{l}\cdot\boldsymbol{S})^2)\} \\ &+\{(2/5)(S(S+1)l(l+1) + (\boldsymbol{l}\cdot\boldsymbol{S}) + 2(\boldsymbol{l}\cdot\boldsymbol{S})^2) - (\boldsymbol{l}\cdot\boldsymbol{S})^2\}].\end{aligned}$$

The first curled bracket is the anisotropic part which is constructed in the form of the spherical harmonic $x^4+y^4+z^4-(3/5)r^4$. We can omit this part and also the constant terms. The remaining part is

$$\mathcal{H}_1' = -(3/4)(\lambda^2/\Delta E)\{2(\boldsymbol{l}\cdot\boldsymbol{S}) - (\boldsymbol{l}\cdot\boldsymbol{S})^2\}.$$

In the same way, we have for (19) and (20) the effective parts

$$\mathcal{H}_2' = (|\lambda|\mu_B/\Delta E)\{6(\boldsymbol{H}\cdot\boldsymbol{S}) - (3/4)((\boldsymbol{H}\cdot\boldsymbol{l})(\boldsymbol{l}\cdot\boldsymbol{S}) + (\boldsymbol{l}\cdot\boldsymbol{S})(\boldsymbol{H}\cdot\boldsymbol{l}))\}$$

and

$$\mathcal{H}_3' = -(3/4)(\mu_B^2/\Delta E)\{4H^2 - (\boldsymbol{H}\cdot\boldsymbol{l})^2\}.$$

The total Hamiltonian effective for the calculation of the powder susceptibility, including the effect of ^4P, is

$$H=-(3/2)(1-(5/3)\beta^2)\lambda \boldsymbol{l}\cdot\boldsymbol{S}+2Jz\langle S\rangle\cdot\boldsymbol{S}+\mu_B\boldsymbol{H}\{2\boldsymbol{S}-(3/2)(1-(5/3)\beta^2\boldsymbol{l}\}$$
$$+(3/4)(1-\beta^2)(\lambda^2/\Delta E)\{(\boldsymbol{l}\cdot\boldsymbol{S})^2-2(\boldsymbol{l}\cdot\boldsymbol{S})\}$$
$$+(1-\beta^2)(|\lambda|\mu_B/\Delta E)\{6(\boldsymbol{H}\cdot\boldsymbol{S})-(3/4)((\boldsymbol{H}\cdot\boldsymbol{l})(\boldsymbol{l}\cdot\boldsymbol{S})+(\boldsymbol{l}\cdot\boldsymbol{S})(\boldsymbol{H}\cdot\boldsymbol{l}))\}$$
$$-(3/4)(1-\beta^2)(\mu_B^2/\Delta E)\{4\boldsymbol{H}^2-(\boldsymbol{H}\cdot\boldsymbol{l})^2\}. \tag{29}$$

In the following, we give the method of calculation of the susceptibility neglecting ${}^4P$; in our actual calculation we took ${}^4P$ into account. We also drop the last term of (29) for a moment, which gives the temperature-independent susceptibility.

We specify the energy of each level by W_{jm}, where the suffix m means the magnetic quantum number of j. To the second power of H, W_{jm} is given by

$$W_{jm}=W_{0j}+2Jz\langle S\rangle\cdot S_{jm}+\mu_B H\{(2+6(|\lambda|/\Delta E))S_{zjm}-(3/2)l_{zjm}+g_{jm}\}$$
$$+\sum_{j'\neq j}'(1/\Delta W_{jj'})[\{2Jz\langle S\rangle+((7/2)+6(|\lambda|/\Delta E))\mu_B H\}^2|S_{zjj'm}|^2$$
$$+\mu_B H\{2Jz\langle S\rangle+((7/2)+6(|\lambda|/\Delta E))\mu_B H\}(g_{jj'm}S_{zj'jm}+S_{zjj'm}g_{j'jm})], \tag{30}$$

where W_{0j} is the energy for vanishing field and g is defined by

$$g=-(3/4)(|\lambda|/\Delta E)\{l_z(\boldsymbol{l}\cdot\boldsymbol{S})+(\boldsymbol{l}\cdot\boldsymbol{S})l_z\}.$$

First, we must determine self-consistently the value of $\langle S\rangle$. $\langle S\rangle$ is calculated by

$$\langle S\rangle=\mathrm{Tr}\left[-\frac{\partial\mathcal{H}}{\partial(2Jz\langle S\rangle)}e^{-\mathcal{H}/kT}\right]\Big/\mathrm{Tr}\,e^{-\mathcal{H}/kT},$$

which becomes, using (30),

$$\langle S\rangle=-(1/\sum_{j,m}'e^{-W_{0j}/kT})\cdot\sum_{j,m}'e^{-W_{0j}/kT}[\{2Jz\langle S\rangle+((7/2)+6(|\lambda|/\Delta E))\mu_B H\}$$
$$\times\{(S_{zjm}^2/kT)+2\sum_{j'\neq j}'(|S_{zjj'm}|^2/\Delta W_{j'j})\}$$
$$-(3/2)\mu_B H(S_{zjm}m/kT)+$$
$$\mu_B H\{(S_{zjm}g_{jm}/kT)+\sum_{j'\neq j}'(S_{zjj'm}g_{j'jm}+g_{jj'm}S_{zj'jm})/\Delta W_{j'j}\}]. \tag{31}$$

From (31), the self-consistent value of $\langle S\rangle$ is obtained as

$$\langle S\rangle=-\mu_B H[\sum_{j,m}'e^{-W_{0j}/kT}\{((7/2)+(6|\lambda|/\Delta E))((S_{zjm}^2/kT)+2\sum_{j'\neq j}'|S_{zjj'm}|^2/\Delta W_{j'j})$$
$$-(3/2)(mS_{zjm}/kT)+(S_{zjm}g_{jm}/kT)$$
$$+\sum_{j'\neq j}'(S_{zjj'm}g_{j'jm}+g_{jj'm}S_{zj'jm})/\Delta W_{j'j}\}]$$
$$\div[\sum_{j,m}'e^{-W_{0j}/kT}+2Jz\sum_{j,m}'e^{-W_{0j}/kT}\{(S_{zjm}^2/kT)+2\sum_{j'\neq j}'|S_{zjj'm}^2|^2/\Delta W_{j'j}\}]. \tag{32}$$

Defining C by the relation,

$$\langle S\rangle=(\mu_B H/2Jz)C, \tag{33}$$

the susceptibility is given by

$$\chi=(N/H)\mathrm{Tr}\left(-\frac{\partial\mathcal{H}}{\partial H}e^{-\mathcal{H}/kT}\right)/\mathrm{Tr}(e^{-\mathcal{H}/kT})+(N\mu_B^2/2Jz)C^2. \tag{34}$$

We must have the second term of (34), because in the molecular field approximation we count the exchange interactions doubly. In the absence of any orbital angular momentum (34) gives the ordinary Curie-Weiss law. More explicitly, (34) is written as

$$\chi=(N\mu_B^2/\sum_{j,m}e^{-W_{0j}/kT})[\sum_{j,m}e^{-W_{0j}/kT}\{((7/2)+C+6(|\lambda|/\Delta E))^2 \times((S_{zjm}^2/kT)+2\sum_{j'\neq j}'|S_{zjj'm}|^2/\Delta W_{j'j}) +2((7/2)+C+6(|\lambda|/\Delta E)) \times((S_{zjm}g_{jm}/kT)+\sum_{j'\neq j}'(S_{zjj'm}g_{j'jm}+g_{jj'm}S_{zj'jm})/\Delta W_{j'j}) -3((7/2)+C+6(|\lambda|/\Delta E))(mS_{zjm}/kT)+(9/4)(m^2/kT)-(3mg_{jm}/kT)\}] +(N\mu_B^2/2Jz)C^2. \tag{35}$$

In our actual calculation we took into account the effect of ^4P and also added to (35) the constant susceptibility due to the last term of (29). The resulting susceptibility is a very complicated function of temperature, as already seen in (35), but numerical calculations give apparently the Curie-Weiss law above 500° K irrespectively of the values of the parameters, $2Jz$ and $\Delta E'$, contained in (35). Roughly speaking, the effect of the residual orbital angular momentum manifests itself in two way: in the first place it gives rise to a change in the apparent g-value through the accompanying magnetic moment, and in the second place it gives rise to a change in the apparent paramagnetic Curie temperature θ through the spin-orbit coupling. The latter effect makes θ smaller than that would be expected from the value of $2Jz$ when orbital momentum is absent. The apparent g-value depends mainly on the value of the parameter, $\Delta E'$, which relates to the previously stated small correction terms of the second order. The apparent paramagnetic Curie temperature θ depends on both parameters, $2Jz$ and $\Delta E'$. We can thus determine these two parameters by comparison with experiment.

§ 4. Comparison of the theory of the preceding paragraph with experiment.

The powder susceptibility of CoO was measured by several authors.[2),14),15)] Their results are consistent with one another within experimental errors; we adopt here the result of La Blanchetais[14)], whose measurements cover the widest temperature range, i. e. 100-750°K. The susceptibility obtained by this experiment was found to follow the Curie-Weiss law

$$\chi_{\text{mol}}=3.0546/(T+280). \tag{36}$$

This corresponds to our

$$2Jz=295\ \text{cm}^{-1} \quad \text{and} \quad \Delta E'=7560\ \text{cm}^{-1}. \tag{37), (37')}$$

With these values it is shown that our calculation agrees with experiment within errors of 0.3 percent above 500°K. The results are given in Fig. 2 in which three curves

corresponding to different sets of two parameters are drawn for comparison. With the value (37′), β, the measure of hybridization of ^4P, is calculated to be $\beta=0.185$.

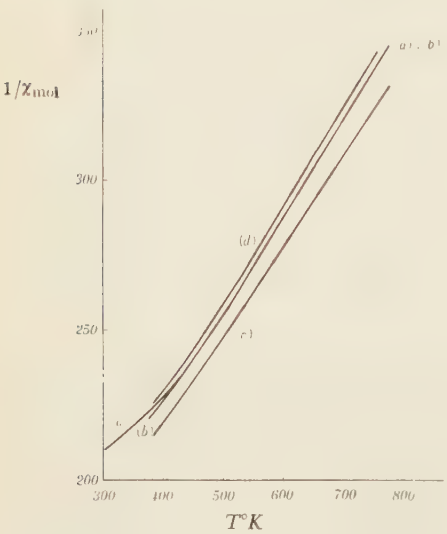

Fig. 2. Susceptibility of CoO above the Néel temperature.
Curve (a): experimental (after La Blanchetais[14]);
curve (b): calculated, assuming $2Jz=294.6\ \mathrm{cm}^{-1}$ and $\Delta E'=7560\ \mathrm{cm}^{-1}$;
curve (c): calculated, assuming $2Jz=297.5\ \mathrm{cm}^{-1}$ and $\Delta E'=6000\ \mathrm{cm}^{-1}$;
curve (d): calculated, assuming $2Jz=289.9\ \mathrm{cm}^{-1}$ and $\Delta E'=8300\ \mathrm{cm}^{-1}$.

The usual simple molecular field theory, which considers only spins, gives

$$\theta=2Jz\,S(S+1)/3k. \qquad (38)$$

And further, if we assume the spin-superstructure of the 2nd kind, which has been supported by neutron diffraction experiments and in which the spins of the next-nearest neighbouring pair of atoms point in the opposite directions, while there are equal number of nearest neighbouring pairs with parallel and antiparallel spins, the Néel temperature is given by

$$T_N=2J_1z_1\,S(S+1)/3k, \qquad (39)$$

where $2J_1z_1$ is the coefficient of the molecular field resulting from the nearest neighbours. The difference $2J_2z_2=2Jz-2J_1z_1$ is the coefficient of the molecular field resulting from the nearest neighbours. Therefore, if this simple theory could be applied to the case of CoO, we would have to interpret $2J_2z_2$ to be negative (ferromagnetic) and to be much smaller in absolute magnitude than $2J_1z_1$, since $\theta/T=280/293$. On the other hand, this simple theory must be applicable to MnO and in this case the value of $2J_2z_2$ was found to be positive (antiferromgnetic) and larger than $2J_1z_1$. Smart[5], comparing these two cases, remarked that CoO cannot be well understood by such a simple theory. However, our value of $2Jz$, given by (37), is much larger than the value 180 cm^{-1} of $2J_1z_1$ to be obtained from the Néel temperature (see the next section) and, therefore, our calculation suggests that $2J_2z_2$ is antiferromagnetic and has a magnitude of (295 − 180 =) 115 cm^{-1}, which is approximately epual to 118 cm^{-1} of MnO.[16]

The electrostatic potential of the cubic symmetry can be written as

$$D(r)\cdot(x^4+y^4+z^4-(3/5)\cdot r^4). \qquad (40)$$

If we assume the surrounding ions to be point charges, $D(r)$ does not depend on r, since the potential satisfies Laplace's equation. In several cases[17,18] it was found that this model can give the correct sign and correct order of magnitude of the level splittings. In our oxides the magnitude of D with this model is given by $D=17.9\ e^2/a^5$ [19,20],

where e is the elementary charge and a the distance between O^{--} and the metallic ion. For CoO $\Delta E'$ is then given by

$$\Delta E' = (16/105) \cdot D \cdot \langle r^4 \rangle, \tag{41}$$

where $\langle r^4 \rangle$ means the average over the density of the d-electrons. For FeO we have

$$\Delta E = (4/21) \cdot D \cdot \langle r^4 \rangle. \tag{42}$$

With these formulas and with the Slater function[21] for the d-electron wave function, we obtain

$$\Delta E' = 6455 \text{ cm}^{-1} \quad \text{for} \quad \text{CoO} \tag{43}$$

and

$$\Delta E = 10885 \text{ cm}^{-1} \quad \text{for} \quad \text{FeO}. \tag{44}$$

The value for CoO is very near that given by (37′). Abragam and Pryce[22] assumed 8800 cm^{-1} for $\Delta E'$ of cobalt tutton salt in order to obtain a good agreement of their calculated g-values and hyperfine splittings with the corresponding experimental results. Considering these situations, our value (37′) appears to be quite reasonable.*

For FeO, we have yet no reliable experimental data, presumably because this oxide is unstable at room temperature. Measurements of the powder susceptibility above the Néel temperature (of about 186°K) up to room temperature by Bizette[23] give

$$\chi_{\text{mol}} = 6.24/(T+570).$$

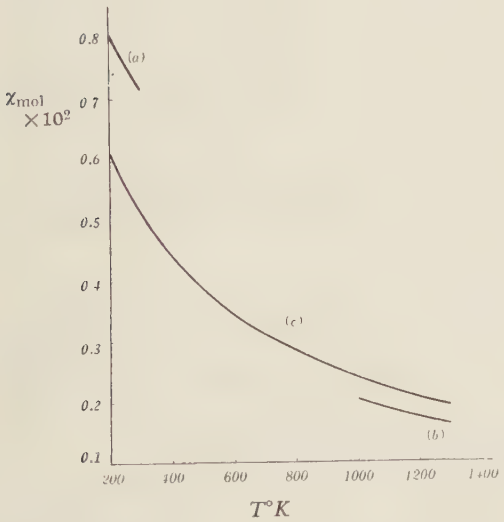

Fig. 3. Susceptibility of FeO above the Néel temperature.
Curve (a): experimental (after Bizette and Tsai[23]);
curve (b): experimental (after Mashiyama et al.[24]);
curve (c): calculated.

However, this value of the Curie constant is unreasonably large. Mashiyama, Uchida and Kondoh[24] obtained

$$\chi_{\text{mol}} = 4.60/(T+190)$$

for the temperature region between 196°K and 460°K, and

$$\chi_{\text{mol}} = 2.42/(T+190)$$

for the temperature region between 973°K and 1320°K. The low temperature data which are not reliable are about 1.5 times as large as those of Bizette.

Since we obtained approximately the same value of $2J_2z_2$ for MnO and CoO, we may assume the same value of $2J_2z_2$ for FeO. Then, adding to this the value of $2J_1z_1$ deduced from the Néel temperature, we obtain a value of

* **Note added in proof.** Recently, the optital absorption experiment on a crystal of MgO containing Ni^{++} has been performed and it shows that the energy separation between Γ_2 and Γ_5 of ^3F of Ni^{++} is about 8600cm^{-1}. (W. Low, Bulletin Amer. Phys. Soc. **1** (1956) No. 8, 398, S11) From this value, the energy separation between Γ_5 and Γ_4, $\Delta E'$, of Ni^{++} can be estimated to be about 6900m^{-1}.

190 cm^{-1} for $2Jz$ of FeO. With this value and with the simplified Hamiltonian

$$\mathcal{H}' = \lambda' \boldsymbol{l} \cdot \boldsymbol{S} + 2Jz\langle \boldsymbol{S} \rangle \cdot \boldsymbol{S} + \mu_B H(2\boldsymbol{S} - \boldsymbol{l}),$$

the susceptibility of FeO can be calculated. The result gives approximately the Curie-Weiss law:

$$\chi_{\text{mol}} = 3.15/(T + 332).$$

As shown in Fig. 3, our results are somewhat below Bizette's data at low temperatures and somewhat above those of Mashiyama, Uchida and Kondoh at high temperatures. It is hoped that more reliable data are published and a better comparison can be made.

Finally, it may be mentioned that if a single crystal specimen were available and measurements of the anisotropy of its paramagnetic susceptibility could be made with it, we would be able to deduce the separation ΔE by applying our theory.

§ 5. Influence of the spin-orbit coupling on the Néel temperature

In this section we investigate the influence of the spin-orbit coupling on the Néel temperature and calculate $2J_1z_1$, i. e. the exchange coupling constant effective in the antiferromagnetic state, from the experimental data of the Néel temperature. The Néel temperature is defined as the temperature at which an infinitesimal antiferromagnetic molecular field sets in. It can be shown that the orbit-orbit interactions cannot affect the Néel temperature in the molecular field approximation, because the change of energies due to them is proportional to the fourth power of the antiferromagnetic molecular field. Then, if in (31) we put H equal to zero and attach minus sign to the left-hand side, we obtain a relation valid at the Néel temperature:

$$\langle S \rangle = 2J_1z_1\langle S \rangle [\sum_{j,m}{}' e^{-W_{0j}/kT_N} \{(S^2_{zjm}/kT_N) + 2\sum_{j' \neq j}{}' |S_{zjj'm}|^2/\Delta W_{j'j}\}]/[\sum_{j,m}{}' e^{-W_{0j}/kT_N}].$$

From this we obtain

$$2J_1z_1 = \sum_{j,m}{}' e^{-W_{0j}/kT_N} / \sum_{j,m}{}' e^{-W_{0j}/kT_N} \{(S^2_{zjm}/kT_N) + 2\sum_{j' \neq j}{}' |S_{zjj'm}|^2/\Delta W_{j'j}\}. \quad (45)$$

Here we neglected the second order term of $\boldsymbol{L} \cdot \boldsymbol{S}$ coupling and also the effect of ^4P in the case of CoO.

In the case of FeO, whose T_N is 186°K[23),24)], we obtain from (45)

$$2J_1z_1 = 68.1 \text{ cm}^{-1}. \quad (46)$$

In the spin-only theory, $2J_1z_1$ is given by (39) and is 65 cm^{-1}. Therefore, we can conclude that the spin-orbit coupling lowers the Néel temperature a little. For CoO ($T_N = 293$°K[14)]), $2J_1z_1$ is obtained as

$$2J_1z_1 = 180.5 \text{ cm}^{-1}. \quad (47)$$

In the spin-only theory we have $2J_1z_1 = 162$ cm^{-1}.

§ 6. State of the ions at the absolute zero of temperature

At absolute zero the spin of each ion is subjected to a strong exchange field and is oriented in its direction. The pseudo-orbital angular momentum $\boldsymbol{l}$, connected with the spin through the spin-orbit coupling, becomes antiparallel to this direction. This is the classical picture of the ground state. Quantum mechanically, we have to start with the following Hamiltonian:

$$\mathcal{H}=-2J_1z_1\langle S\rangle\cdot S_z+\lambda'\boldsymbol{l}\cdot\boldsymbol{S}, \tag{48}$$

neglecting smaller terms.* In the case of CoO, we must take account of the effect of ^4P on λ'. $\langle S\rangle$ is the average value of the spin which we assume to point in the positive z-direction. The state given by $S_z=S$ and $l_z=-1$, which corresponds to the classical ground state, is connected with the states given by $S_z=S-1$, $l_z=0$ and $S_z=S-2$, $l_z=1$ through the x, y components of $\lambda'\boldsymbol{l}\cdot\boldsymbol{S}$. Therefore we must solve a secular equation of the third order and determine $\langle S\rangle$ self-consistently. Adopting the value of $2J_1z_1$ obtained from the Néel temperature and the spectroscopic data of λ', we calculate the wave function of the ground state to

$$\psi_g=0.909\,\psi_{2,-1}-0.395\,\psi_{1,0}+0.135\,\psi_{0,1} \quad \text{for} \quad \text{FeO}, \tag{49}$$

$$\psi_g=0.875\,\psi_{3/2,-1}-0.446\,\psi_{1/2,0}+0.188\,\psi_{-1/2,1} \quad \text{for} \quad \text{CoO}. \tag{50}$$

Here $\psi_{2,-1}$ etc., mean the wave functions of the states $S_z=2$ and $l_z=-1$, etc..

In (48) the deformation of the crystal in antiferromagnetic state is not considered. Actually we must consider the influence of the deformation on the crystalline field and thus on the wave function. We shall develop the theory of the deformation in the next paper. Here, using the results of it in advance, we assume that, in the case of CoO, the effect is taken into account by adding a term $-cl_z^2$ to (48), where the constant c has the magnitude of about $100\,\text{cm}^{-1}$. Then the wave function for CoO becomes

$$\psi_g'=0.900\,\psi_{3/2,-1}-0.401\,\psi_{1/2,0}+0.169\,\psi_{-1/2,1}. \tag{50'}$$

From the data of the forward scattering of neutrons by the ions we can deduce the magnitude of the magnetic moment of the ions. Using the wave functions obtained above, we can calculate this quantity, m_0, theoretically. For CoO, using (50'), we have

$$m_0'=\mu_B\langle 2S_z-(3/2)\,(1-(5/3)\beta^2)l_z\rangle_{\text{Av.}}=3.69\,\mu_B.$$

More exactly, considering the small correction due to (19), we have

$$m_0=\mu_B\langle 2S-(3/2)\,(1-(5/3)\beta^2)l_z+(15/4)\,(|\lambda|/\Delta E)\,\{4S_zl_z^2-l_z(\boldsymbol{l}\cdot\boldsymbol{S})-(\boldsymbol{l}\cdot\boldsymbol{S})l_z\}\rangle_{\text{Av.}}$$
$$=3.83\,\mu_B. \tag{51}$$

If we use (50) instead of (50'), we obtain $m_0=3.68\,\mu_B$. From the experimental data obtained by Shull et al[2], we have $m_0\simeq 3.7\,\mu_B$. The agreement is therefore very good.

* The validity of neglecting the orbit-orbit interaction will be discussed in the next paper.

For FeO, experimental data are not available. Assuming the value of ΔE obtained from the point charge model and using (49) we calculate to $m_0 = 4.44 \mu_B$.

§ 7. Discussion of the origins of the crystalline field

We have hitherto developed our theory with the assumption that, except for the spin-dependent exchange energy, the influence of the surrounding ions could be represented by an electrostatic potential included in the one-ion Hamiltonian. Though this assumption seems to be generally acceptable, we have yet no conclusive theory of the origins of the crystalline field. It may, therefore, be worth while to discuss here the probable origins in some detail and thereby investigate the validity of our theory.

Our estimation of the crystalline field, (43) or (44), was made with the assumption of the point charge model. However, this model is far from the reality, since it disregards the overlap of the electron clouds between neighbouring ions. In the first place, the overlap effect makes $D(r)$ of (40) depend on r, since the potential no longer satisfies Laplace's equation, but it satisfies Poisson's equation. Attempts to calculate this 'classical' effect of the overlap were made by several authors.[12),25)] In the case of cubic symmetry, it reduces or reverses the level splitting of the orbital states obtained from the point charge model because of an imperfect screening of the attractive potential arising from the nuclei of the neighbouring negative ions. In particular, Kleiner[25)] has shown in his calculation of Cr alum that the potential obtained from such an improved calculation gives the wrong sign to the level splitting, whereas the point charge model can give the correct sign and a correct order of the magnitude of the splitting. Recently, Tanabe and Sugano[26)] pointed out that the non-orthogonality of the electron orbits belonging to neighbouring ions has an important effect to the level splitting of the orbital state. To the first approximation this effect could be understood as the Heisenberg exchange interaction between the non-orthogonal orbits. Since the sign of this exchange integral is negative because of the attractive potential between the electron and the nucleus of the neighbouring ion, it works as a repulsive potential between the negative ion and the electron. These authors assert that in Cr alum the exchange part of the interaction of the electrons with the surrounding ions exceeds the coulomb part so that the level scheme originally obtained from the point charge model comes nearer to the truth. The corresponding exchange integral in our oxides is presumably of the order of 1000 cm^{-1}, and therefore the exchange effect will in any case have an appreciable contribution to the level splitting.

There is another mechanism which will cause the splitting of the orbital levels. Owen[27)] has pointed out that the weak covalent bond between the anion and the cation can explain the discrepancy between the optical data and the magnetic data concerning the level splitting of the cation orbital state of some hydrated salts. Also in our case, as is well known, there exists a partial covalency between the anion and the cation which is an essential origin of the superexchange mechanism.[28)] Since its degree depends on the orbital state of the cation, the energy arising from this effect will contribute to the level splitting.

Its qualitative nature is the same as that of the point charge interaction and also that of the exchange effect mentioned above, because this effect lowers the energy of the state of the cation with the least overlap, since then there are vacant orbits which have large overlaps and available for electron jumps from the anions. As we shall show later, this energy amounts in our case to several thousands inverse centimeters.

Therefore, the value of $\Delta E'$ given by (37′) should be interpreted as the combined result of these complicated effects. It will be questionable whether we can treat these effects in the form of an electrostatic potential in the one-ion Hamiltonian as in the case we have calculated the effect of ^4P. In other words, we have to ask whether we can adopt the one-electron approximation for the effective Hamiltonian of these effects. If we can do it, we can write the effective Hamiltonian in the form (40) and determine the coefficient D so as to get the actual level splitting of the d-orbits.

The effect of the non-orthogonality relates only to the energy of each d-orbit within the approximation of the Heisenberg model. On the other hand, the effect of the covalency can be treated in the scheme of the perturbation theory which will give the relevant energy in the second order. The degree of the covalency depends both on the transition probability of the electron from the p-orbit of the neighbouring oxygen ion to one of the five d-orbits and on the excitation energy required for this transition. We assume here that the potential field which causes the transition is well approximated by the Hartree field which does not depend on the orbital state of the cation, or in other words, the multipole part of the potential arising from the cation electron cloud is negligible. Then, as we shall show explicitly later, the transition probability is related only to the availability of a specific d-orbit into which an oxygen electron can jump. So assuming that we can neglect the effect of the electron correlation on the excitation energy of this jump, it can be concluded that the effect of the covalency is represented by an effective one-electron Hamiltonian. The mentioned assumption is valid, if the excitation energy is large compared with the internal coulomb energy among the electrons belonging to that cation, which has the magnitude of about 1 ev.. On the other hand, the excitation energy is given approximately by the following formula :[30]

$$\Delta W = (4\alpha - 1)\, e^2/a + E - I, \tag{52}$$

where α is the Madelung constant for the lattice of the NaCl type, a the distance between the anion and the cation, E the electron affinity of O^{--}, I the second ionization energy of the metal atom. Adopting the value of -9 ev. for E[30], we obtain from (52)

$$\Delta W = 14.3 \text{ ev.} \quad \text{for both oxides.}$$

The actual value of ΔW would be smaller, since polarisation effects are not considered in (52). According to soft X-ray emission data[31], ΔW for MgO is estimated to be $10 \sim 15$ ev.. Considering that the second ionization energy of Mg is smaller by about 1.5 ev. or 2.3 ev. than those of Fe or Co, it may reasonably be concluded that ΔW of our oxides is about 10 ev.. Thus the difference in excitation energy due to the internal

coulomb energy and other effects for different orbital states of the cation may approximately be neglected.

An idea of the magnitude of the exchange and covalent effects can be got by making use of the theory of superexchange. Since the d-shells of Fe^{++} and Co^{++} are 'more-than-half' full, the mechanism originally proposed by Anderson[28] works here most effectively, i. e., the exchange integral J_1 between the next-nearest neighbours is given by

$$J_1=(t/\Delta W)^2 J'/(2S+1), \tag{53}$$

where $(t/\Delta W)^2$ is the degree of covalency, t the transfer integral, J' the average exchange integral between the p-orbit of the oxygen ion and the d-orbits of the cation. If we make an approximate estimation of J' by a relatation,

$$J'=ts/S,$$

where s means the overlap integral, we can obtain a rough estimate of t and thus of J' and $t^2/\Delta W$ (depression of energy due to the covalent bond formation) with the use of the experimental value of J_1 given in § 5, the value of ΔW estimated above and a reasonable value of s. Assuming thus $s=0.1$ and $\Delta W=10$ ev., we obtain

$$t^2/\Delta W=\begin{matrix}3000\ \mathrm{cm}^{-1} & \text{for} & \text{FeO},\\ 4100\ \mathrm{cm}^{-1} & \text{for} & \text{CoO},\end{matrix}\qquad J'=\begin{matrix}780\ \mathrm{cm}^{-1} & \text{for} & \text{FeO},\\ 1200\ \mathrm{cm}^{-1} & \text{for} & \text{CoO}.\end{matrix} \tag{54}$$

Since both effects depend largely on the different degrees of overlap between different d-orbits of the cation and the oxygen orbit, the values of (54) themselves represent also the order of magnitude of their contributions to the level splitting. Thus we see that these effects are as large as those of the classical electrostatic field of the point charges and the finite spread of the oxygen electron clouds. In the following, we shall discuss in detail the different degrees of overlaps and the consequent contribution of the covalency to the level splitting.

We define the five d-orbits, $\phi_i\,(i=1,\cdots,5)$, by the following symmetry properties:

$$\phi_1\sim zx,\ \phi_2\sim xy,\ \phi_3\sim yz,\ \phi_4\sim(3z^2-r^2),\ \phi_5\sim(x^2-y^2). \tag{55}$$

ϕ_i have the same symmetry properties as φ'_i of the ferrous ion. We consider a cation at the origin and an oxygen ion situated on the positive z-axis and denote its $2p$ and $2s$ orbits by p_x, p_y, p_z and s. There are three kinds of the electron transition involved in covalency: from p_z to ϕ_1, from p_x or p_y to ϕ_1 or ϕ_2, and from s to ϕ_4. Other transitions are small owing to their symmetry properties. We define t_1, t_2 and t_3 by

$$t_1=(p_z|\mathcal{H}|\phi_4),\ t_2=(p_x|\mathcal{H}|\phi_1)=(p_y|\mathcal{H}|\phi_2),\ t_3=(s|\mathcal{H}|\phi_4). \tag{56}$$

If a relevant d-orbit is vacant, the energy of the covalency effect in the second order perturbation is given respectively by

$$-t_1^2/\Delta W,\ -t_2^2/\Delta W,\ -t_3^2/\Delta W'.$$

Here we distinguish the excitation energy $\Delta W'$ for the $2s$-electron transition from that

of the $2p$-electron transition, ΔW. Since the promotion energy of the $2s$-electron to one of the $2p$-orbits is about 15 ev.[13] in the neutral oxygen atom, $\Delta W'$ is larger by a factor of 2 or more than ΔW, given by (52).

For the covalent effect with an oxygen ion on the x-axis, we notice that

$$\phi_4 = (1/2)\phi_4' + (\sqrt{3}/2)\phi_5', \qquad \phi_5 = (\sqrt{3}/2)\phi_4' - (1/2)\phi_5', \tag{57}$$

where the primed orbits are those which can be obtained from the unprimed orbits by replacing the z-axis with the x-axis and therefore ϕ_4' and ϕ_5' are given by

$$\phi_4' \sim (3x^2 - r^2), \qquad \phi_5' \sim (y^2 - z^2).$$

From (57) we easily see that the availability of ϕ_4 for the electron jump from the oxygen ion on the x-axis is $1/4$ of that from the oxygen on the z-axis, since only ϕ_4' is available for this jump. Similarly, the availability of ϕ_5 for the electron jump from the oxygen ion on the x-axis is $3/4$. Therefore, if ϕ_4 is vacant, the relevant total energy of the covalency effect with six oxygen ions surrounding the cation is given by

$$\begin{aligned} E(\phi_4) &= -\frac{2t_1^2}{\Delta W} - (1/4)\times\frac{4t_1^2}{\Delta W} - \frac{2t_3^2}{\Delta W'} - (1/4)\times\frac{4t_3^2}{\Delta W'} \\ &= -3t_1^2/\Delta W - 3t_3^2/\Delta W'. \end{aligned} \tag{58}$$

If ϕ_5 is vacant, the corresponding energy is

$$E(\phi_5) = -(3/4)\times\frac{4t_1^2}{\Delta W} - (3/4)\times\frac{4t_3^2}{\Delta W'} = -\frac{3t_1^2}{\Delta W} - \frac{3t_3^2}{\Delta W'}. \tag{58'}$$

As for the covalent energy for ϕ_1, ϕ_2 and ϕ_3, we notice that

$$\phi_1 = \phi_3', \qquad \phi_2 = \phi_1', \qquad \phi_3 = \phi_2'.$$

Therefore the total energy of the covalency effect of these orbits is given by

$$E(\phi_{1,2,3}) = -4t_2^2/\Delta W. \tag{59}$$

In the case of FeO, only one of the five d-orbits is doubly occupied: in Γ_5', one of the three orbits ϕ_1, ϕ_2 and ϕ_3 is doubly occupied, and in Γ_3', ϕ_4 or ϕ_5 is doubly occupied. So the contribution of the covalency to the level separation between Γ_3' and Γ_5' is the difference between (59) and (58), that is,

$$\Delta E = (3t_1^2 - 4t_2^2)/\Delta W + 3t_3^2/\Delta W'. \tag{60}$$

The case of CoO is a little more complicated because of the configuration mixing arising from the presence of ^4P. For example, the orbital wave function, ψ_2, is obtained by antisymmetrizing the product,

$$(2/\sqrt{5})\phi_1(\mathrm{I})\phi_3(\mathrm{II}) + (1/\sqrt{5})\phi_2(\mathrm{I})\phi_5(\mathrm{II}), \tag{61}$$

and the corresponding product for ψ_5 is given by

$$\phi_2(\mathrm{I})\phi_4(\mathrm{II}). \tag{62}$$

Here we treat the case of Co^{++} as the two-electron case. Then the energy of the covalency effect with an oxygen on the z-axis is given by

$$-\frac{t_1^2}{\Delta W}-(2/5)\frac{t_2^2}{\Delta W}-\frac{t_3^2}{\Delta W'}, \quad \text{for the state given by } \psi_2, \tag{63}$$

and

$$-\frac{2t_2^2}{\Delta W} \quad \text{for} \quad \psi_5. \tag{64}$$

The corresponding energy arising from the covalency with an oxygen on the x-axit is, using the relation (57), obtained as

$$-(17/20)\frac{t_1^2}{\Delta W}-\frac{t_2^2}{\Delta W}-(17/20)\frac{t_3^2}{\Delta W'} \quad \text{for} \quad \psi_2. \tag{65}$$

and

$$-(3/4)\frac{t_1^2}{\Delta W}-\frac{t_2^2}{\Delta W}-(3/4)\frac{t_3^2}{\Delta W'} \quad \text{for} \quad \psi_5. \tag{66}$$

Using these relations, the total contribution of the covalency effect to the energy separation between Γ_5 and Γ_4 of ^4F is obtained as

$$\Delta E'=\frac{12t_1^2-16t_2^2}{5\Delta W}+\frac{12t_3^2}{5\Delta W'}.$$

Since the wave functions have the largest overlap for the transition given by t_1, t_1^2 is probably larger than t_2^2 by a factor of 2 or more. If the former is much larger than the latter and if the covalency effect with the $2s$ electrons, specified by t_3, is ineffective in the superexchange because of a large excitation energy, we can identify t_1 with t given in (53). Then we see that the level splitting arising from the covalency effect has an order of magnitude comparable with those from other effects.

Finally, one might question the validity of the fact that we disregarded in the calculation of § 3 the effect of the configuration mixing arising from our covalency effect. As Owen points out, this configuration mixing leads to a smaller absolute value of the coefficient of the spin-orbit coupling, especially for the non-diagonal part of the spin-orbit coupling with respect to the levels split by the crystalline field potential. The latter is due to the fact that the non-diagonal parts connect the orbital states of different degrees of covalency. On the other hand, however, Tanabe and Sugano[26] obtained by a rigorous treatment of non-orthogonality a larger coefficient of the spin-orbit coupling, the orthogonalization making the effective weight of the d-orbits larger than unity. It is therefore difficult to infer a plausible value of the coefficient of the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling. We estimate that our ΔE and $\Delta E'$, which are affected by the non-diagonal parts of the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling, have errors of about 10 percent.

Another question is whether the residual orbital angular momentum is partially quenched by the covalency effect. Since the orbits used by the predominant bond specified

by t_1 are almost unoccupied by the electrons in the lowest sublevel of both Fe^{++} and Co^{++}, the three states can have an equal covalency with all the neighbouring oxygen ions. It can therefore be concluded that, even in the state in which an electron has jumped into one of these unoccupied orbits, the orbital momentum is still 'alive' and has approximately the same magnitude as in non-covalent states.

§ 8. Summary

In this paper we stressed the importance of the residual orbital angular momentum for the interpretation of the magnetic properties of FeO and CoO. In § 2 we derived the effective Hamiltonian of the Zeeman and $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling for degenerate orbital states. In § 3 and § 4 we discussed the paramagnetic susceptibility and obtained reasonable results especially for CoO. A difficulty pointed out by Smart in interpreting the relation between θ and T_N of CoO have been removed. In § 5 we discussed the Néel temperature and in § 6 we calculated the wave functions of the ground states of the cations at absolute zero. Using these wave functions, we calculated the magnetic moment of these ions and obtained a good agreement with experiment for cobalt ion. Finally, in § 7 we discussed the origins of the crystalline field.

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Theory of the Magnetic Properties of Ferrous and Cobaltous Oxides, II

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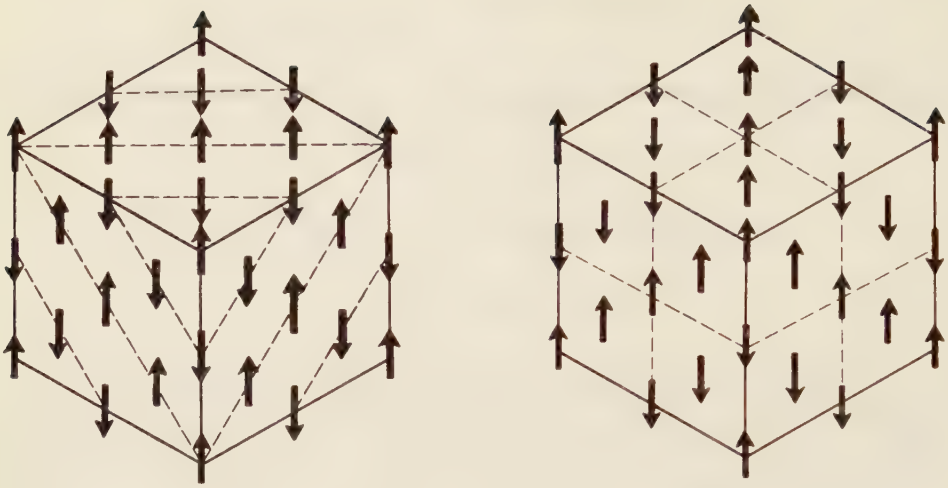
The origins of the magnetic anisotropy energies of the crystals of FeO and CoO in the state of antiferromagnetic ordering and the mechanism of the deformation they suffer when the antiferromagnetic ordering sets in are investigated from the atomistic point of view, following the line developed in the preceding paper. It is concluded that the most effective part of the magnetic anisotropy energy in the deformation-free state of the crystals originates from orbital multipole (or van Vleck's orbital valence) interactions arising from both the coulomb and exchange interactions between cations. The deformation of the crystals, however, is caused mainly by magnetostriction arising from linearly strain-dependent terms of the crystalline field energies. It is shown that the theoretical determination of the axis and magnitude of the deformation, using the point charge model for the calculation of the crystalline field, can give results which are semi-quantitatively consistent with the experimental results. It is shown that in CoO the deformation-dependent anisotropy energy overcomes other anisotropy energies and determines the direction of the magnetization to be the direction of the tetragonal axis of deformation, [001], while in FeO the deformation-independent anisotropy energies are predominant in determining that direction, which coincides with the crystalline trigonal axis, [111]. Discussions are also given for other kinds of deformations which do not depend on the orientation of the magnetic moments, that is, the volume striction and the trigonal deformation suggested by Greenwald and Smart. A brief comment is given to the magnetic anisotropies of MnO, MnS and NiO.

§ 1. Introduction

In this paper we investigate the origins of the magnetic anisotropy energies of FeO and CoO in their antiferromagnetic state and look for the mechanism of their deformation below the Néel temperature. MnO, MnS and NiO will also be discussed briefly.

FeO, MnO, MnS and NiO deform into trigonal symmetry[1),2)] below their Néel temperatures. In the case of FeO, the angle α that specifies the fundamental rhombohedron is smaller than 60°, whereas for MnO and NiO α is greater than 60°. CoO[2),3)] deforms into tetragonal symmetry and a contraction occurs along the c-axis. X-ray data for this oxide show that the contraction takes place gradually from the Néel temperature with decreasing temperature and its magnitude amounts to about 1 percent at the lowest temperature of the measurements.

According to neutron diffraction experiments,[1)] the antiferromagnetic spin arrangement in FeO is that proposed by Néel,[5)] Fig. 1 (a), which we shall call structure [A] after Li.[6)] This arrangement can be obtained by attaching alternately + spins and − spins to hexagonal close-packed layers which compose the cubic face-centred lattice of the cations. It has trigonal symmetry. In the following, we shall take [111] as the trigonal



(a) trigonal (b) cubic

Fig. 1

axis. The orientation of the magnetic moments in FeO was definitely determined to be [111] by neutron diffraction. For CoO, MnO and NiO, the neutron diffraction experiments with powder specimens could not decide whether the spin arrangement has structure [A] or another cubic structure [B], Fig. 1 (b). Li[6] showed that for structure [A] the orientation consistent with neutron diffraction data is [001] and for structure [B] the orientation cannot be determined at all. As will be shown in later sections, the tetragonal deformation of CoO can reasonably be interpreted as a magnetostriction originating in the effect of the crystalline field on the residual orbital angular momentum of Co^{++}, so that it is highly probable that the orientation of magnetic moments in CoO is [001].* For MnS, neutron diffraction experiment[7] determined the spin arrangement to be the structure [A] and the orientation of the magnetic moments to be in the (111) plane. In NiO, recent magnetic measurements indicate the orientation of magnetic moments to be likely in the (111) plane, as will be referred to in the last section.

Theoretically speaking, we can distinguish between two kinds of the mechanisms of deformation : one is exchange striction which arises from the dependence of the exchange energies on inter-ionic distances and does not depend on the orientation of the magnetic moments ; another is magnetostriction which arises from the anisotropy energies and depends on the orientation of the magnetic moments. In structure [A], the exchange striction can cause a deformation of trigonal symmetry, as well as a volume striction, as suggested by Greenwald and Smart.[8] In structure [B], it causes only a volume striction. The present investigation will show that, in both FeO and CoO, the magnetostriction predominates due to the presence of a residual orbital angular momentum. In FeO, the

* **Note added in proof.** Recently, Roth has found by neutron diffraction that the orientation of the magnetic moments in MnO, CoO and NiO is probably in the (111) plane. (W. L. Roth, Annual Meeting, American Crystallographic Association, June, 1956 (unpublished).) His result for CoO is at variance with the conclusion of this study. However, the result of Singer's[23] magnetic measurements on a single crystal of CoO is favourable to our conclusion.

magnetostriction elongates the crystal along the trigonal axis, overcoming the exchange striction which contracts the crystal along the same trigonal axis.

So far, the studies of the origins of the anisotropy energies of antiferromagnetics[9] have been limited to the non-degenerate orbital case. In cubic crystals of FeO and CoO, however, the orbital state is degenerate. From a new standpoint, we can divide the origins of their magnetic anisotropy energies into four categories: 1. the magnetic dipole-dipole interaction; 2. the spin-orbit interactions; 3. the orbit-orbit interactions; 4. the anisotropy energy arising from deformation.

In the first category, we include intra-ionic magnetic spin-spin interaction as well as inter-ionic magnetic dipole-dipole interaction. The latter was calculated for structure [A] by Kaplan[10]; it makes (111) the easy plane of magnetization. In cubic structure [B], it makes no first order contribution to the anisotropy energy. The intra-ionic spin-spin interaction in a free atom or an ion was discussed by Araki[11] and Pryce[12]; modified by cubic crystalline field, it gives rise to an anisotropy energy of cubic symmetry.

The spin-orbit coupling energy modified by cubic crystalline field was calculated in the preceding paper,* using the second order perturbation theory. Anisotropy energies of other origins, to be included in this second category are such as anisotropic exchange energy derived by Moriya and Yosida,[13] but they are negligible in our case.

The anisotropy energies belonging to the third category were discussed generally by van Vleck[14] for the case of ferromagnetism, but we shall attempt to make more specific calculations in later sections. These energies arise from that the cations act as multipoles on their surrounding cations which are also multipoles. In the case of degenerate orbital level, in which the residual orbital angular momentum drags the spin moment, these energies contribute to the anisotropy energy in the first order of perturbation. Classically they can be calculated as the multipole part of the coulomb interactions between the cations. In addition to it, however, there are contributions arising from the exchange part. Also, two cations intervened by an oxygen ion can interact via covalent electron configuration and this produces an anisotropy energy (§ 5).

The magnetostriction is caused by anisotropy energies which depend linearly on strain components. The anisotropy energies belonging to the above three categories have parts which are linearly dependent on strain components, but it can be shown that they cannot produce as large deformations as those observed in FeO and CoO. On the other hand, the crystalline field energy arising from deformation acts directly on the residual orbital angular momentum and produces spontaneous deformations of a right order of magnitude.

Since the change of energy by magnetostriction depends on the orientation of the magnetic moments, it contributes also to the anisotropy energy. This anisotropy energy is important in the present case, particularly in CoO. There is also an anisotropy energy arising from a combination of the exchange striction and magnetostriction. These anisotropy energies are included in the fourth category.

The calculations of the following sections are based on the molecular field approxi-

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* Cited as [I], in the following.

mation, or the one-ion approximation, in which the Hamiltonian of each ion is derived from the total Hamiltonian by averaging the operators relating to other ions. We mostly confine ourselves to the absolute zero of temperature, so that the average is taken over the zero-point motion of the spins. We treat the anisotropy Hamiltonians as perturbations and calculate their first order contributions using the wave functions calculated in § 6 of [I] as the unperturbed wave functions. The orbital multipole interactions turn out to have the magnitude not very small compared with the exchange energy and spin-orbit coupling energy, and one might question if it is allowable to treat them as perturbations. It can be shown, however, that their non-diagonal elements are small within the scheme of the one-ion approximation and therefore this procedure is allowable.

In performing these calculations, we meet difficulties for precise evaluations, and some ambiguities as regards the numerical magnitudes will inevitably be introduced. Nevertheless, it is believed that all the conceivable important origins of the anisotropy energies and deformations are considered in the present paper and an essentially correct picture of the antiferromagnetic oxides is obtained.

In § 2, magnetic dipole-dipole interactions and spin-orbit interactions are investigated; in § 3, 4 and 5, orbit-orbit interactions are calculated; in § 6 theory of magnetostriction is developed; in § 7, calculation is made of the magnetostriction using the point charge model for the crystalline field; in § 8 exchange striction is considered; § 9 summarises the anisotropy energies including that which arises from deformation; the last section is a brief discussion of the magnetic anisotropy energies and deformations of MnO, MnS and NiO.

§ 2. Magnetic dipole-dipole interaction and spin-orbit coupling energy

The contribution of the inter-ionic magnetic dipole-dipole interaction energy in structure [A] to the total energy can be written in the following form of uniaxial symmetry;

$$E_D = T_D(\alpha_x\alpha_y + \alpha_y\alpha_z + \alpha_z\alpha_x), \tag{1}$$

where α_x, α_y and α_z are the direction cosines of the sublattice magnetizations in antiferromagnetic state referred to the cubic principal axes. According to Kaplan,[10] T_D at absolute zero is given by

$$T_D = (114.82/a_0^3)\mu^2 \text{ erg per ion}, \tag{2}$$

where μ is the magnitude of the magnetic moment of the cation, a_0 the lattice constant of the magnetic unit cell.

For FeO, assuming the value $4.44\mu_B$ of μ calculated in § 6 of [I] and 8.664 Å for a_0, we obtain

$$T_D = 1.5 \text{ cm}^{-1} \text{ per ion}. \tag{3}$$

For CoO, using the value $3.7\mu_B$ of μ deduced from the neutron scattering data and 8.50 Å for a_0, we obtain

$$T_D = 1.1 \text{ cm}^{-1} \text{ per ion}. \tag{4}$$

These energies stabilize the magnetization in the plane (111).

The effective Hamiltonian of the intra-ionic spin-spin interaction in the Russel-Saunders scheme of coupling is given by

$$\mathcal{H}_{SS}=-\rho[\,(\boldsymbol{L}\cdot\boldsymbol{S})^2+(1/2)\,(\boldsymbol{L}\cdot\boldsymbol{S})\,]\,. \tag{5}$$

According to Pryce,[12] the value of ρ * is 0.95 cm^{-1} for Fe^{++} and 1.5 cm^{-1} for Co^{++}. We rewrite this Hamiltonian using the following relations between the orbital angular momentum and the pseudo-orbital angular momentum $\boldsymbol{l}$ in subspace l'_4 (for FeO) or l'_4 (for CoO) :

$$L_x^2=4-3l_x^2,\ L_xL_y+L_yL_x=3\,(l_xl_y+l_yl_x),\ \text{etc. for FeO}\,; \tag{6}$$

$$L_x^2=6l_x^2,\ L_xL_y+L_yL_x=-\,(3/2)\,(l_xl_y+l_yl_x),\ \text{etc. for CoO}. \tag{7}$$

Then (5) becomes

$$\mathcal{H}_{SS}=+\rho[\,6\,(l_x^2S_x^2+l_y^2S_y^2+l_z^2S_z^2)-(3/2)\,(\boldsymbol{l}\cdot\boldsymbol{S})-3\,(\boldsymbol{l}\cdot\boldsymbol{S})^2]\ \text{for FeO}\,; \tag{8}$$

$$\mathcal{H}_{SS}=-\rho[\,(15/2)\,(l_x^2S_x^2+l_y^2S_y^2+l_z^2S_z^2)-(3/4)\,(\boldsymbol{l}\cdot\boldsymbol{S})-(3/2)\,(\boldsymbol{l}\cdot\boldsymbol{S})^2]\ \text{for CoO}. \tag{9}$$

$\mathcal{H}_{SS}$ gives an anisotropy energy of cubic symmetry, since $\boldsymbol{l}$ is antiparallel to $\boldsymbol{S}$ in antiferromagnetic state. In numerically evaluating the anisotropy constant, we average $\mathcal{H}_{SS}$ using the wave function of the ground state, (49) and (50) of [I], and assuming an arbitrary direction of the axis of magnetization. The part dependent on this direction can be written in the form,

$$E_{SS}=K_{SS}\,(\alpha_x^2\alpha_y^2+\alpha_y^2\alpha_z^2+\alpha_z^2\alpha_x^2)\,.$$

K_{SS} is determined from the difference of E_{SS} for the direction [100] and the direction [111]. The results are

$$K_{SS}=-1.9\ cm^{-1}\ \text{per ion for FeO}\,; \tag{10}$$

$$K_{SS}=0.78\ cm^{-1}\ \text{per ion for CoO}. \tag{11}$$

Further, as calculated in § 2 of [I], the anisotropic spin-orbit coupling energy is given by the following Hamiltonian :

$$\mathcal{H}_{LS}=(\lambda^2/\Delta E)[\,6\,(l_x^2S_x^2+l_y^2S_y^2+l_z^2S_z^2)-(\boldsymbol{l}\cdot\boldsymbol{S})-(\boldsymbol{l}\cdot\boldsymbol{S})^2]\ \text{for FeO}\,; \tag{12}$$

$$\mathcal{H}_{LS}=-\,(1-\beta^2)\,(\lambda^2/\Delta E)[\,(15/2)\,(l_x^2S_x^2+l_y^2S_y^2+l_z^2S_z^2)-(15/4)\,(\boldsymbol{l}\cdot\boldsymbol{S})^2]\ \text{for CoO}. \tag{13}$$

These have the same form as the intra-ionic spin-spin interaction. By a similar calculation, the anisotropy energies are obtained as

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* As Pryce points out, these values of ρ include second order effect arising from non-diagonal elements of the spin-orbit coupling energy between multiplets of $3d^n$ configuration. This effect may be larger than the spin-spin interaction in Fe^{++} and Co^{++}. Therefore, in the cubic field, the value of ρ will be somewhat modified due to the change of the energy separations between multiplets. In Co^{++}, another modification comes from the hybridization of ^4F and ^4P. However, we neglect these minor effects in the following evaluation of the anisotropy energy.

$$E_{LS}=K_{LS}(\alpha_x^2\alpha_y^2+\alpha_y^2\alpha_z^2+\alpha_z^2\alpha_x^2),$$

with

$$K_{LS}=-1.85\ \mathrm{cm}^{-1}\ \text{per ion for FeO,} \tag{14}$$

$$K_{LS}=2.0\ \mathrm{cm}^{-1}\ \text{per ion for CoO.} \tag{15}$$

In the actual calculations, use was made of the spectroscopic data of λ in both ions, experimental values of ΔE and β in CoO and the value of ΔE calculated with the point charge model in FeO.

If we evaluated K_{SS} and K_{LS} by using the wave function corresponding to $S_z=S$, $l_z=-1$, we would have obtained considerably larger absolute values for them. Since the states $S_z=S-1$, $l_z=0$ and $S_z=S-2$, $l_z=1$ are mixed by spin-orbit coupling, the interference effects reduce the values of K_{SS} and K_{LS} to those given above.

§ 3. Orbital multipole interactions I: the coulomb interactions

In this section, we discuss the coulomb part of the multipole interactions between cations. The coulomb interaction between two cations can be expanded in a series of multipole interactions, of which only the quadrupole-quadrupole, quadrupole-hexadecapole and hexadecapole-hexadecapole interactions appear in the first order, since the multipole clouds are formed of d-electrons. The dipole-dipole interaction gives the van der Waals force in the second order, which, however, can be neglected in our case since the polarizabilities of the cations are small.

First we derive formulas for the multipole interactions between two charge clouds which do not overlap with each other. We denote by 1 and 2 the lattice points at which the two charge clouds are located and specify by $\boldsymbol{r}_i$ the position of electron i belonging to charge cloud 1, the origin being taken at the lattice point 1, and by $\boldsymbol{r}_j$ the position of electron j belonging to charge cloud 2, the origin being taken at the lattice point 2. The coulomb interaction between these electrons can be expanded in a power series of $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$. The part proportional to the product $r_i^2 r_j^2$ represents the quadrupole-quadrupole interaction, the part proportional to $r_i^2 r_j^4$ or $r_i^4 r_j^2$ the quadrupole-hexadecapole interaction and the part proportional to $r_i^4 r_j^4$ the hexadecapole-hexadecapole interaction. If we adopt the coordinate system whose z-axis is the line joining the lattice points 1 and 2, the general forms of these interactions are as follows;

the quadrupole-quadrupole interaction

$$(e^2 r_i^2 r_j^2/R_{12}^5)\,[6P_2(\cos\vartheta_i)\,P_2(\cos\vartheta_j)-(4/3)\,P_2^1(\cos\vartheta_i)\,P_2^1(\cos\vartheta_j)\cos(\varphi_i-\varphi_j) + (1/12)\,P_2^2(\cos\vartheta_i)\,P_2^2(\cos\vartheta_j)\cos 2(\varphi_i-\varphi_j)]; \tag{16}$$

the quadrupole-hexadecapole interaction

$$(e^2 r_i^2 r_j^4/R_{12}^7)\,[15P_2(\cos\vartheta_i)\,P_4(\cos\vartheta_j)-2P_2^1(\cos\vartheta_i)\,P_4^1(\cos\vartheta_j)\cos(\varphi_i-\varphi_j) + (1/12)\,P_2^2(\cos\vartheta_i)\,P_4^2(\cos\vartheta_j)\cos 2(\varphi_i-\varphi_j)] \tag{17}$$

and a similar expression with i and j interchanged;

the hexadecapole-hexadecapole interaction

$$(e^2 r_i^4 r_j^4/R_{12}^9)\left[70 P_4(\cos\vartheta_i) P_4(\cos\vartheta_j) - (28/5) P_4^1(\cos\vartheta_i) P_4^1(\cos\vartheta_j)\cos(\varphi_i-\varphi_j)\right.$$
$$+ (7/45) P_4^2(\cos\vartheta_i) P_4^2(\cos\vartheta_j)\cos 2(\varphi_i-\varphi_j) - (1/315) P_4^3(\cos\vartheta_i) P_4^3(\cos\vartheta_j)$$
$$\times\cos 3(\varphi_i-\varphi_j)$$
$$\left.+ (1/2016) P_4^4(\cos\vartheta_i) P_4^4(\cos\vartheta_j)\cos 4(\varphi_i-\varphi_j)\right]. \qquad (18)$$

In these formulas, R_{12} is the distance between 1 and 2, and e the elementary charge.

To calculate the anisotropy energy, we rewrite the above formulas referring to the cubic axes and sum the results over all the pairs of cation. Further, we make the approximation of molecular field, that is, we assume that all the cations have the same average spacial distribution of electrons around each of their nuclei. It is convenient to replace the spacial coordinates of a representative electron of each cation by the total orbital angular momentum of that cation, which in our case, is further replaced by the pseudo-orbital angular momentum $\boldsymbol{l}$, using relations (6), (7) and the relations connecting the 4th order polynomials of $\boldsymbol{L}$ with the quadratic forms of $\boldsymbol{l}$. Details are given in App. I. In terms of the pseudo-orbital angular momentum $\boldsymbol{l}$, all these interactions become effectively quadrupole-quadrupole interactions, since, in the lowest triplet, the hexadecapole reduces to quadrupole. Thus we obtain the following expression for the effective Hamiltonian of these interactions:

$$\mathcal{H}_{LL} = \kappa_1\left[l_{x1}^2 l_{x2}^2 + l_{y1}^2 l_{y2}^2 + l_{z1}^2 l_{z2}^2\right] + (\kappa_2/2)\left[(l_{x1}l_{y1}+l_{y1}l_{x1})(l_{x2}l_{y2}+l_{y2}l_{x2})\right.$$
$$\left.+ (l_{y1}l_{z1}+l_{z1}l_{y1})(l_{y2}l_{z2}+l_{z2}l_{y2}) + (l_{z1}l_{x1}+l_{x1}l_{z1})(l_{z2}l_{x2}+l_{x2}l_{z2})\right]. \qquad (19)$$

For the details of the calculation of κ_1 and κ_2 the reader is referred to Appendix I. The contribution from this to the anisotropy energy is obtained by averaging (19) with the wave function of the ground state. As easily be shown, we can replace the operators by the direction cosines α_i of the magnetization through the following relations:

$$(l_x^2 - 2/3) = (1/2)(a^2+c^2-2b^2)(\alpha_x^2 - 1/3), \text{ etc., and}$$
$$l_x l_y + l_y l_x = (a^2+c^2-2b^2)\alpha_x\alpha_y, \qquad (20)$$

where a, b and c are defined by the relation,

$$\psi_g = a\psi_{S,-1} + b\psi_{S-1,0} + c\psi_{S-2,1}.$$

The factor $(a^2+c^2-2b^2)$ represents the reducing effect of the spin-orbit coupling on the anisotropy constant. This factor tends approximately to zero for increasing strength of the spin-orbit coupling. This factor is 0.531 in FeO and 0.402 in CoO.

In quantitatively estimating the anisotropy energy, however, we meet some difficulties. Firstly, the average values of r^2 and r^4, $\langle r^2\rangle$ and $\langle r^4\rangle$, which enter into the calculation through (16), (17) and (18), depend sensitively on the radial distribution function and in particular, on its tail at far distances from the ion core. Secondly, there are overlap regions of charge clouds between nearest neighbouring cations, which, though include only a few percent of the total density, give appreciable effects on the final results, since r^2 and r^4 have large values there and thus formulas (16), (17) and (18) are not justified.

To take these features into account precisely is not easy. Therefore, in order to obtain a semi-quantitative idea of these effects, we made two calculations: in one calculation we ignored the fact of the overlaps but still used in (16), (17) and (18) the values of $\langle r^2 \rangle$ and $\langle r^4 \rangle$ of the free ion calculated with the Slater function[15]; in the other calculation we cut off, for the interactions between nearest neighbours, the charge clouds outside the spheres of a radius equal to half the nearest neighbour distance drawn around each of the lattice points, and assumed for other interactions the free ion values of $\langle r^2 \rangle$ and $\langle r^4 \rangle$ calculated with the Slater function. These free ion values of $\langle r^2 \rangle$ and $\langle r^4 \rangle$ are

$$\begin{matrix} \langle r^2 \rangle = 0.903 \text{ Å}^2 \\ \langle r^4 \rangle = 1.310 \text{ Å}^4 \end{matrix} \text{ for FeO, and } \begin{matrix} \langle r^2 \rangle = 0.741 \text{ Å}^2 \\ \langle r^4 \rangle = 0.882 \text{ Å}^4 \end{matrix} \text{ for CoO;} \tag{21}$$

while the cut-off values are

$$\begin{matrix} \langle r^2 \rangle_c = 0.759 \text{ Å}^2 \\ \langle r^4 \rangle_c = 0.853 \text{ Å}^4 \end{matrix} \text{ for FeO, and } \begin{matrix} \langle r^2 \rangle_c = 0.669 \text{ Å}^2 \\ \langle r^4 \rangle_c = 0.664 \text{ Å}^4 \end{matrix} \text{ for CoO.} \tag{22}$$

The first calculation gives for the anisotropy constant

$$\begin{aligned} K_{LL} &= -72.9 \text{ cm}^{-1} \text{ per ion for FeO,} \\ K_{LL} &= -18.0 \text{ cm}^{-1} \text{ per ion for CoO,} \end{aligned} \tag{23}$$

and the second calculation gives

$$\begin{aligned} K_{LL} &= -33.7 \text{ cm}^{-1} \text{ for FeO,} \\ K_{LL} &= -11.1 \text{ cm}^{-1} \text{ for CoO.} \end{aligned} \tag{24}$$

As shown in Appendix I, it is found that in both oxides, the quadrupole-hexadecapole interactions make the largest contribution to the anisotropy energy. The quadrupole-quadrupole interactions favour the directions $\{100\}$ in both oxides but they are overcome by the quadrupole-hexadecapole interactions which favour $\{111\}$. The hexadecapole-hexadecapole interactions make a much smaller contribution.

It might be remarked that a contribution to the one-ion Hamiltonian for atom 1 is obtained from (19) by replacing the operators having suffix 2 by their average values, but these average values have to be determined self-consistently since it has non-diagonal elements between those states which are determined from the one-ion Hamiltonian consisting only of exchange energy and spin-orbit energy. However, it can be shown that these non-diagonal elements are small compared with the unperturbed energy separations and therefore we can neglect them. A crude estimation shows that these non-diagonal elements affect the value of the anisotropy constant by about 10 percent.

To conclude, we obtain from the multipole interactions a large anisotropy energy favouring the directions $\{111\}$ and overcoming K_{SS} and K_{LS}. This is consistent with the neutron diffraction result in FeO, but not so with that in CoO. The latter discrepancy will be removed in a later section (§ 9).

§ 4. Orbital multipole interactions II: the direct exchange part

In this section we shall investigate the direct exchange interactions between the cations. In [I], it was concluded from the analysis of the paramagnetic susceptibility data that there are significant exchange interactions between nearest neighbouring cations. Whether these exchange interactions are of the direct type or of an indirect type, similar to superexchange interactions, is a question to be settled before treating the anisotropy problem. If they arise by an indirect mechanism, we may expect that the magnitude of the exchange integral will change proportionally to the magnitude of the superexchange interaction constant between second neighbours when we go from MnO to CoO or from MnO to MnS.[16] In going from MnO to MnS, θ decreases from 610°K to 528°K and the Néel temperature T_N increases from 122°K to 165°K, θ corresponding to a certain sum of the superexchange interaction constant and the exchange integral in question and T_N to the superexchange interaction constant alone. Thus the exchange interaction in question does not behave in the same way as the superexchange. It decreases with increasing interionic distance. As regards MnO and CoO, we concluded in paper [I] from the analysis of the paramagnetic susceptibility of CoO that the magnitude of the exchange integral in CoO is about the same as that in MnO, while the magnitude of the superexchange is at variance, the Néel temperature being 122°K and 293°K for MnO and CoO, respectively. One may therefore reasonably assume that the exchange interactions between nearest neighbours are mainly of the direct type.

The exchange interaction can be divided into the spin-independent part and the spin-dependent part. We define the latter as $-2J_2\mathbf{S}_1\cdot\mathbf{S}_2$, where J_2 is the average exchange integral (J_2 has the opposite sign to that defined in [I]). We shall first discuss the spin-independent part.

We consider a cluster composed of one central cation and its twelve nearest neighbours (see Fig. 2), and denote the doubly occupied d-orbits of the central cation by suffix μ, those of the nearest neighbours by suffix ν, and the singly occupied d-orbits of the central cation by suffix ρ, those of the nearest neighbours by suffix σ. Then the spin-independent part of the exchange energy between the central cation and one of its nearest neighbours is given by

$$-(1/2)\sum_{\rho,\sigma}J_{\rho\sigma}-\sum_{\rho,\nu}J_{\rho\nu}-\sum_{\mu,\sigma}J_{\mu\sigma}-2\sum_{\mu,\nu}J_{\mu\nu}\,.$$

We have to pick up only the part of the exchange energy which depend simultaneously on the doubly occupied orbits, since the sum of the exchange integrals taken over all the d-orbits of one of the participant cations depends only on the d-orbit belonging to the other cation and further when this sum is summed over the cations surrounding the latter cation, the result becomes isotropic, i.e. independent of the kind of that d-orbit. Therefore the anisotropic part can be expressed as a certain sum of the interaction between one of the electrons in each μ and one of the electrons in each ν:

$$\mathcal{E}_{ex}=-(1/2)\cdot(1/2)\cdot\sum_{\mu,\nu}J_{\mu\nu}\,, \tag{25}$$

where the factor 1/2 is inserted in order to get the energy per one ion. In FeO, only one d-orbit is doubly occupied, so we shall at first investigate this case in detail and then go over to the case of CoO, making use of the results obtained for FeO.

We assume that the wave function of the doubly occupied orbit μ is given by

$$\phi_\mu = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 \tag{26}$$

and that of the orbit ν by

$$\phi_\nu = b_1\phi_1' + b_2\phi_2' + b_3\phi_3', \tag{26'}$$

where ϕ_i are the wave function given in § 7 of |I| and a prime is attached to the orbits of the surrounding cations. Then the exchange energy (25) between the central cation and one of the nearest neighbours is written as

$$\varepsilon_{ex} = -(1/4)\sum_{k,l,m,n} a_k^* a_l b_m^* b_n (km\,|\,\mathcal{H}\,|\,nl), \tag{27}$$

where

$$(km\,|\,\mathcal{H}\,|\,nl) = \int \phi_k^*(\mathrm{I})\,\phi_m'^*(\mathrm{II})\,\mathcal{H}\,\phi_n'(\mathrm{I})\,\phi_l(\mathrm{II})\,d\tau_1 d\tau_{\mathrm{II}}, \tag{28}$$

$\mathcal{H}$ being the interacting potential. (27) can be written in operator form by the following relations:

$$\begin{aligned} a_k^* a_l &= -l_{1l}l_{k1}, \quad b_m^* b_n = -l_{n2}l_{m2}, \\ a_k^* a_k &= 1 - l_{k1}^2, \quad b_m^* b_m = 1 - l_{m2}^2; \end{aligned} \tag{29}$$

$k, l = 1, 2, 3$; $m, n = 1, 2, 3$ and $1, 2, 3 = y, z, x$, where the suffices 1 and 2 of l's specify respectively the central ion and the participant nearest neighbour. When the summation is taken over all the nearest neighbours, we obtain two different anisotropy Hamiltonians: one has cubic symmetry and is similar to that for the coulomb interaction, (19), and the other is uniaxial. However, lack of our knowledge on the magnitudes of the individual integrals forbids us to make a precise estimation. Here a rough estimate of their order of magnitude will be given. We first discuss the cubic anisotropy Hamiltonian.

We consider the nearest neighbours in the xy-plane. The cubic anisotropy Hamiltonian represents effectively a kind of quadrupole-quadrupole interaction. Thus, integrals which do not change its sign when we go from the nearest neighbours in the first quadrant to that in the second quadrant contribute to it. The most effective integral in (27) will be $(22\,|\,\mathcal{H}\,|\,22)$, because the pertinent wave function has the largest overlap, and the corresponding term in (27) will be $-(1/4)\,(22\,|\,\mathcal{H}\,|\,22)\,(1-l_{z1}^2)\,(1-l_{z2}^2)$. Neglecting other integrals and summing this over all the twelve nearest neighbours of the cation 1, we obtain for the total cubic anisotropy Hamiltonian

$$\mathcal{H}'_{ex} = -(22\,|\,\mathcal{H}\,|\,22)\,(l_{x1}^2 l_{x2}^2 + l_{y1}^2 l_{y2}^2 + l_{z1}^2 l_{z2}^2), \tag{30}$$

the suffix 2 now indicating any representative one of the nearest neighbours. If we assume the magnitude of $(22\,|\,\mathcal{H}\,|\,22)$ to be about 6 times the magnitude of the average exchange integral, which can be estimated from the data of MnO and CoO, we obtain

$(22\,|\,\mathcal{H}\,|\,22) = -30\text{cm}^{-1}$.* Then the anisotropy constant K_{ex} becomes about $-4.2\ \text{cm}^{-1}$.

The uniaxial anisotropy Hamiltonian represents a kind of dipole-dipole interaction. It comes from such integrals that change their sign when we go from the nearest neighbour in the first quadrant of the xy-plane to that in the second quadrant. In (27), terms which contribute to this anisotropy energy are

$$-(1/4)[(21\,|\,\mathcal{H}\,|\,23)\,(a_2^*a_3b_1^*b_2+a_3^*a_2b_2^*b_1+a_1^*a_2b_2^*b_3+a_2^*a_1b_3^*b_2)$$
$$+(22\,|\,\mathcal{H}\,|\,13)\,(a_2^*a_3b_2^*b_1+a_3^*a_2b_1^*b_2+a_1^*a_2b_3^*b_2+a_2^*a_1b_2^*b_3)],$$

which can also be written as

$$-(1/8)[\{(21\,|\,\mathcal{H}\,|\,23)+(22\,|\,\mathcal{H}\,|\,13)\}\{(a_2^*a_3+a_3^*a_2)\,(b_1^*b_2+b_2^*b_1)$$
$$+(a_1^*a_2+a_2^*a_1)\,(b_2^*b_3+b_3^*b_2)\}$$
$$+\{(21\,|\,\mathcal{H}\,|\,23)-(22\,|\,\mathcal{H}\,|\,13)\}\{(a_2^*a_3-a_3^*a_2)\,(b_1^*b_2-b_2^*b_1)$$
$$+(a_1^*a_2-a_2^*a_1)\,(b_2^*b_3-b_3^*b_2)\}]. \tag{31}$$

The first term of (31) makes no contribution, because it vanishes when summed over all the nearest neigobours. The second term can be written in the operator form as

$$+(1/8)\{(21\,|\,\mathcal{H}\,|\,23)-(22\,|\,\mathcal{H}\,|\,13)\}\,(l_{y1}l_{x2}+l_{x1}l_{y2}).$$

For spin-structure [A], this energy gives a uniaxial anisotropy, since, if we go from the nearest neighbour of plus spin to that of minus spin, the direction of $\boldsymbol{l}$ and the sign of the integrals are simultaneously reversed. Adding similar terms coming from the nearest neighbours in the yz- and zx-plane and summing over twelve nearest neighbours, we obtain

$$(1/2)\{(21\,|\,\mathcal{H}\,|\,23)-(22\,|\,\mathcal{H}\,|\,13)\}\{(l_{x1}+l_{y1}+l_{z1})\,(l_{x2}+l_{y2}+l_{z2})$$
$$-(\boldsymbol{l}_1\cdot\boldsymbol{l}_2)\}. \tag{32}$$

Here we took the cation 2 in the first quadrant of the xy-plane as the representative nearest neighbour. In (32), the integral $(21\,|\,\mathcal{H}\,|\,23)$ is much larger than $(22\,|\,\mathcal{H}\,|\,13)$ in absolute magnitude and its sign is positive, since the pertinent wave functions, ϕ_1' and ϕ_3 have different signs. Assuming that the absolute magnitude of $(21\,|\,\mathcal{H}\,|\,23)$ is equal to that of the average exchange integral, we obtain the uniaxial anisotropy given by

$$E_{ex}''=T_{ex}'(\alpha_x\alpha_y+\alpha_y\alpha_z+\alpha_z\alpha_x),$$

with

$$T_{ex}'=-4\ \text{cm}^{-1}\ \text{per ion}. \tag{33}$$

It is the present author's opinion that this anisotropy energy as well as that obtained from the spin-dependent exchange energy to be discussed later are most important in

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* This estimation is based on the following consideration. If we define ϕ_4, ϕ_5 after the definition of § 7 of [I], the effective integrals contained in the average exchange integral will be $(44|\mathcal{H}|44)$, $(22|\mathcal{H}|22)$, $(42|\mathcal{H}|24)$ and $(24|\mathcal{H}|42)$. Assuming that they are approximately equal, and considering the weight of the state, 2, in paramagnetic state, we obtain this value.

determining the favourable direction of the aligned magnetic moments in FeO, since, had we not these anisotropy energies, the strong cubic anisotropy derived in the preceding section favours four directions $\{111\}$ equally and the weak magnetic dipole-dipole interaction, E_D, calculated by Kaplan excludes the trigonal axis itself and leaves only the remaining three $\{111\}$ as favourable directions, whereas the experiment shows that the magnetic moments are in $[111]$.

In the case of CoO, two of the three d-orbits, ϕ_1, ϕ_2 and ϕ_3 are almost doubly occupied. If, moreover, the remaining one were also doubly occupied, there would be no contribution to the anisotropy energy. Therefore the anisotropic exchange interaction can be interpreted as the interaction arising from a hole which occupies one of the three orbits. Thus the Hamiltonian takes the same form as (27). However, the absolute magnitude of the anisotropy constants would be smaller than in FeO, because, for the cubic anisotropy constant, the factor $(1/2)(a^2+c^2-2b^2)$ is smaller in CoO and also because, influenced by ${}^4P$, the electrons tends to occupy the three d-orbits equally in order to lower the internal coulomb energy.

Next, we shall discuss the spin-dependent part of the direct exchange interactions. This part is usually considered as being cancelled out in antiferromagnetic state, since there are equal numbers of nearest neighbours with parallel and antiparallel spins (see Fig. 2). However, if the spin arrangement is of structure $[A]$, there is a possibility that this exchange energy gives rise to an anisotropy energy of trigonal symmetry. Qualitatively speaking, this comes out in the following way. Since the pseudo-orbital angular momentum $\boldsymbol{l}$ points antiparallel to the spin, the shape of the cation charge cloud depends on the orientation of its spin, and since the lines connecting parallel spin-pairs and those connecting antiparallel spin-pairs have different directions (the former being perpendicular to the trigonal axis and the latter oblique), the sum of the exchange integrals for parallel spin-pairs are different from those for antiparallel spin-pairs, this difference depending on the orientation of the magnetic moments. Further, since the exchange energy changes its sign for these two kinds of pairs, the total exchange energy is proportional to this difference and so it has trigonal symmetry.

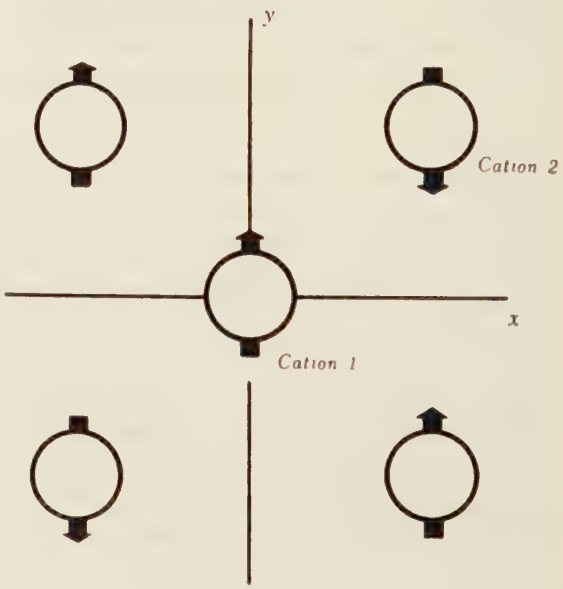

Fig. 2

Quantitative calculation of this anisotropy energy is difficult, because it requires the knowledge of various complicated integrals. We attempt, however, to estimate its order

of magnitude by a crude calculation similar to that already made in this section.

First we discuss the case of FeO. If J_{km} is the exchange integral between the k-th orbit of the central cation and the m-th orbit of one of the nearest neighbours and suffices μ and ν mean, as before, the electron paired orbits, the average exchange integral J_2 is written as

$$J_2=(1/16)\{\sum_{k,m}J_{km}-\sum_{m}J_{\mu m}-\sum_{k}J_{k\nu}+J_{\mu\nu}\}, \tag{34}$$

where the sums are taken over all the five d-orbits. The first sum does not contribute to the anisotroby energy, since it does not depend on the orbital states of the participant cations.

The other sums can be expanded in a way similar to (27). When summing the exchange energy over the four nearest neighbours in the xy-plane, it is clear that only those integrals that change their signs in going from plus spin to minus spin give finite contributions. Such integrals, referred to the cation in the first quadrant of the xy-plane and the central cation, are

$$(1m\,|\,\mathcal{H}\,|\,m3),\ (22\,|\,\mathcal{H}\,|\,13),\ (21\,|\,\mathcal{H}\,|\,23),$$

and those integrals which become equivalent to these by interchanges of the indices. So the sum of the exchange energy over the four nearest neighbours in the xy-plane is expressed as follows:

The part arising from the sum, $\sum_{m}J_{\mu m}$, is given by

$$(1/2)\,(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\,(a_1^*a_3+a_3^*a_1)\sum_{m}(1m\,|\,\mathcal{H}\,|\,m3), \tag{35}$$

the part arising from $J_{\mu\nu}$ is given by

$$\begin{aligned}-(1/2)\,(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\big[&(12\,|\,\mathcal{H}\,|\,23)b_2^*b_2(a_1^*a_3+a_3^*a_1)+(11\,|\,\mathcal{H}\,|\,13)\,(b_1^*b_1+b_3^*b_3)\\&\times(a_1^*a_3+a_3^*a_1)\\&+(1/4)\{(22\,|\,\mathcal{H}\,|\,13)+(21\,|\,\mathcal{H}\,|\,23)\}\\&\times\{(a_2^*a_3+a_3^*a_2)\,(b_1^*b_2+b_2^*b_1)+(a_1^*a_2+a_2^*a_1)\,(b_2^*b_3+b_3^*b_2)\}\big].\end{aligned} \tag{35'}$$

In (35′) we have omitted the term corresponding to the second term of (31), because it cancels out at the summation. Also terms which are obtained from (35) and (35′) by interchanging a and b are omitted; they make the same contribution as that of (35) and (35′) and can be dropped for the calculation of the anisotropy energy per ion. As before, we express a and b in terms of the components of angular momenta $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and obtain the anisotropy Hamiltonian for the cluster in the xy-plane as

$$\begin{aligned}-(1/2)\,(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\big[&\{\sum_{m}(1m\,|\,\mathcal{H}\,|\,m3)-(12\,|\,\mathcal{H}\,|\,23)\}\,(l_{x1}l_{y1}+l_{y1}l_{x1})\\&+\{(12\,|\,\mathcal{H}\,|\,23)-(11\,|\,\mathcal{H}\,|\,13)\}\,(l_{x1}l_{y1}+l_{y1}l_{x1})\,l_{z2}^2\\&+(1/4)\{(22\,|\,\mathcal{H}\,|\,13)+(21\,|\,\mathcal{H}\,|\,23)\}\{(l_{z1}l_{x1}+l_{x1}l_{z1})\,(l_{y2}l_{z2}+l_{z2}l_{y2})\\&+(l_{y1}l_{z1}+l_{z1}l_{y1})\,(l_{z2}l_{x2}+l_{x2}l_{z2})\}\big].\end{aligned} \tag{36}$$

In (36), the first term makes the largest contribution. Neglecting the other terms and adding to it two similar expressions with x, y, z cyclically interchanged, we obtain for the total anisotropy Hamiltonian of the central cation

$$\mathcal{H}'''_{ex} = -(1/2)(\mathbf{S}_1 \cdot \mathbf{S}_2)\{\sum_{m'''}(1m|\mathcal{H}|m3) - (12|\mathcal{H}|23)\} \times (l_x l_y + l_y l_x + l_y l_z + l_z l_y + l_z l_x + l_x l_z). \quad (37)$$

In all the above formulas $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are antiparallel spins. In order to estimate the order of magnitude of the anisotropy energy, we replace the sum $\{\sum_{m'''}(1m|\mathcal{H}|m3) - (12|\mathcal{H}|23)\}$ by half the average exchange integral, $J_2/2$, and use the value of $-5\ \mathrm{cm}^{-1}$ for J_2. Then we obtain the trigonal (uniaxial) anisotropy energy as

$$E'''_{ex} = T''_{ex}(\alpha_x\alpha_y + \alpha_y\alpha_z + \alpha_z\alpha_x),$$
$$T''_{ex} = -3\ \mathrm{cm}^{-1}\ \text{per ion}. \quad (38)$$

The argument above contains quantitatively somewhat dubious points, but it can safely be said that the uniaxial anisotropy energies, (33) and (38), can overcome the interionic dipole-dipole interactions.

The case of CoO is much more difficult because of its more complicated electron configuration. Even the sign of the anisotropy constant cannot be inferred. Roughly speaking, however, the magnitude of the anisotropy energy obtainable from the spin-dependent part will be reduced considerably as compared with that of FeO. We may suppose that this anisotropy energy is not so much important in CoO to determine the direction of the magnetic moments.

Finally, it may be remarked that there is a possibility that the second order perturbation energy, arising from the transition matrix elements between the ground state and those excited states in which an electron is transferred from the cation to one of its nearest neighbours, makes some contribution to the anisotropy energy. But this energy can be expressed in the same form as the exchange energy, so that the above calculation effectively includes this energy.

§ 5. Orbital multipole interactions III: the indirect interactions; summary of the results of this and previous sections

Under the heading of the indirect multipole interactions, we include those orbit-orbit interactions which are related to the covalent electron configuration between one of the cations and an intervening oxygen ion, as in superexchange mechanism. We consider for instance two cations intervened by an oxygen ion which is located midway on the line joining the cations. We take this line as the z-axis. Since the interaction energies are obtained in the third order process of the perturbation scheme, they consist of two quantities of which one is the degree of the covalency between the oxygen and one of the cations and the other is the interaction between the other cation and the oxygen in covalent state with the first cation. As discussed in § 7 of [I], the degree of covalency

depends on the orbital state of the participant cation via the magnitude of the excitation energy of the electron transfer, the availability of the bonding orbit and the magnitude of the transfer integral. The interaction between the covalently bonded oxygen ion and the other cation consists of coulomb and exchange interactions. Though precise calculations are not feasible, we shall show in the following that these interactions make no important contributions to the anisotropy energy.

In both oxides, the dependence of the covalent degree on the orbital state of the cation is considered to be small, since the predominant bond, specified by t_1 (see § 7 of [I]), can be formed almost irrespectively of the orbital state of the cation and also the dependence of the excitation energy on the orbital state is small. On the other hand, a large contribution to the interaction between O^- and the non-covalently bonded cation will arise from the coulomb interaction between them. We may assume that the dependence of the covalent degree on the cation orbital state is given by

$$(t/\Delta W)^2(1+c_1 l_{z1}^2) \quad (t \approx t_1)$$

and that the interaction of the oxygen with other cation is given by $c_2 l_{z2}^2$. Somewhat arbitrarily $|c_1|$ will be assumed to be 10 percent, while $|c_2|$ is roughly estimated to be at most $3000\,\mathrm{cm}^{-1}$. Then we obtain a contribution of about $1 \sim 2\,\mathrm{cm}^{-1}$ to the cubic anisotropy constant. This is negligible in both oxides.

Summarizing the calculations made in this and the previous sections, we obtain the following conclusion as to the origins of the anisotropy energy which are effective in the deformation-free state of the crystal. In FeO, a large cubic anisotropy energy arising mainly from orbit-orbit coulomb interactions favours the directions {111}, and a uniaxial anisotropy energy pertinent to structure [A] arising mainly from direct exchange interactions favours the trigonal axis [111]. The latter overcomes the uniaxial anisotropy energy arising from the magnetic dipole-dipole interactions, which favours the plane (111). In this way we obtained a result consistent with the experiment in the case of FeO. In CoO, the cubic anisotropy energies arising from intra-ionic spin-spin energy and spin-orbit coupling energy in cubic crystalline field favour the directions {100}, but a larger orbit-orbit coulomb energy makes the directions {111} more favourable. The uniaxial anisotropy energy arising from the dipole-dipole interactions favours the plane (111), but it was not investigated in detail whether another uniaxial anisotropy arising from the direct exchange interactions makes the direction [111] more favourable or not. The latter is possibly smaller than in FeO. However, we shall find in § 9 that magnetostrictive anisotropy energy is most effective in CoO and that this energy deforms crystal tetragonally in one of the {100} directions and makes the magnetic moments align along this tetragonal axis. In FeO, such does not occur.

§ 6. Theory of the magnetostriction

In the following sections, we shall investigate the deformation of the crystal in antiferromagnetic state. In the first place, we discuss the mechanism of the magnetostriction of FeO and CoO.

The magnetostriction is caused by an anisotropy energy which depends linearly on the strain components of the crystal. Its general form for a cubic crystal can be written[17] as

$$E_{ME}=B_1(\alpha_x^2 e_{xx}+\alpha_y^2 e_{yy}+\alpha_z^2 e_{zz})+B_2(\alpha_x\alpha_y e_{xy}+\alpha_y\alpha_z e_{yz}+\alpha_z\alpha_x e_{zx}), \qquad (39)$$

neglecting terms of higher powers of the direction cosines, α_x, α_y, α_z, of the magnetization. One might suspect that the anisotropy energies discussed in previous sections would also make contributions to this magnetoelastic energy. However, the observed deformations of FeO and CoO amount to as large as about 1 percent, which requires that B_1 or B_2 is at least about 10^3 cm^{-1} per ion, since the elastic constants have a magnitude of $10^{12}\sim10^{13}$ ergs per c.c. or $10^5\sim10^6$ cm^{-1} per ion. Such a large value of B_1 or B_2 cannot be expected from the origins hitherto investigated. We shall show that the crystalline field energy dependent on deformation is the predominant origin of the magnetostriction and that it can give the required mangitudes of the coupling constants B_1 and B_2. The residual orbital angular momentum $\boldsymbol{l}$ of magnitude 1 of each cation can freely take any direction if only the cubic field acting on it is present and the exchange interactions are neglected; a deformation will, however, remove the three-fold degeneracy and stabilize a certain direction or substate of the momentum, or inversely, a spontaneous deformation in a certain direction, a Jahn-Teller effect, will occur.* In the present case, however, the spins of the cations are coupled with one another by exchange interactions and each spin is coupled to the residual orbital angular momentum of the same cation. The crystalline field acts directly on this residual angular momentum. A deformation in a certain direction will here also stabilize the crystal and this direction will further determine the direction of the antiparallel magnetic moments. We shall show that this actually occurs in CoO, the axis of the deformation and magnetization being one of the cubic axes. In FeO, it seems that the deformation-independent anisotropy energy first determines the direction of the magnetic moments and the deformation then follows.

Other conceivable origins, such as spin-dependent indirect exchange interactions, whose strength is anisotropic since it depends on the orbital state of the cation, or also certain complicated second order effects, are of minor importance, and we shall neglect them and focus here our attention to the crystalline field effect.

As discussed in § 7 of [I], the crystalline field appears as a combined result of several effects which cause splittings of the orbital state. If we expand its potential equivalent in a power series of strain components, we can write its first order terms as

$$\begin{aligned} V_1=&K_1(e_{xx}+e_{yy}+e_{zz})+D_1(e_{xx}+e_{yy}+e_{zz})(x^4+y^4+z^4-3/5\cdot r^4)\\ &+A_1\{e_{xx}(3x^2-r^2)+e_{yy}(3y^2-r^2)+e_{zz}(3z^2-r^2)\}\\ &+C_1\{e_{xy}xy+e_{yz}yz+e_{zx}zx\}+F_1\{e_{xx}(7x^4-6x^2y^2+3/5\cdot r^4)\end{aligned}$$

* However, such a deformation does not occur in paramagnetic state, because orbital level splitting due to the deformation is smaller than kT.

$$+e_{yy}(7y^4-6y^2r^2+3/5\cdot r^4)+e_{zz}(7z^4-6z^2r^2+3/5\cdot r^4)\}$$
$$+H_1\{e_{xy}(x^3y+y^3x-6xyz^2)+e_{yz}(y^3z+z^3y-6yzx^2)$$
$$+e_{zx}(z^3x+x^3z-6zxy^2)\}. \tag{40}$$

As done in the calculation of orbital multipole interactions, the spacial coordinates in this formula can be replaced by the corresponding components of the pseudo-orbital angular momentum, $\boldsymbol{l}$, of the cation. Terms quartic in $\boldsymbol{l}$ reduce to quadratic terms. Thus, the general form of the magnetoelastic Hamiltonian can be expressed as

$$\mathcal{H}_{ME}=b_1\{e_{xx}l_x^2+e_{yy}l_y^2+e_{zz}l_z^2-(2/3)(e_{xx}+e_{yy}+e_{zz})\}$$
$$+b_2\{e_{xy}(1/2)(l_xl_y+l_yl_x)+e_{yz}(1/2)(l_yl_z+l_zl_y)$$
$$+e_{zx}(1/2)(l_zl_x+l_xl_z)\}, \tag{41}$$

where we dropped those terms which do not affect the cubic symmetry of the crystal.

The relations between b_1, b_2 and A_1, C_1, F_1, H_1 can be obtained by a calculation similar to that made in the case of orbital multipole interactions (see Appendix I, (Ap. 4) and (Ap. 5)). The results are

$$b_1=N\{(6/7)A_1\langle r^2\rangle-(8/21)F_1\langle r^4\rangle\},$$
$$b_2=N\{-(2/7)C_1\langle r^2\rangle+(8/21)H_1\langle r^4\rangle\}, \quad \text{for FeO,} \tag{42}$$

$$b_1=N[-\{(12/35)+(72/35)\alpha\beta-(54/35)\beta^2\}A_1\langle r^2\rangle$$
$$+\{(2/7)+(8/21)\alpha\beta-(2/7)\beta^2\}F_1\langle r^4\rangle],$$
$$b_2=N[\{(1/35)+(16/35)\alpha\beta+(13/35)\beta^2\}C_1\langle r^2\rangle \quad \text{for CoO.} \tag{43}$$
$$-\{(4/7)-(4/21)\alpha\beta-(4/7)\beta^2\}H_1\langle r^4\rangle].$$

Here N is the total number of the cation per c.c.

From (41), the magnetoelastic coupling constants B_1 and B_2 are calculated to be

$$B_1=(1/2)(a^2+c^2-2b^2)b_1 \text{ and } B_2=(1/2)(a^2+c^2-2b^2)b_2$$

by taking necessary averages with respect to the wave function of the ground state.

The equilibrium strain components e_{ij} are

$$e_{ii}=B_1\{(1/3)-\alpha_i^2\}/(c_{11}-c_{12}), \tag{44}$$
$$e_{ij}=-B_2\alpha_i\alpha_j/c_{44}, \quad (i,\ j=x,\ y,\ z). \tag{44'}$$

§ 7. Calculation of A_1, C_1, F_1 and H_1

The calculation A_1, C_1, F_1 and H_1 presents the same difficult problem as that we encountered in the calculation of the strength of the crystalline field in § 7 of [I]. In the case of cubic field, we saw that the simplest model of point charges located on lattice sites gave a good result. According to the discussion in § 7 of [I], this comes out from the compensation of various other effects. Though the same situation cannot necessarily

be expected here, we shall still use this model with a hope of obtaining a semiquantitative idea of A_1, etc.

Deferring the details of the calculation to Appendix II, we describe here the results. The following values of A_1, C_1, F_1 and H_1 are obtained:

$$A_1=-5.96(e^2/a^3),\quad C_1=11.93(e^2/a^3),$$
$$F_1=-16.61(e^2/a^5),\quad H_1=10.21(e^2/a^5), \tag{45}$$

where a is the distance between neighbouring cation and oxygen. Using these values and the values (21) of $\langle r^2\rangle$ and $\langle r^4\rangle$, the coupling constants B_1 and B_2 are obtained as

$$B_1=-8660\ \mathrm{cm}^{-1}\ \text{per ion or}\ -8.42\times10^{10}\ \text{erg per c.c.},$$
$$B_2=-6050\ \mathrm{cm}^{-1}\ \text{per ion or}\ -5.89\times10^{10}\ \text{erg per c.c.},\quad \text{for FeO}, \tag{46}$$

$$B_1=3930\ \mathrm{cm}^{-1}\ \text{per ion or}\ 4.07\times10^{10}\ \text{erg per c.c.},$$
$$B_2=200\ \mathrm{cm}^{-1}\ \text{per ion or}\ 0.21\times10^{10}\ \text{erg per c.c.},\quad \text{for CoO}. \tag{47}$$

These results predict in CoO a contraction along one of the $\{100\}$ directions if the magnetization is assumed to point in this direction, because B_1 is positive, and in FeO an elongation along $[111]$ if the magnetization is assumed to point in this direction, because B_2 is negative. These are in agreement with the experimental results. For quantitative calculations, we need the elastic constants of the crystals. Since they are not available, we use the following data of MgO[18]:

$$1/(c_{11}-c_{12})=s_{11}-s_{12}=4.94\times10^{-13}\ \text{c.c./erg},$$
$$1/c_{44}=s_{44}=6.46\times10^{-13}\ \text{c.c./erg},$$
$$1/(c_{11}+2c_{12})=s_{11}+2s_{12}=2.15\times10^{-13}\ \text{c.c./erg}. \tag{48}$$

Then our theory gives $\epsilon_{xy}=1.3$ percent for FeO and $(a-c)/a=2.0$ percent for CoO. Experiment on FeO at 95°K gives a rhombohedral angle $\alpha=59^\circ 32'$,[2] which gives $\epsilon_{xy}=0.705$ percent $(\partial\alpha=-2/\sqrt{3}\,\epsilon_{xy})$. Data for CoO[2),3)] are available at several temperature; extrapolating them to 0°K, we obtain about 1.2 percent for $(a-c)/a$.

We see that the point charge model gives fairly good results again in this case. Though the present calculation contains quantitatively ambiguous points, still it might not be useless to point out that the point charge approximation overestimates the coupling constants. As discussed in § 7 of [I], the covalent effect makes an important contribution to the strength of the cubic field in the direction cancelling the overlap effect of the electron clouds, but this effect will make a less important contribution to magnetoelastic coupling, because the predominant bond specified by t_1 can be formed almost irrespectively of the orbital state which the cation can take, and thus it cannot produce a large crystalline field of lower symmetry than cubic in the deformed crystal. Therefore, in the case of the magnetoelastic coupling, the point charge effect will considerably be reduced by the electron overlap effect. The anisotropy energy arising from magnetostriction will be discussed in § 9.

§ 8. Discussion of the exchange striction

As mentioned in introduction, there is another mechanism of the deformation, i.e. that arising from the exchange interactions between the spins. We shall estimate its order of magnitude in this section.

We shall first discuss the volume striction which arises from the exchange interactions between next-nearest neighbouring cations. In the molecular field approximation, the respective total energy is given by

$$-6N|J_1|\langle S\rangle^2,$$

of which the strain-dependent part can be written

$$-2N\langle S\rangle^2(\partial|J_{1z}|/\partial e_{zz})(e_{xx}+e_{yy}+e_{zz}), \tag{49}$$

J_{1z} being the exchange integral between next-nearest neighbouring cations which lie along the z-direction. The volume striction arising from this elasto-exchange coupling is given by

$$\Delta V/V=(e_{xx}+e_{yy}+e_{zz})=6N\langle S\rangle^2(\partial|J_{1z}|/\partial e_{zz})/(c_{11}+2c_{12}). \tag{50}$$

From this formula we obtain a contribution to the thermal expansion coefficient as

$$\beta=6N(\partial|J_{1z}|/\partial e_{zz})(\partial\langle S\rangle^2)/\partial T)/(c_{11}+2c_{12}). \tag{51}$$

If we assume the simple spin-only molecular field theory, $(\partial\langle S\rangle^2/\partial T)$ at the Néel temperature is given by

$$(\partial\langle S\rangle^2/\partial T)_{T=T_N}=-(k/6|J_1|)[40S^2(S+1)^2/\{(2S+1)^4-1\}]. \tag{52}$$

Using this relation, the anomalous expansion coefficient at the Néel temperature is given by

$$\beta|_{T=T_N}=-(Nk/c_{11}+2c_{12})[40S^2(S+1)^2/\{(2S+1)^4-1\}]\times(\partial|J_{1z}|/\partial e_{zz})/|J_1|. \tag{53}$$

M. Foex[19] measured the thermal expansion coefficients of the monoxides of Iron group elements in the temperature range between $-200°$C and $400°$C which covers the Néel temperatures of these oxides. The anomaly at Néel temperature is about 5×10^{-5} for MnO, 3×10^{-5} for FeO, 2×10^{-5} for CoO and 1.5×10^{-5} for NiO. Using the value of $(c_{11}+2c_{12})$ for MgO, we can deduce the values of $(\partial|J_{1z}|/\partial e_{zz})/|J_1|$ from these data. The results range from -5 for NiO to -15 for MnO. On the other hand, if we assume for the exchange integral and for the square of the transfer integral between cation and oxygen the same exponential dependence on distance as that we know in Born-Mayer's repulsive potential (see ref. 20), we obtain for this quantity a value of about -6, since the covalent degree and the exchange integral will contribute respectively about -3 to it.

In the second place, we shall discuss the deformation arising from the direct exchange interactions between nearest neighbouring cations. As Greenwald and Smart[8] suggested,

this mechanism will produce a deformation of trigonal symmetry in structure [A]. However, it might be mentioned that, if the interaction were of indirect nature similar to superexchange, it would not produce such a deformation, because a trigonal strain would not produce any large difference between the exchange integral relating parallel spins and that relating antiparallel spins.

When a shear strain e_{xy} is present, the change of distance between antiparallel spins is given by (see Fig. 2)

$$\partial r = (e_{xy}/2)\sqrt{2}\,a\,,$$

and the change of distance between parallel spins is $-\partial r$. The corresponding change of the total exchange energy is

$$-2N\langle S\rangle^2 e_{xy}\sqrt{2}\,a\,(\partial|J_2|/\partial r)\,,$$

and therefore the total elasto-exchange energy is

$$-2N\langle S\rangle^2\sqrt{2}\,a\,(\partial|J_2|/\partial r)\,(e_{xy}+e_{yz}+e_{zx})\,. \tag{54}$$

From (54), the equilibrium shear strains become

$$e_{xy}=e_{yz}=e_{zx}=-B_3/c_{44}\,, \tag{55}$$

where $B_3=-2N\langle S\rangle^2\sqrt{2}\,a\,(\partial|J_2|/\partial r)$. Assuming $\sqrt{2}\,a\,(\partial|J_2|/\partial r)/|J_2|=-3$* and $J_2=-5\ \mathrm{cm}^{-1}$, the following equilibrium values of e_{xy} are obtained for the four oxides:

$$e_{xy}=\begin{matrix}-11\times10^{-4}\ \text{for MnO},\quad -6\times10^{-4}\ \text{for FeO},\\ -3\times10^{-4}\ \text{for CoO},\quad -2\times10^{-4}\ \text{for NiO}.\end{matrix} \tag{56}$$

The observed rhombohedral deformation of NiO[21] at room temperatures corresponds to $e_{xy}=-1.2\times10^{-3}$, which is larger by a factor of 6 than the value above. The discrepancy might be due to ambiguities about the assumed values of $(\partial|J_2|/\partial r)$ and c_{11}. However, it is reported[22] that the deformation of NiO increases at lower temperatures to $e_{xy}=-3\times10^{-4}$ at -183°C. Such a large temperature dependency cannot be expected by the present calculation, since the Néel temperature of NiO is as high as 520°K.

§ 9. Discussion of the total anisotropy energies

Using (44), (44′) and (55), the energy gain arising from deformation is obtained as

$$\Delta\mathcal{E}=-(1/2)\,(B_1^2/(c_{11}-c_{12}))\sum_x\{(1/3)-\alpha_x^2\}^2-(1/2)\,(B_2^2/c_{44})\sum_{x,y}\alpha_x^2\alpha_y^2$$
$$-(B_2B_3/c_{44})\sum_{x,y}\alpha_x\alpha_y-(3/2)\,(B_3^2/c_{44})\,. \tag{57}$$

Consequently, the anisotropy energy arising from the deformation is given by

$$E_{ME}=K_{ME}(\alpha_x^2\alpha_y^2+\alpha_y^2\alpha_z^2+\alpha_z^2\alpha_x^2)+T_{ME}(\alpha_x\alpha_y+\alpha_y\alpha_z+\alpha_z\alpha_x)\,,$$

with

*) In going from MnO to MnS, the change $|J_2|$ is smaller than that calculated from this ratio under the assumption of the exponential form. However, the superexchange-type interaction will make some contribution to $|J_2|$. The increase of this interaction will compensate to some extent the decrease of the direct exchange interaction. Considering these circumstances, we have tentatively assumed the value of $\partial|J_2|/\partial r$ mentioned in the text.

$$K_{ME}=B_1^2/(c_{11}-c_{12})-B_2^2/2c_{44} \text{ and } T_{ME}=-B_2B_3/c_{44}. \quad (58)$$

From experimental data for CoO, an estimate of $B_1=2.4\times10^{10}$ erg per c.c.* is obtained. Using this value, $B_1^2/(c_{11}-c_{12})$ is calculated to 29.2×10^7 erg per c.c. or 28.2 cm^{-1} per ion. In the calculation of § 7 we obtained a much smaller value of B_2 than that of B_1 for CoO. Though we are not perfectly certain of this, we may neglect B_2 and obtain for the anisotropy constant $K_{ME}=28.2$ cm^{-1}. This corresponding anisotropy energy can easily overcome the anisotropy energy investigated in § 2–5, which amounted to values of the anisotropy constants of $-8\sim-15$ cm^{-1}, though minor neglected effects will affect these values to some extent. In particular, if the values of $\langle r^2\rangle$ and $\langle r^4\rangle$ estimated by using the Slater function were too large compared with their actual values, these values would have been somewhat reduced. We may conclude, however, that the magnetic moments orient along one of the directions {100} in CoO, these directions being favoured by K_{ME} and other opposing anisotropy energies being overcome by it.

In FeO, we obtain $B_2^2/2c_{44}=34.6\times10^7$ erg/c.c. or 35.5 cm^{-1} per ion, using experimental data. If we use this value and assume the calculated value for the ratio, B_1/B_2, we obtain $K_{ME}=75.7$ cm^{-1}. This value favours the directions {100} and is comparable in absolute magnitude with the largest value obtained for the anisotropy constant in deformation-free state. However, since experiment shows that the magnetic moments are in [111], the total anisotropy energy must favour the direction |111|. Thus the above value of K_{ME} would have been overestimated. Nevertheless, considering the reliability of our numerical estimations, we might conclude that the anisotropy energy in deformation-free state is more important in determining the direction of the magnetic moments in FeO.

In (58), there is a uniaxial anisotropy term arising from a combined action of exchange striction and magnetostriction. In FeO, this anisotropy energy favours the plane (111), since the sign of B_2 is opposite to that of B_3. If we assume the exchange striction to be $e_{xy}=-0.1$ percent and the magnetostriction to be $e_{xy}=0.7$ percent, the magnitude of T_{ME} will be about 3 cm^{-1}. This energy, as well as the energy given by T_D, is overcome by the anisotropy energy arising from the exchange energy.

If we assume spin-structure [A] for CoO, there must be a small deviation of the easy axes from the cubic axes. This arises from a combined action of the trigonal and cubic anisotropies. The angle of deviation ϑ is given by $T/\sqrt{2}K$ in the first order. Adopting $K=20$ cm^{-1} and $T=1$ cm^{-1}, ϑ turns out to be about 2°. It would be interesting to verify this experimentally, though perhaps very difficult.

§ 10. Remarks about the magnetic anisotropy of MnO, MnS and NiO

In this final section we shall briefly discuss the magnetic anisotropies of MnO, MnS

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* This value leads to $c=140$ cm^{-1}, where c is defined as the coefficient of the tetragonal crystalline field in § 6 of [I]. There we assumed a value $c=100$ cm^{-1}, which was obtained assuming a contraction of 1 percent. The difference does not affect the results obtained there.

and NiO. Since orbital degeneracy is completely removed in these substances, their magnetic anisotropy energies must be considerably smaller as compared with those of FeO and CoO. Though several origins of the anisotropy energy are conceivable, one concludes in the present stage of the theory that the inter-ionic magnetic dipole-dipole interactions make the largest contribution to it if structure |A| is valid for them. The lowering of the energy by an amount of the order of 0.1 cm^{-1} per ion due to a rhombohedral exchange striction in structure |A| favours this structure. Further, if the spins orient in the direction favoured by the inter-ionic magnetic dipole-dipole interactions, a further energy reduction of 0.9 cm^{-1} per ion occurs in the case of MnO. Also, by using the spin-wave theory, the present author has found that the zero-point energy of the spins prefers structure |A| to structure |B|, though the energy difference is very small. Therefore, it can theoretically be concluded that these oxides have spin-structure |A| and their spins lie in the plane (111). This conclusion might appear to contradict with the experimental results of Shull et al.[4] for MnO and NiO. However, a recent neutron diffraction experiment on MnS supports the above conclusion.* Furthermore, Singer's susceptibility measurements[27] on a single crystal of NiO, annealed by applying a pressure along one of the trigonal axes, appear to indicate that the spins lie in the plane perpendicular to this axis, i.e., in the (111) plane. Also, torque measurements made by Uchida and Kondoh[24] on a single crystal of NiO in magnetic field and an analysis of their results exclude the possibilities {100} and {111} for the direction of the spin axis.

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Appendix I. Calculation of the anisotropy energy arising from the classical orbital multipole interactions

The quadrupole-quadrupole interaction, (16), can be written in cartesian coordinates as

$$(e^2/R_{12}^3)\left[(3/2)\,(3z_i^2-r_i^2)\,(3z_j^2-r_j^2)+12z_i^2z_j^2-12z_iz_j(\boldsymbol{r}_i\cdot\boldsymbol{r}_j)\right.$$
$$\left.+(3/4)\,\{(r_i^2-z_i^2)\,(r_j^2-z_j^2)-2(x_iy_j-y_ix_j)^2\}\right]. \qquad \text{(Ap. 1)}$$

We define n_1, n_2 and n_3 as the direction cosines of the z-axis (which we take along the line joining the two cations) referred to the cubic axes, and denote the coordinates of the electrons referred to the same cubic axes by X, Y and Z. Then summation of (Ap. 1) over all the equidistant neighbours of the cation 1 yields

$$(e^2/R_{12}^5)\left[\{-(315/4)\sum n_1^2n_2^2+(63/4)\sum n_1^2\}\,(X_i^2X_j^2+Y_i^2Y_j^2+Z_i^2Z_j^2)\right.$$
$$+\{(105/2)\sum n_1^2n_2^2-(21/2)\sum n_1^2\}\,(2X_iY_iX_jY_j+2Y_iZ_iY_jZ_j+2Z_iX_iZ_jX_j)$$
$$\left.+\{(105/4)\sum n_1^2n_2^2-(21/4)\sum n_1^2\}\,R_i^2R_j^2\right]. \qquad \text{(Ap. 2)}$$

As mentioned in the text, we do not distinguish between the coordinates of the electrons

* See also the footnote on page 2 of this paper.

belonging to different neighbouring cations, since they have the same spatial distribution relative to each of their nuclei. Further, summation must be taken over all the pairs of the *d*-electrons belonging to the interacting cations, but we dispensed with this summation in the above formulas for the sake of simplicity.

The quadrupole-hexadecapole interaction can be transformed in a similar way. Except for isotropic terms, the result is given by

$$\begin{aligned}
&15(e^2/R_{ij}^5)\big[\{(693/16)\textstyle\sum n_1^2 n_2^4-(63/4)\sum n_1^2 n_2^2-(9/16)\sum n_1^2\}\\
&\times\{Z_i^2(X_j^4+Y_j^4-6X_j^2Y_j^2)+X_i^2(Y_j^4+Z_j^4-6Y_j^2Z_j^2)+Y_i^2(Z_j^4+X_j^4-6Z_j^2X_j^2)\}\\
&+\{(231/2)\textstyle\sum n_1^2 n_2^4-42\sum n_1^2 n_2^2-(3/2)\sum n_1^2\}\{X_iY_i(X_j^3Y_j+Y_j^3X_j-6X_jY_jZ_j^2)\\
&+Y_iZ_i(Y_j^3Z_j+Z_j^3Y_j-6Y_jZ_jX_j^2)+Z_iX_i(Z_j^3X_j+X_j^3Z_j-6Z_jX_jY_j^2)\}\big].
\end{aligned}\tag{Ap. 3}$$

We can obtain a similar formula for the hexadecapole-hexadecapole interactions, but we shall not write it down here.

The operator-equivalence relations[24] between spacial coordinates and orbital angular momenta are well known in the calculation of crystalline field splittings. In our case, the orbital momenta are further replaced by pseudo-orbital angular momenta $\boldsymbol{l}_i$. The relations between them are easily obtained by calculating the matrix elements. Thus, the spacial coordinates in (Ap. 2) and (Ap. 3) are replaced by the following relations:

for FeO,

$$\begin{aligned}
&Z^2=(2/7)\langle r^2\rangle l_z^2,\ XY=-(1/7)\langle r^2\rangle(l_xl_y+l_yl_x),\\
&X^4+Y^4-6X^2Y^2=(8/21)\langle r^4\rangle l_z^2,\\
&X^3Y+Y^3X-6XYZ^2=(4/21)\langle r^4\rangle(l_xl_y+l_yl_x),
\end{aligned}\tag{Ap. 4}$$

and for CoO,

$$\begin{aligned}
&Z^2=\{-(4/35)-(24/35)\alpha\beta+(18/35)\beta^2\}\langle r^2\rangle l_z^2,\\
&XY=\{(1/70)+(8/35)\alpha\beta+(13/70)\beta^2\}\langle r^2\rangle(l_xl_y+l_yl_x),\\
&X^4+Y^4-6X^2Y^2=\{-(2/7)-(8/21)\alpha\beta+(2/7)\beta^2\}\langle r^4\rangle l_z^2,\\
&X^3Y+Y^3X-6XYZ^2=\{-(2/7)+(2/21)\alpha\beta+(2/7)\beta^2\}\langle r^4\rangle(l_xl_y+l_yl_x).
\end{aligned}\tag{Ap. 5}$$

The constant terms which are not essential are neglected in the above formulas.

As examples, the interactions between nearest neighbouring cations in FeO are given by the following formulas:

the quadrupole-quadrupole interaction is

$$\begin{aligned}
\mathcal{H}_{QQ}=(e^2\langle r^2\rangle^2/R_0^5)\big[&-(9/14)(l_{x1}^2l_{x2}^2+l_{y1}^2l_{y2}^2+l_{z1}^2l_{z2}^2)\\
&+(3/14)\{(l_{x1}l_{y1}+l_{y1}l_{x1})(l_{x2}l_{y2}+l_{y2}l_{x2})\\
&+(l_{y1}l_{z1}+l_{z1}l_{y1})(l_{y2}l_{z2}+l_{z2}l_{y2})\\
&+(l_{z1}l_{x1}+l_{x1}l_{z1})(l_{z2}l_{x2}+l_{x2}l_{z2})\big];
\end{aligned}\tag{Ap. 6}$$

the hexadecapole-quadrupole interaction is

$$\begin{aligned}\mathcal{H}_{HQ}=(e^2\langle r^2\rangle\langle r^4\rangle/R_0^7)[&(585/98)(l_{x1}^2l_{x2}^2+l_{y1}^2l_{y2}^2+l_{z1}^2l_{z2}^2)\\&+(195/49)\{(l_{x1}l_{y1}+l_{y1}l_{x1})(l_{x2}l_{y2}+l_{y2}l_{x2})\\&+(l_{y1}l_{z1}+l_{z1}l_{y1})(l_{y2}l_{z2}+l_{z2}l_{y2})\\&+(l_{z1}l_{x1}+l_{x1}l_{z1})(l_{z2}l_{x2}+l_{x2}l_{z2})\}];\end{aligned}\qquad\text{(Ap. 7)}$$

and the hexadecapole-hexadecapole interaction is

$$\mathcal{H}_{HH}=(e^2\langle r^4\rangle^2/R_0^9)(495/56)(l_{x1}^2l_{x2}^2+l_{y1}^2l_{y2}^2+l_{z1}^2l_{z2}^2).\qquad\text{(Ap. 8)}$$

In these formulas, R_0 is the distance between the nearest neighbours, and each interaction was divided by 2 for the purpose of estimating the anisotropy energy per ion.

Using the values of $\langle r^2\rangle$ and $\langle r^4\rangle$ given by (21) or (22), the ratios between the contributions from multipole interactions (Ap. 6), (Ap. 7) and (Ap. 8) to the anisotropy constant are calculated to be

$$K'_{HQ}=\begin{matrix}-2.0\,K'_{QQ}\\-1.55\,K'_{QQ}\end{matrix},\quad K'_{HH}=\begin{matrix}-0.20\,K'_{QQ}\\-0.12\,K'_{QQ}\end{matrix},\qquad\text{(Ap. 9)}$$

where the upper values are obtained using (21) and the lower ones using (22). The value of K'_{QQ} is given by

$$K'_{QQ}=\begin{matrix}+53.7\ \mathrm{cm}^{-1}\\+37.9\ \mathrm{cm}^{-1}\end{matrix}.\qquad\text{(Ap. 10)}$$

Adding the contributions from other surrounding cations, which are mainly due to quadrupole-quadrupole interactions and which amount to $-8.3\ \mathrm{cm}^{-1}$ in total, we obtained the final results of the anisotropy constant (23) and (24).

In the case of CoO, the relations (Ap. 9) between the magnitudes of the interactions are approximately the same as in the case of FeO. Since the Hamiltonians $\mathcal{H}_{QQ}$, etc., can easily be obtained from those of FeO, taking account of different factors appearing in (Ap. 5), they will not here be reproduced. In this case, we obtain smaller values of the anisotropy constant, because $\langle r^2\rangle$ and $\langle r^4\rangle$ have a smaller value and also the factors appearing in (Ap. 4) and (Ap. 5) are different.

Appendix II. Calculation of A_1, C_1, F_1, H_1 with the point charge model

In this model, in which the ions surrounding a given cation are approximated by point charges, the electrostatic potential for the electrons can be expanded in a series of spherical harmonics:

$$\begin{aligned}V=&A(3z^2-r^2)+Cxy+F(7z^4-6z^2r^2+3/5\cdot r^4)\\&+H(x^3y+y^3x-6z^2xy)+\cdots\cdots\end{aligned}\qquad\text{(Ap. 11)}$$

The coefficients, A, C, F and H, are given by the summations,[26]

$$A=-(1/4)e\sum_j{}'(e_j/R_j^5)(3Z_j^5-R_j^2),\quad C=-3e\sum_j{}'(e_j/R_j^5)X_jY_j,$$

$$F=-(5/64)e\sum_j{}'(e_j/R_j^9)[5(7Z_j^4-6Z_j^2R_j^2+(3/5)R_j^4)-7(X_j^4+Y_j^4-6X_j^2Y_j^2)],$$

$$H=-(75/2)e\sum_j{}'(e_j/R_j^9)(X_j^3Y_j+Y_j^3X_j-6X_jY_jZ_j^2), \qquad \text{(Ap. 12)}$$

where X_j, etc., are the coordinates of the j-th lattice point, e_j the point charge on it. A_1, C_1, F_1 and H_1 are the derivatives of these coefficients with respect to strain components;

$$A_1=(\partial A/\partial e_{zz})=-(1/4)\sum_j{}'e_jZ_j\cdot\partial/\partial Z_j\cdot[(3Z_j^2-R_j^2)/R_j^5]$$
$$=(3e/4)\sum_j{}'(e_j/R_j^7)(5Z_j^4-R_j^4),$$

$$C_1=(\partial C/\partial e_{xy})=-(3e/2)\sum_j{}'e_j(X_j\cdot\partial/\partial Y_j+Y_j\cdot\partial/\partial X_j)X_jY_j/R_j^5=-2A_1,$$

$$F_1=-(35/2)(7e/8)\sum_j{}'(e_j/R_j^9)(5Z_j^4-R_j^4)$$
$$+(135e/16)\sum_j{}'(e_j/R_j^{11})(7Z_j^6-R_j^6),$$

$$H_1=25(7e/8)\sum_j{}'(e_j/R_j^9)(5Z_j^4-R_j^4)-(45e/4)\sum_j{}'(e_j/R_j^{11})(7Z_j^6-R_j^6).$$

(Ap. 13)

These formulas are obtained by considering the cubic symmetry of the lattice. The sum $D=-(7e/8)\sum_j{}'(e_j/R_j^9)(5Z_j^4-R_j^4)$ is the coefficient of the cubic crystalline field, which was calculated by Bethe[27] to be

$$D=7.16\,(e^2/a^5),$$

a being the distance between the nearest neighbouring cation and anion. Other two sums appearing in (Ap. 13) are calculated by the direct summation method to be

$$\sum_j{}'(ee_j/R_j^7)(5Z_j^4-R_j^4)=-7.95\,(e^2/a^3),$$
$$\sum_j{}'(ee_j/R_j^{11})(7Z_j^6-R_j^6)=-16.819\,(e^2/a^5). \qquad \text{(Ap. 14)}$$

Using these values, (45) of the text is obtained.

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Antiferromagnetism of Zn-Ferrite

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Assuming a reasonable magnetic superstructure of tetragonal symmetry, the magnetic anisotropy of Zn-ferrite is discussed. Magnetic moment carriers are Fe^{3+} ions in this substance. Therefore, the main source of the anisotropy energy is considered to be the magnetic dipole-dipole interaction between spins of Fe^{3+} ions, which gives the anisotropy energy depending upon the crystallographic orientation of the common axis of + and − spins below the Néel temperature, whereas above this temperature it does not give any anisotropy on account of the cubic structure of this crystal. The calculated anisotropy energy below the Néel temperature takes a minimum value when the direction of the magnetic moment lies in the plane perpendicular to the tetragonal c-axis. If the anisotropy in this plane is sufficiently small, the powder susceptibility would remain constant below the Néel temperature. This is supported by the experimental result.

Further, the short range order is introduced by a method somewhat like that used by Li. The behaviors of the susceptibility and the specific heat near the Néel temperature are investigated by this method. The effect of the short range order on the susceptibility is shown to be considerably remarkable, but the constancy of the perpendicular susceptibility with respect to temperature is expected to remain almost unchanged.

§ 1. Introduction

According to Néel, the ferromagnetism of the ferrites of Mn, Fe, Co and Ni occurs by a strong antiferromagnetic coupling between spins of the paramagnetic ions on the tetrahedral sites and those on the octahedral sites. Néel's analysis of the experimental data on the paramagnetic susceptibility of magnetite and Ni-ferrite clarified the fact that there exists, though weaker, an interaction between spins of the octahedral sites. This interaction is ferromagnetic in magnetite and antiferromagnetic in Ni-ferrite. In Zn-ferrite, since Zn ions which have been believed to prefer tetrahedral sites are non-magnetic, the exchange interactions between spins of Fe^{3+} ions occupying the octahedral sites have a direct influence on the magnetic behavior of this substance, in contrast with the case of other ferromagnetic ferrites. These interactions should operate antiferromagnetically, if one infers from the case of Ni-ferrite. (In magnetite, there would exist between Fe^{3+} and Fe^{2+} ions of the octahedral sites ferromagnetic double exchange coupling in addition to antiferromagnetic superexchange coupling.)

According to the preliminary short reports recently published by Arrott and Goldman[1] and Friedberg and Burk[2], the susceptibility of Zn-ferrite is constant up to 30°K and above this temperature it decreases obeying the Curie-Weiss law, the paramagnetic Curie temperature being about 40°K. An anomaly in its specific heat is also observed at 9.5°K,

this corresponding to the Néel temperature. The main features of the antiferromagnetic properties of this substance discovered by the above authors are the following:
1) The ratio of the paramagnetic Curie temperature to the antiferromagnetic Néel temperature is considerably large, being equal to 4.2. 2) The susceptibility of powder specimens is constant at low temperatures. 3) The effect of the short range order is remarkable in the behavior of the susceptibility and specific heat above the Néel temperature.

Since the determination of the magnetic superstructure of this substance has not been completed though the complex pattern has been reported by Corliss and Hastings[3], we assume tentatively a probable superstructure with which we calculate the anisotropy energy and magnetic susceptibility and compare them with the experimental results, focusing our major interest on the above mentioned three features.

§ 2. Molecular field approximation

As seen from the crystal structure of Zn-ferrite shown by Fig. 1, each Fe^{3+} ion has six nearest neighbors of Fe^{3+} ions and twelve second nearest neighbors. The interaction with the first neighbor ions seems weak, judging from Anderson's theory of superexchange, because these ions are situated in the perpendicular direction to each other with respect to the nearest oxygen ions. As for the second neighbor pair of Fe^{3+} ions, the

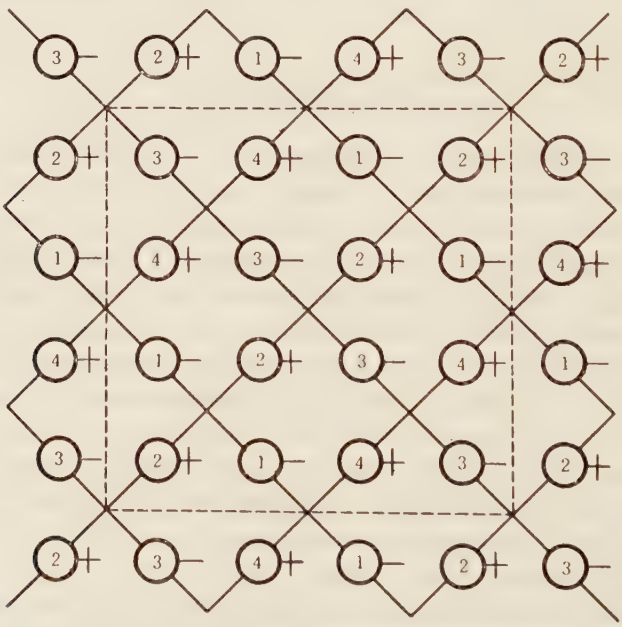

Fig. 1. Projection of the magnetic structure of the octahedral sites of Zn-ferrite on the plane perpendicular to the c-axis. The number means the height of Fe^{3+} ion in the unit of one fourth of the lattice parameter. + and − signs show two sublattices.

angle of Fe-O-Fe is about 125° but the relevant distance is $\sqrt{3}$ times as large as that between the nearest neighbors. Therefore, the second neighbors interaction is probably also weak. However, if we compare this crystal structure with that of MnO, the first neighbor pair in Zn-ferrite corresponds to the pair of the first neighbor Mn^{2+} ions in MnO and the second neighbor pair to the third neighbor pair in MnO. In antiferromagnetic crystals with NaCl structure such as MnO, FeO, etc., the interaction between the first neighbor pair is considerably large, though each ion is in the perpendicular direction from the oxygen ion. In the case of MnO, this exchange integral for the nearest pair is twice as large as that of the superexchange interaction for the second neighbor pair. Therefore we consider here that the main antiferromagnetic coupling operates between the first neighbor Fe^{3+} ions and the rather weak antiferromagnetic coupling exists between the second neighbor ions.

Assuming this, the most stable superstructure of + and − spins in Zn-ferrite would be such that the spins of ions on a layer perpendicular to one of the cubic axes are parallel to each other and layers of + and − spins are alternately piled up. This kind of superstructure is the same as the ordered arrangement of Fe^{2+} and Fe^{3+} ions in magnetite below its transition temperature, which has been proposed by Verwey, Haayman and Romeijn[4]. In this superstructure, as shown in Fig. 1, a plus spin has four nearest minus spins, two nearest plus spins and eight next nearest minus spins and four next nearest plus spins and vice versa. Therefore, this structure is stable with regard to the nearest and next to nearest antiferromagnetic interactions. Recently Anderson[5] showed that in the octahedral sites of ferrites the nearest neighbor interaction alone does not produce any long range order. Therefore in the ordering mechanism of the present superstructure the second neighbor interaction would play an indispensable role. Besides the second neighbor interaction, also the magnetic dipole-dipole interaction may contribute something to the formation of the long range order. The knowledge of the actual superstructure of Zn-ferrite must await a further analysis of neutron diffraction patterns.* However, we shall investigate what kinds of magnetic properties can be expected from the superstructure assumed here.

First we shall deal with the molecular field approximation. In our superstructure, a spin on the + sublattice has four nearest and eight next nearest spins on the − sublattice and two nearest and four next nearest spins on the + sublattice, and vice versa, so that the molecular fields H^+ and H^- acting on + and − spins are expressed by

$$\boldsymbol{H}^{\pm} = -\tfrac{1}{2}A\boldsymbol{M}^{\pm} - A\boldsymbol{M}^{\mp}, \qquad (2\cdot1)$$

where $\boldsymbol{M}^+$ and $\boldsymbol{M}^-$ represent respectively the magnetizations of the + and − sublat-

*) Recently, Hastings and Corliss have published the result of the neutron diffraction analysis on Zn-ferrite (Phys. Rev. **102** (1956), 1460). The superstructure assumed by them is very complicated and evidently differs from ours. However, this may not be a final decision and our calculation developed in the following sections can be applied to other superstructures in which any tetrahedron contains two plus spins and two minus spins.

tices and A is the so-called molecular field coefficient equal to $16(J_1+2J_2)/Nf^2\mu_B^2$, where J_1 and J_2 mean the exchange integrals for the nearest pair and the next nearest pair, and N is the number of Fe^{3+} ions. Above the Néel temperature the magnetization induced by the external field H is given by

$$\boldsymbol{M}^{\pm}=\frac{C}{2T}\left(-\tfrac{1}{2}A\boldsymbol{M}^{\pm}-A\boldsymbol{M}^{\mp}+\boldsymbol{H}\right), \tag{2·2}$$

where C means the Curie constant which is equal to $Nf^2\mu_B^2S(S+1)/3k$. (2·2) gives rise to the following value of the susceptibility:

$$\chi=\frac{C}{T+\frac{3}{4}AC}. \tag{2·3}$$

The Néel temperature becomes

$$T_N=AC/4. \tag{2·4}$$

Therefore the molecular field approximation gives a value of $\Theta/T_N=3$, which comes out to be rather near the experimental value of 4.2.

Since the assumed structure has tetragonal symmetry, a slight tetragonal deformation similar to a trigonal deformation found in MnO and NiO[6] is expected below the Neel temperature. Now we shall denote the strain component in the direction of the tetragonal c-axis by e_{zz}. By this strain the exchange interaction between spins in the different planes perpendicular to c-axis undergoes some change but that between spins in the same c-plane does not change. The total change of exchange energy neglecting the second neighbor interaction becomes

$$\Delta E_{ex}=-2\sum_{(i,j)}(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)\cdot\frac{dJ_1(r_{ij})}{dr_{ij}}\frac{z_{ij}^2}{r_{ij}}e_{zz},$$

where the nearest neighbor exchange interaction depends only upon the distance between two spins. Using the classical approximation, this becomes

$$\Delta E_{ex}=-\frac{1}{\sqrt{2}}NS^2\frac{d|J_1|}{dr}a\cdot e_{zz}, \tag{2·5}$$

where a means the lattice parameter and N is the total number of spins. On the other hand, the increase of the elastic energy due to the strain e_{zz} has the quadratic form of $\frac{1}{2}c_{11}e_{zz}^2$ where c_{11} is the usual elastic stiffness constant in the cubic crystal. The spontaneous strain which is expected below the Neel temperature is determined by minimizing the exchange and elastic energies with respect to the strain e_{zz}. The change of energy and the spontaneous strain obtained in this manner are as follows:

$$\Delta E=-\tfrac{1}{2}c_{11}e_{zz}^2, \qquad e_{zz}=\frac{1}{c_{11}}\frac{N}{\sqrt{2}}S^2\frac{d|J_1|}{dr}a. \tag{2·6}$$

If we tentatively put $c_{11}\sim10^{12}$ dyne/cm^2 and $\frac{d|J_1|}{dr}a\sim-10^{-15}\sim10^{-14}$ erg, then e_{zz}

and ΔE become $-10^{-3}\sim10^{-4}$ and $-1\ \mathrm{cm}^{-1}\sim10^{-2}\ \mathrm{cm}^{-1}$ per spin. Therefore, the lowering of the energy for the present superstructure due to the spontaneous deformation is about one percent of the exchange energy, assuming the latter energy as about $10\ \mathrm{cm}^{-1}$ per spin.

The anisotropy energy of this antiferromagnetic substance arises from the magnetic dipole-dipole interaction between spins and from the anisotropy of the crystalline field. The former part of anisotropy appears when the superstructure is formed and has tetragonal symmetry and vanishes above the Néel temperature. The latter part has cubic symmetry and is expected to have a smaller value less than $10^{-2}\ \mathrm{cm}^{-1}$ per ion. Therefore, the main part of the anisotropy energy can be considered to come from the former part. This situation remains unchanged in Zn-chromite too, because a Cr^{3+} ion has the spin of 3/2 and its anisotropy from the cubic crystalline field identically vanishes.

The numerical calculation of the magnetic dipole-dipole interaction by means of Evjen's direct sum method gives the following result, putting the η-value of Fe^{3+} equal to 2 and using the lattice parameter $a=8.416$ A:[7]

$$E_A=0.935\ \gamma^2\ \mathrm{cm}^{-1}\ \text{per ion}, \tag{2·7}$$

where γ is the direction cosine of the common axis of the spins with respect to the c-axis. The sign of (2·7) shows that the preferred direction of spins lies in the plane perpendicular to the c-axis. Therefore, if we assume the anisotropy energy in this plane to be of the order of less than $10^{-2}\ \mathrm{cm}^{-1}$, the critical field for the rotation of spins becomes less than 4000 oersteds, and hence the susceptibility of powder samples (or any other samples) below the Néel temperature becomes constant for external field exceeding 4000 oersteds, being equal to the constant perpendicular susceptibility given by $1/A$ which is independent of temperature. This is consistent with the experimental result.

(2·7) also shows that the magnetic dipole-dipole energy stabilizes the present superstructure by an amount of $-0.31\ \mathrm{cm}^{-1}$ per spin, this being about 3 percent of the exchange energy.

Above the Néel temperature the anisotropy from the magnetic dipole-dipole interaction vanishes as mentioned above and only a smaller cubic anisotropy arising from the crystalline field remains. Therefore we can neglect the anisotropy energy for the discussion of the susceptibility above the Néel temperature. The susceptibility above the Néel temperature measured by Arrott and Goldman scarcely decreases from the Néel temperature to about 30°K. They have pointed out that this might be due to a strong short range order remaining above the Néel temperature. In the next section we shall introduce the short range order and investigate this effect on the susceptibility above the Néel temperature.

§ 3. Introduction of the short range order

The theory of ferromagnetism which takes into consideration the short range order has been developed by P. R. Weiss[8]. Li[9] extended the Weiss theory to the case of antiferromagnetism and discussed the effect of the short range order on the magnetic

properties of antiferromagnets. However, Li's method when applied to the present case would become very troublesome because of a large spin value and of the complex crystal structure. We shall adopt a simpler and more adequate model for the present magnetic structure.

We shall divide the whole octahedral sites into linked clusters each consisting of four Fe^{3+} ions and forming a tetrahedron. We shall approximate the exchange interactions between each spin within one cluster and the spins outside it by an effective molecular field acting on the former spin. This effective molecular field will be determined by a condition of self-consistency similar to but simpler than that used by Li. In this approximation, a weak second neighbor interaction scarcely influences the result as seen from the subsequent calculation, so that we neglect such an interaction in the following. In the antiferromagnetic ordered state, two of the four spins belonging to this cluster occupy the + sublattice and the other two the − sublattice. Then we have the following Hamiltonian for this four spin system:

$$\begin{aligned}\mathcal{H} = {} & 2J(\boldsymbol{S}_1\cdot\boldsymbol{S}_2+\boldsymbol{S}_1\cdot\boldsymbol{S}_3+\boldsymbol{S}_1\cdot\boldsymbol{S}_4+\boldsymbol{S}_2\cdot\boldsymbol{S}_3+\boldsymbol{S}_2\cdot\boldsymbol{S}_4+\boldsymbol{S}_3\cdot\boldsymbol{S}_4)\\ & +g\mu_B H^+(S_{1z}+S_{2z})+g\mu_B H^-(S_{3z}+S_{4z})\\ & +g\mu_B H_0(S_{1z}+S_{2z}+S_{3z}+S_{4z}),\end{aligned} \tag{3·1}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ represent spins of the + sublattice and $\boldsymbol{S}_3$ and $\boldsymbol{S}_4$ those of the − sublattice and the external magnetic field $\boldsymbol{H}_0$ and the molecular fields $\boldsymbol{H}^+$ and $\boldsymbol{H}^-$ acting on the + and − sublattices are supposed to be both in the direction of the z-axis. If we put

$$\begin{aligned}&\boldsymbol{S}=\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{S}_3+\boldsymbol{S}_4,\\ &\boldsymbol{S}^+=\boldsymbol{S}_1+\boldsymbol{S}_2,\ \boldsymbol{S}^-=\boldsymbol{S}_3+\boldsymbol{S}_4\end{aligned} \tag{3·2}$$

and include $g\mu_B$ into H^+, H^- and H_0, (3·1) can be written as

$$H=JS^2+S_z^+H^++S_z^-H^-+S_zH_0, \tag{3·3}$$

where constant terms are omitted. Here, $S_z^+H^+$ and $S_z^-H^-$ are, respectively, introduced approximately in place of

$$2J(\boldsymbol{S}_1\cdot\textstyle\sum_i'\boldsymbol{S}_i+\boldsymbol{S}_2\cdot\sum_i'\boldsymbol{S}_i) \text{ and}$$

$$2J(\boldsymbol{S}_3\cdot\textstyle\sum_i'\boldsymbol{S}_i+\boldsymbol{S}_4\cdot\sum_i'\boldsymbol{S}_i),$$

where $\boldsymbol{S}_i$'s represent nearest neighboring spins of $\boldsymbol{S}_1$, etc., if they appear in the term $\boldsymbol{S}_1\cdot\sum_i'\boldsymbol{S}_i$, etc., which are not included in the cluster. The molecular field approximation means to replace $\boldsymbol{S}_i$ by its average value $\frac{1}{2}\cdot\langle S_z^+\rangle$ or $\frac{1}{2}\cdot\langle S_z^-\rangle$ according as it belongs to the + sublattice or the − sublattice. Therefore the molecular field can be expressed by

$$\begin{aligned}&H^+=2J(\tfrac{1}{2}\cdot\langle S_z^+\rangle+\langle S_z^-\rangle),\\ &H^-=2J(\tfrac{1}{2}\langle S_z^-\rangle+\langle S_z^+\rangle),\end{aligned} \tag{3·4}$$

because one nearest neighbor of $\boldsymbol{S}_1$ or $\boldsymbol{S}_2$, for example, not belonging to that cluster is

situated on the + sublattice and the other two on the − sublattice.

For the purpose of obtaining the eigen-values of Hamiltonian (3·3), we adopt the representation which diagonalizes the magnitudes of $\boldsymbol{S}$, $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$, and the z componet of the total spin S_z. In this scheme the z components of $\boldsymbol{S}$ and $\boldsymbol{S}^-$ have non-diagonal elements. Therefore we shall divide $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$ into the diagonal part which is proportional to $\boldsymbol{S}$ and the non-diagonal part which is proportional to $\boldsymbol{S}'$ as follows:

$$\boldsymbol{S}^+ = a\boldsymbol{S} - \boldsymbol{S}', \quad \boldsymbol{S}^- = b\boldsymbol{S} + \boldsymbol{S}', \tag{3·5}$$

where

$$a = \frac{1}{2}\left(1 + \frac{S^+(S^+ + 1) - S^-(S^- + 1)}{S(S+1)}\right),$$

$$b = \frac{1}{2}\left(1 + \frac{S^-(S^- + 1) - S^+(S^+ + 1)}{S(S+1)}\right). \tag{3·6}$$

Using (3·5) in (3·3), we get

$$\mathcal{H} = (JS^2 + S_z H_0 + aS_z H^+ + bS_z H^-) + S_z'(H^- - H^+). \tag{3·7}$$

The first term represents the diagonal part, while the second term represents the non-diagonal part. Since above the Néel temperature H^- and H^+ are proportional to the external field H_0 and just below it they are small, we can calculate the eigen-values of Hamiltonian (3·7) by the perturbation method. The non-vanishing matrix elements of S_z' are as follows:

$$\langle S+1, M|S_z'|S, M\rangle = \rho(S+1, S^+, S^-)\sqrt{(S+M+1)}\,(S-M+1),$$

$$\langle S-1, M|S_z'|S, M\rangle = \rho(S, S^+, S^-)\sqrt{(S+M)\,(S-M)}, \tag{3·8}$$

where

$$\rho(S, S^+, S^-) = \left[\frac{\{(S^+ + S^- + 1)^2 - S^2\}\,\{S^2 - (S^+ - S^-)^2\}}{4S^2(4S^2 - 1)}\right]^{\frac{1}{2}}. \tag{3·9}$$

The eigen-values of Hamiltonian calculated with the use of (3·8) and (3·9) in the second order approximation with respect to H^+ or H^- become

$$E = JS(S+1) + MH_0 + aMH^+ + bMH^- - c(H^- - H^+)^2, \tag{3·10}$$

where

$$c(S, S^+, S^-, M) = \frac{1}{2J}\left\{\frac{1}{(S+1)}\rho^2(S+1, S^+, S^-)\,(S+M+1)\,(S-M+1)\right.$$

$$\left. - \frac{1}{S}\rho^2(S, S^+, S^-)\,(S+M)\,(S-M)\right\}. \tag{3·11}$$

Correspondingly, the partition function of the cluster can be written as

$$P(H_0, H^+, H^-, T) = \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} \exp\left[-\frac{1}{kT}\{MH_0 + aMH^+ + bMH^- - C(H^- - H^+)^2\}\right]. \tag{3·12}$$

The average values of S_z^+ and S_z^- are given by the following relations:

$$\langle S_z^+ \rangle P = -kT \frac{\partial}{\partial H^+} P,$$

$$\langle S_z^- \rangle P = -kT \frac{\partial}{\partial H^-} P. \tag{3·13}$$

H^+ and H^- are determined by these equations together with (3·4).

If we develop both sides of (3·13) with respect to H^+ and H^- up to their first order and put the external field equal to zero, we obtain the following relations which determine $\langle S_z^+ \rangle$ and $\langle S_z^- \rangle$ in the absence of external field:

$$\sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} \left\{1 + 2J\left[\frac{M^2}{2kT} a(1+b) - c\right]\right\} \langle S_z^+ \rangle + \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} 2J\left[\frac{M^2}{2kT} a(1+a) + c\right] \langle S_z^- \rangle = 0,$$

$$\sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} 2J\left[\frac{M^2}{2kT} b(1+b) + c\right] \langle S_z^+ \rangle + \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} \left\{1 + 2J\left[\frac{M^2}{2kT} b(1+a) - c\right]\right\} \langle S_z^- \rangle = 0. \tag{3·14}$$

The Néel temperature can be determined by the condition that the determinant formed by the coefficients of $\langle S_z^+ \rangle$ and $\langle S_z^- \rangle$ in (3·14) vanishes. This condition is equivalent to the following equation:

$$\sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} \left(1 + 2J\left[\frac{a(b-a)}{2kT} M^2 - 2c\right]\right) = 0. \tag{3·15}$$

The numerical solution for (3·15) can be obtained as follows:

$$J/kT_N = 0.142.$$

On the other hand the simple molecular field approximation gives us the value of $J/kT_N' = 0.0857$. Therefore, $T_N'/T_N = 1.66$. According to Li's calculation for $S = 1/2$ and the simple cubic lattice, the value of the above ratio is 1.5 and for the body centered lattice it is 1.25. Though the degree of approximation is different between Li's case and ours, the lowering of the Néel temperature due to the short range order found in our case seems to be reasonable.

§ 4. Susceptibility above the Néel temperature

In this section we shall calculate the susceptibility above the Néel temperature. In this temperature range, the difference between the + and − sublattices vanishes and $\langle S_z^+ \rangle$ and $\langle S_z^- \rangle$ become equal to each other and so we can put $\langle S_z^+ \rangle = \langle S_z^- \rangle = \frac{1}{2}\langle S_z \rangle$ and $H_i = H^+ = H^-$. Then Hamiltonian (3·3) can be written as

$$\mathcal{H} = J\boldsymbol{S}^2 + S_z(H_0 + H_i). \tag{4·1}$$

The eigen-values of (4·1) are

$$E = JS(S+1) + (H_0 + H_i)M.$$

Therefore, the partition function P is obtained as

$$P = \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-} \exp\left(-\frac{M(H_0+H_i)}{kT}\right). \tag{4·2}$$

The relation (3·4), which determines the internal field H_i, is written as

$$H_i = \tfrac{3}{2} J\langle S_z \rangle. \tag{4·3}$$

On the other hand $\langle S_z \rangle$ can be calculated by the following relation:

$$\langle S_z \rangle P = -kT\frac{\partial P}{\partial H_0}. \tag{4·4}$$

Expanding this equation with respect to H_0 and H_i, the linear part gives rise to

$$\langle S_z \rangle \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} \left(1 + \frac{3J}{2kT}M^2\right)$$

$$= -(H_0/kT) \cdot \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} M^2. \tag{4·5}$$

Since the susceptibility is given by

$$\chi = \frac{Ng^2 \mu_B^2}{4H_0} \langle S_z \rangle, \tag{4·6}$$

where N is the number of Fe^{3+} ions, χ has the following form:

$$\chi = \frac{C(T)}{T + \theta(T)}. \tag{4·7}$$

$C(T)$ and $\theta(T)$ being calculated from (4·5) as

$$C(T) = \frac{Ng^2 \mu_B^2}{4k} \frac{B}{A}, \quad \theta(T) = \frac{3J}{2k} \frac{B}{A}, \tag{4·8}$$

where

$$A = \sum_S \exp\left(-\frac{JS(S+1)}{kT}\right) \sum_{S^+, S^-, M} 1, \tag{4·9}$$

$$B=\sum_{S}\exp\left(-\frac{JS(S+1)}{kT}\right)\sum_{S^+,S^-,M}M^2.$$

A and B can be calculated numerically. The susceptibility obtained in this way is shown in Fig. 2 and Fig. 3 together with the value by the molecular field approximation. As seen from these figures the susceptibility given by (4·7) decreases more slowly with temperature

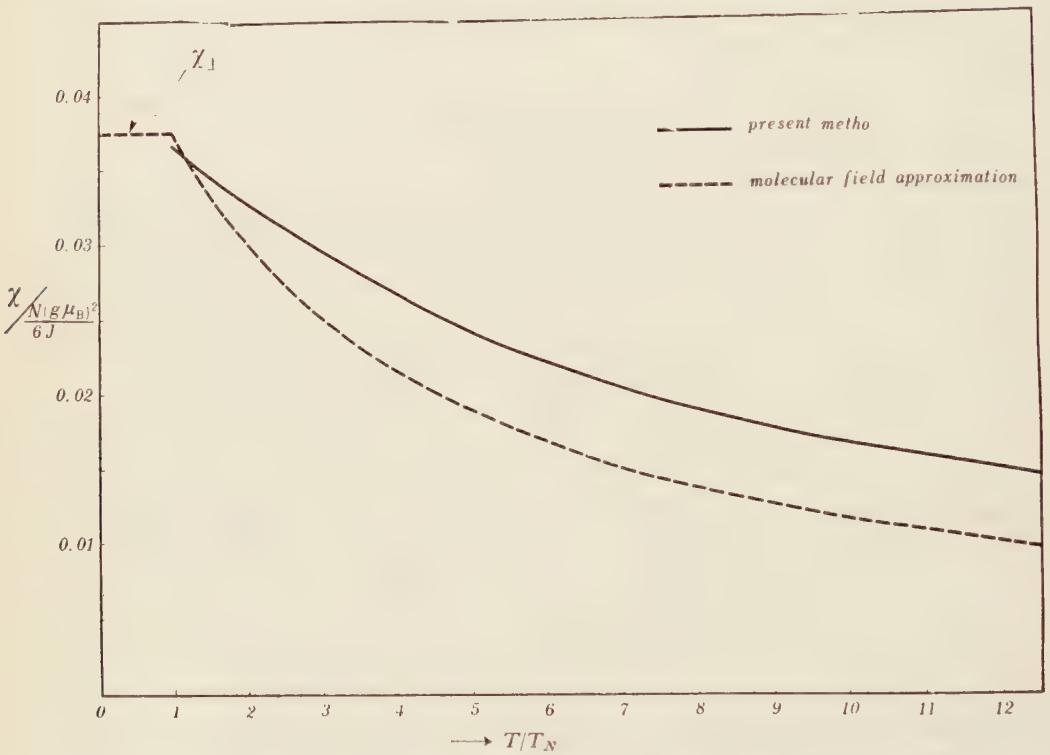

Fig. 2. Susceptibility versus reduced temperature curve.

than by the molecular field approximation. θ/T_N obtained by extrapolating the linear part in the high temperature side is about 4.9. But the present method of introducing the short range order is still too insufficient judging from the behavior of the susceptibility near the Néel temperautre.

As mentioned in Sec. 2, the susceptibility below the Néel temperature under a considerably high magnetic field becomes equal to $\chi_\perp$ which is temperature independent in the molecular field approximation. To calculate the perpendicular susceptibility on the present model is very difficult. $\chi_\perp$ obtained by the molecular field approximation is only a little larger than the susceptibility at the Néel temperature calculated here, as shown by the dotted line in Fig. 2. The perpendicular susceptibility at the absolute zero of temperature is somewhat smaller than the value by the molecular field approximation $\chi_\perp$, because of the quantum effect which may be small in the case of $S=5/2$. On the

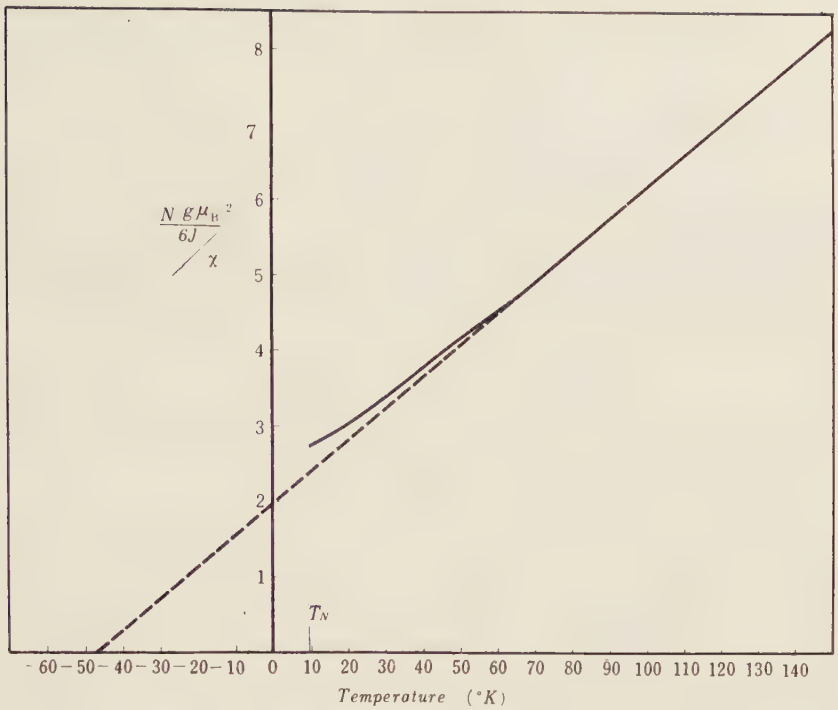

Fig. 3. Reciprocal susceptibility versus temperature curve.

other hand, the spin wave theory[10] shows that the perpendicular susceptibility decreases with rising temperature. Therefore, the change of the perpendicular susceptibility from absolute zero to the Néel temperature would be expected to be smaller than the difference between $\chi_{\perp}$ by the molecular field approximation and χ at the Néel temperature.

§ 5. Specific heat

Now we shall turn to the consideration on the specific heat above the Néel temperature. In the paramagnetic state the internal field vanishes in the absence of the external field. Therefore, Hamiltonian of the spin system becomes as simple as

$$\mathcal{H} = J\,\boldsymbol{S}^2. \tag{5·1}$$

Hence, the specific heat per ion is given by

$$C_v = \tfrac{1}{4}\left(\frac{J^2}{kT^2}\right)\frac{1}{\{\sum_S \exp(-JS(S+1)/kT)\sum_{S^+,S^-,M} 1\}^2}$$
$$\times\left[\left\{\sum_S S^2(S+1)^2\exp\left(-\frac{JS(S+1)}{kT}\right)\sum_{S^+,S^-,M} 1\right\}\left\{\sum_S \exp\left(\frac{-JS(S+1)}{kT}\right)\sum_{S^+,S^-,M} 1\right\}\right.$$
$$\left.-\left\{\sum_S S(S+1)\exp\left(\frac{-JS(S+1)}{kT}\right)\sum_{S^+,S^-,M} 1\right\}^2\right]. \tag{5·2}$$

Using A and B defined by (4·9) and C newly defined below, this can be rewritten as

$$C_v/k=\frac{1}{4}\left(\frac{J}{k}\right)^2\left[\frac{C}{A}-\left(\frac{3B}{A}\right)^2\right]\frac{1}{T^2}, \tag{5·3}$$

$$C=\sum_S S^2(S+1)^2\exp\left(-\frac{JS(S+1)}{kT}\right)\sum_{S^+,S^-,M}{}' 1. \tag{5·4}$$

C_v/k is plotted in Fig. 4.

For comparison we shall calculate here the specific heat by the moment expansion method. The specific heat by this method in the lowest order is given by the following formula :

$$C_v=\frac{1}{kT^2}|\overline{\mathcal{H}^2}-(\overline{\mathcal{H}})^2|, \quad \overline{H^n}=\frac{\mathrm{T}_r\mathcal{H}^n}{\mathrm{T}_r 1}. \tag{5·5}$$

Here $\mathcal{H}$ means Hamiltonian of the total system, namely

$$\mathcal{H}=2J\sum_{(i,j)}{}' \boldsymbol{S}_i\cdot\boldsymbol{S}_j, \tag{5·6}$$

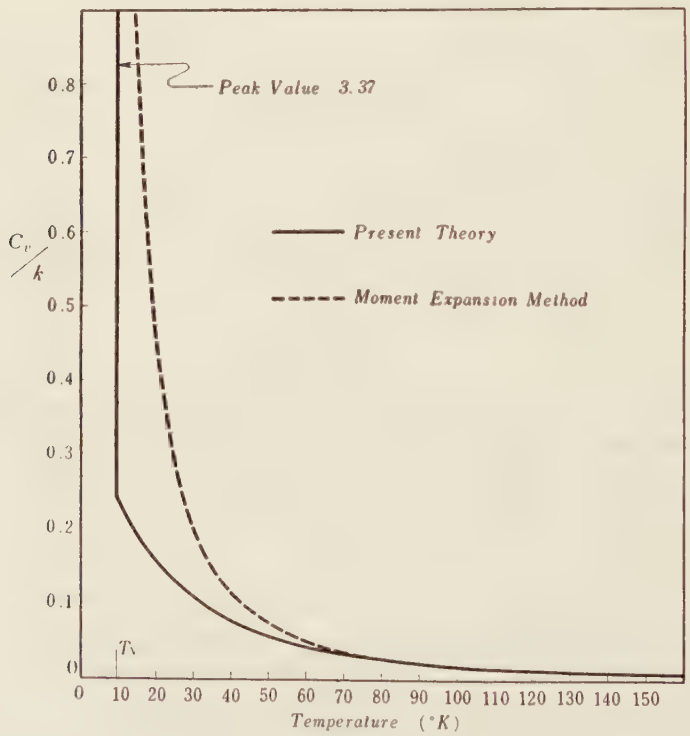

Fig. 4. Specific heat per Fe^{3+} ion.

where summation is taken over all the nearest spin pairs. Using $\overline{\mathcal{H}}=0$ and $\overline{\mathcal{H}^2}=\frac{4}{3}NJ^2S^2(S+1)^2$, we obtain as the specific heat per ion

$$\frac{C_v}{k}=\frac{4}{3}\left(\frac{J}{k}\right)^2 S^2(S+1)^2\frac{1}{T^2}, \tag{5·7}$$

where S is the magnitude of the spin of an ion, namely 5/2. This result is also plotted with a dotted line in Fig. 4. At higher temperatures these two curves agree with each other very well.

Next we shall turn to the calculation of the discontinuity of the specific heat at the Néel temperature. For this purpose we must calculate the specific heat just below the Néel temperature. This calculation necessitates the expansion of the energy eigen-value up to the fourth order with respect to H^+ and H^-. Hamiltonian in this case is

$$\mathcal{H}=[JS^2+(a-b)S_z H^+]-2S_z' H^+, \tag{5·8}$$

because we can put $H^-=-H^+$ below the Néel temperature. We put the eigen-value of this Hamiltonian in the following form:

$$E=a_0+a_1H^++a_2H^{+2}+a_3H^{+3}+a_4H^{+4}, \tag{5·9}$$

a_i's can be calculated by the perturbation theory and they are functions of S, S^+, S^- and M. In (5·8) and (5·9) the interactions between ions belonging to the cluster and those outside it have been introduced in the form of the internal field, so this part of the exchange interactions will be accounted twice if we take (5·8) and (5·9) as the energy per cluster. Therefore, as the energy per cluster, we must take the following quantity:

$$\begin{aligned} h&=JS^2+\tfrac{1}{2}(S_z^+-S_z^-)H^+ \\ &=\mathcal{H}-\tfrac{1}{2}H^+(\partial\mathcal{H}/\partial H^+). \end{aligned} \tag{5·10}$$

Accordingly, the eigen-value of h becomes, by inserting (5·9) into (5·10),

$$\varepsilon=a_0+\tfrac{1}{2}a_1H^+-\tfrac{1}{2}a_3H^{+3}-a_4H^{+4}. \tag{5·11}$$

The average value of ε can be obtained from the expression,

$$\langle\varepsilon\rangle=\frac{\mathrm{T}_r[h\exp(-\mathcal{H}/kT)]}{\mathrm{T}_r[\exp(-\mathcal{H}/kT)]}.$$

Expanding this expression with regard to H^+/kT and using the relation (3·4), namely $H^+=-J\langle S_z^+\rangle$, $\langle\varepsilon\rangle$ can be written as

$$\langle\varepsilon\rangle=\overline{a_0}-(J/2)\{x(\overline{a_1^2}+2\overline{a_0a_2}-2\overline{a_0}\cdot\overline{a_2})-(x^2/J)(\overline{a_0a_1^2}-\overline{a_0}\cdot\overline{a_1^2})\}\langle S_z^+\rangle^2, \tag{5·12}$$

where

$$\bar{a}_i=\frac{\sum a_i\exp(-a_0/kT)}{\sum\exp(-a_0/kT)}, \quad x=\frac{J}{kT}. \tag{5·13}$$

The peak value of the specific heat per ion at the Néel temperature C_{T_N} becomes as follows:

$$C_{T_N}=\frac{1}{4}\left(\frac{\partial\overline{a_0}}{\partial T}\right)_{T_N-0}-\frac{J}{2}\left\{x_N(\overline{a_1^2}+2\overline{a_0a_2}-2\overline{a_0}\cdot\overline{a_2})\right.$$

$$-\frac{x_N^2}{J}(\overline{a_0 a_1^2}-\overline{a_0}\cdot\overline{a_1^2})\Big\}\frac{1}{4}\Big(\frac{\partial\langle S_z^+\rangle^2}{\partial T}\Big)_{T_N-0}, \tag{5·14}$$

where a symbol T_N-0 means the limiting value as the temperature approaches the Néel temperature from the lower side and x_N is the value at $T=T_N$.

The next step is to calculate $(\partial\langle S_z^+\rangle^2/\partial T)_{T_N-0}\cdot\langle S_z^+\rangle^2$ can be obtained as follows. Expanding the partition function

$$P=\sum_S\exp\Big(-\frac{a_0}{kT}\Big)\sum_{S^+,S^-,M}\exp\Big\{-\frac{1}{kT}(a_1H^+ +a_2H^{+2}+a_3H^{+3}+a_4H^{+4})\Big\}, \tag{5·15}$$

into powers of H^+/kT and inserting it into the self-consistent relation

$$\langle S_z^+\rangle P=-(kT/2)\ \partial P/\partial H^+, \tag{5·16}$$

we obtain the following expression for $\langle S_z^+\rangle^2$:

$$\begin{aligned}\langle S_z^+\rangle^2=&-\{1+J\overline{a_2}-(x/2)\ \overline{a_1^2}\}\cdot[(-xJ\ \overline{a_2}+\tfrac{1}{2}x^2\ \overline{a_1^2})\\&-2\{-\overline{a_4}J^3+\tfrac{1}{2}xJ^2(\overline{a_2^2}+2\overline{a_1a_3})-(x^2/2)\ J\overline{a_1^2a_2}+(x^3/24)\ \overline{a_1^4}\}]^{-1}.\end{aligned} \tag{5·17}$$

The condition for $\langle S_z^+\rangle^2$ to vanish, namely

$$1+J\overline{a_2}-\tfrac{1}{2}x\ \overline{a_1^2}=0, \tag{5·18}$$

gives the Néel temperature and this is of course equivalent to (3·15).

If we differentiate (5·17) with the temperature and simplify it with the use of (5·18) we obtain

$$\begin{aligned}&\Big(\frac{\partial\langle S_z^+\rangle^2}{\partial T}\Big)_{T_N-0}\\&=-\frac{\Big[\dfrac{\partial}{\partial T}(J\overline{a_2}-(x/2)\overline{a_1^2})\Big]_{T_N-0}}{[2J^3\overline{a_4}+x\{1-J^2(\overline{a_2^2}+2\overline{a_1a_3})\}+x^2J\overline{a_1^2a_2}-(x^3/12)\overline{a_1^4}]_{T_N-0}}.\end{aligned} \tag{5·19}$$

The numerator of this equation is calculated as

$$\begin{aligned}\Big[\frac{\partial}{\partial T}(J\overline{a_2}-\tfrac{1}{2}x\overline{a_1^2})\Big]_{T_N-0}=&\frac{k}{2J}x_N^2\Big[(\overline{a_1^2}+2\overline{a_0a_2}-2\overline{a_0}\cdot\overline{a_2})\\&-(x/J)(\overline{a_0a_1^2}-\overline{a_0}\cdot\overline{a_1^2})\Big]_{T_N-0}.\end{aligned} \tag{5·20}$$

With the use of (5·19) and (5·20) in (5·14), we obtain the following expression for C_{T_N}:

$$\begin{aligned}C_{T_N}=&\frac{1}{4}\Big(\frac{\partial\overline{a_0}}{\partial T}\Big)_{T_N-0}\\&+\frac{k}{16}x_N^3\ \frac{[\overline{a_1^2}+2\overline{a_0}\cdot\overline{a_2}+2/J\ \overline{a_0}-x/J\overline{a_0a_1^2}]^2_{T_N-0}}{[2J^3\overline{a_4}+x\{1-J^2(\overline{a_2^2}+2\overline{a_1a_3})\}+x^2J\overline{a_1^2a_2}-(x^3/12)\overline{a_1^4}]_{T_N-0}}.\end{aligned} \tag{5·21}$$

For the calculation of the constants $a_0, a_1\cdots, a_4$, the following perturbation formula

can be used:

$$\begin{aligned}
\Delta E=&\sum_{p\neq n}\frac{\langle n|V|p\rangle\langle p|V|n\rangle}{E_n-E_p}-\sum_{p\neq n}{}'\frac{\langle n|V|p\rangle\langle p|V|n\rangle}{(E_n-E_p)^2}\cdot\sum_{q\neq n}{}'\frac{\langle n|V|q\rangle\langle q|V|n\rangle}{E_n-E_q}\\
&+\sum_{p,q,r\neq n}\frac{\langle n|V|p\rangle\langle p|V|q\rangle\langle q|V|r\rangle\langle r|V|n\rangle}{(E_n-E_p)(E_n-E_q)(E_n-E_r)},\\
V=&-2S_z'H^+.
\end{aligned}\tag{5·22}$$

Here we used the fact that S_z' has finite matrix elements only between the states of S and $S\pm1$. From (5·22), we obtain the following results for a_i's:

$$\begin{aligned}
a_0=&JS(S+1),\quad a_1=\frac{\alpha\cdot\beta}{S(S+1)}M,\\
a_2=&\frac{1}{2J}\left\{\frac{h(S)}{S}-\frac{h(S+1)}{S+1}\right\},\\
a_3=&\frac{1}{2J^2}\left\{\frac{h(S)\cdot g(S)}{S}-\frac{h(S+1)\cdot g(S+1)}{S+1}\right\}M,\\
a_4=&\frac{1}{8J^3}\left[4\left\{\frac{h(S)\cdot g^2(S)}{S}-\frac{h(S+1)g^2(S+1)}{S+1}\right\}M^2\right.\\
&+\left\{\frac{h^2(S+1)}{(S+1)^3}-\frac{h^2(S)}{S^3}+\frac{h(S)\cdot h(S+1)}{S^2(S+1)^2}\right\}\\
&\left.+\left\{\frac{h(S)\cdot h(S-1)}{S^2(2S-1)}-\frac{h(S+1)h(S+2)}{(S+1)^2(2S+3)}\right\}\right]
\end{aligned}\tag{5·23}$$

and for $S=0$

$$\begin{aligned}
&a_0(0)=a_1(0)=a_3(0)=0,\\
&a_2(0)=-h(1)/2J,\\
&a_4(0)=(1/8J^3)[h^2(1)-\tfrac{1}{3}h(1)h(2)],
\end{aligned}\tag{5·24}$$

where $h(S)$ and $g(S)$ have been defined by

$$\begin{aligned}
h(S,S^+,S^-,M)=&\frac{(\alpha^2-S^2)(S^2-\beta^2)}{S^2(4S^2-1)}\quad(S^2-M^2),\\
g(S,S^+,S^-)=&\frac{\alpha\cdot\beta}{(S-1)S^2(S+1)}\quad(S\neq1),\\
g(1,S^+,S^-)=&-\alpha\cdot\beta/4,\\
\alpha=&S^++S^-+1,\quad\beta=S^+-S^-.
\end{aligned}\tag{5·25}$$

Inserting (5·23), (5·24) and (5·25) into (5·21) and summing over M, we obtain

$$C_{T_N}=\frac{1}{4}\left(\frac{\overline{\partial a_0}}{\partial T}\right)_{T_N-0}+\frac{5}{48}kx_N^3\frac{1}{P_0}\frac{\{\sum_S{}'e^{-x_NS(S+1)}(b_0+x_Nb_1)\}^2}{\sum_S e^{-x_NS(S+1)}(c_0+c_1x_N+c_2x_N^2+c_3x_N^3)}.\tag{5·26}$$

Here several quantities have been introduced in order to simplify the description, namely

$$\begin{aligned} p_0 &= \sum_S e^{-\varepsilon_N S(S+1)} (2S+1) w(S), \\ b_0 &= 7S(S+1)(2S+1) w(S), \\ b_1 &= -(2S+1) \sum_{S^+, S^-} \alpha^2 \cdot \beta^2, \end{aligned} \tag{5·27}$$

where $w(S)$ means the number of states having the same values of S and M and

$$\begin{aligned} C_0 &= \left\{ \frac{A(S)}{S^2(S-1)(S+1)} - \frac{A(S+1)}{(S+1)^2 S(S+2)} \right\} - \left\{ \frac{B(S)}{S} - \frac{B(S+1)}{S+1} \right\} \\ &\quad + \frac{1}{2} \left\{ \frac{C(S)}{S^2(S+1)^2} + \frac{C(S-1)}{2S^2(2S-1)} - \frac{C(S+1)}{2(S+1)^2(2S+3)} \right\}, \\ C_1 &= -\frac{1}{S(S+1)} \{A(S) - A(S+1)\} - \{B(S) + B(S+1)\} \\ &\quad + \frac{C(S)}{S(S+1)} + 15(2S+1) w(S), \\ C_2 &= \frac{(S-1)}{2(S+1)} A(S) - \frac{S+2}{2S} A(S+2), \\ C_3 &= -\frac{1}{12} (2S+1) \left\{ \frac{3}{S^2(S+1)^2} - \frac{1}{S^3(S+1)^3} \right\} \sum_{S^+, S^-} \alpha^4 \cdot \beta^4 \quad \text{for } S \neq 0, \end{aligned} \tag{5·28}$$

where, for $S=1$, the first term in C_0 must be put zero and $A(S)$, $B(S)$ and $C(S)$ are given by

$$\begin{aligned} A(S) &= \frac{1}{S^4} \sum_{S^+, S^-} \alpha^2 \beta^2 (\alpha^2 - S^2)(S^2 - \beta^2), \\ B(S) &= \frac{4S^2+1}{4S^5(4S^2-1)} \sum_{S^+, S^-} (\alpha^2 - S^2)^2 (S^2 - \beta^2)^2, \\ C(S) &= \frac{1}{S(S+1)(2S+1)} \sum_{S^+, S^-} (\alpha^2 - S^2)(S^2 - \beta^2) \{\alpha^2 - (S+1)^2\} \{(S+1)^2 - \beta^2\}, \end{aligned} \tag{5·29}$$

with exceptions of

$$\begin{aligned} C_0(0) &= -\frac{5}{12} \sum_{S^+, S^-} (\alpha^2 - 1)^2 (1 - \beta^2)^2 - \frac{1}{36} \sum_{S^+, S^-} (\alpha^2 - 1)(1 - \beta^2)(\alpha^2 - 2^2)(2^2 - \beta^2), \\ C_1(0) &= -\frac{5}{12} \sum_{S^+, S^-} (\alpha^2 - 1)^2 (1 - \beta^2)^2, \\ C_2(0) &= C_3(0) = 0. \end{aligned} \tag{5·30}$$

The numerically calculated value of the specific heat at the Néel temperature by using the above expressions becomes

$$C_{T_N}=3.37\ k\,.$$

The corresponding value by the molecular field approximation is $2.36\ k$. The ratio of these two values is 1.43, showing that the short range order raises the peak value to a considerable extent.

Experiments for Zn-ferrite shows that the specific heat anomaly at the Néel temperature is not so sharp. However, in the Zn-chromite with the same crystal structure a sharp peak of the specific heat has been observed at its Néel temperature. This peak value amounts to $3.5\ k$ per Cr^{3+} ion and is comparable to the above calculated value. The reason why the anomaly of the specific heat at the Néel temperature in Zn-ferrite is not so sharp remains unclear.

§ 6. Summary

The susceptibility of Zn-ferrite both below and above the Néel temperature has been discussed on the assumed superstructure. The calculated value is consistent with the observed magnetic behavior of this substance. In this assumed superstructure, the magnetic anisotropy energy has tetragonal symmetry, its axis coinciding with one of the cubic axes, and the antiferromagnetic preferred axis is in the plane perpendicular to c-axis. This anisotropy energy has an order of $1\ \mathrm{cm}^{-1}$ per ion and arises from the magnetic dipole-dipole interaction, while the anisotropy in this preferred plane would be much smaller and would be less than $10^{-2}\ \mathrm{cm}^{-1}$. Therefore, the critical field strength for the rotation of the spin axis is considerably smaller than in other antiferromagnetic substances. The field dependence of the susceptibility below the Néel temperature is expected to be remarkable at field strength of the order of 4000 oersteds.

The effect of the short range order has also been investigated with the use of a simplified Li's model taking a cluster which matches the present superstructure. The Néel temperature, the specific heat and the susceptibility have been calculated on this model. In this approximation, the ratio of the paramagnetic Curie temperature θ to the Néel temperature T_N is $4\cdot 9$ and the peak value of the specific heat and the discontinuity at the Néel temperature are respectively $3.37\ k$ and $3.13\ k$ per ion.

The present method of introducing the short range order is insufficient to explain the constant susceptibility above the Néel temperature. Further, though the peak value of the specific heat at the Néel temperature obtained here is in good agreement with the experimental result for chromite, the remarkable discrepancy between its calculated and experimental values remains unexplained in the case of ferrite.

Moreover, it might be worth while to mention about the present statistical theory itself. In Sec. 2, we mentioned that the second neighbor exchange interaction is important for establishing the long range order. Nevertheless, the Néel temperature calculated by the approximate method used here is almost independent of this second neighbor interaction. In our treatment this interaction plays only a catalystic role for the forma-

tion of the long range order. On account of this situation, the question might occur whether the Néel temperature obtained here is an illusory one due to the present statistical approximation or an actual one. This problem would be very difficult to be solved completely. However, the present approximate method does not give any Néel temperature for Ising model and for the case of one half spin, so that the Néel temperature obtained here can be considered to occur as a result of large spins of magnetic Cr^{3+} and Fe^{3+} ions.

With the use of spin wave method for this special crystal structure, the excitation energy of the most of spin waves is determined by the nearest neighbor exchange interaction, but for waves propagating to special directions it becomes zero unless the second neighbor interactions are taken into account as remarked by Anderson[5]. Under such a circumstance, to what extent the Néel temperature depends on the second neighbor interaction seems to be an interesting problem.

In conclusion the authors wish to express their sincere thanks to Professor T. Nagamiya and Mr. J. Kanamori for their valuable discussions. Thanks are also due to Professor Friedberg who has sent us his experimental results prior to publication.

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Hydrodynamical Treatment of Multiple Meson Production in High Energy Nucleon-Nucleus Collisions

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The multiple meson production in high energy nucleon-nucleus collisions is treated by extending Landau's hydrodynamical model which is supported not only from experimental data on cosmic ray, but also from some theoretical point of view. As we have much data on the nucleon-nucleus collisions rather than on elementary collisions, we may get some clue to the future theory by comparing our results with the data on cosmic ray, though the latter are very complicated and our theory is somewhat phenomenological. Theoretical results are derived focusing on the dynamical properties of secondary particles, such as the effective inelasticity, energy and angular distributions and transverse momenta; and their dependence on the size of nucleus is considered.

§ 1. Introduction

Cosmic ray experiments so far tried have revealed several features of the multiple meson production in high energy nuclear interactions. First, the air shower data[1] on the distribution of shower cones indicate very narrow angular distribution of the secondary energy. The nuclear emulsion data[2], in which the angular distribution of particle number is more important, also favour its strong anisotropy in the center of mass system. Secondly, the production rate of new heavy particles and anti-protons seems rather small compared with Fermi's thermodynamical theory[3]. The situation is not so clear for the inelasticity, i.e., the ratio of energy transferred to secondary particles to the incident energy, but several authors suggest that in most case it lies between 0.2 to 0.7.[4]

Unfortunately the cosmic ray data in question are almost due to meson production in high energy nucleon-nucleus or pion-nucleus collisions, and although there are some phenomenological theories of Roesler and McCusker[5], G. Cocconi,[6] and others[7], it is desirable to get the theory which could explain these facts more quantitatively.

Recent analyses of lower energy phenomena provide us with some clue to this subject. That is, as suggested by several people[8], meson-meson interaction may play an important role to explain some Gev experiments.* If it is true, the final interaction between created particles, which flow out in a very narrow cone and have high density, may also play an important role in meson production in high energy collisions. If we assume that the interaction cross section between high energy particles is almost geometrical, the critical temperature T_c[10], which corresponds to such a state of the many meson system that the mean free path of particles is comparable with the size of the system and thus marks the breaking-off of the interaction, becomes very low ($T_c \sim m_\pi c^2$)**, so that interactions between them will be continued long after spurting out from Fermi's Lorentz contracted volume. (According to our estimate they cease to intereact after travelling more than thousand times of Fermi's contracted length, for high incident energies.) As Watson[11] once pointed out, the final state interaction sometimes causes a great distortion to the particle system, especially on the angular and energy distributions. Moreover there is some support from the ratio between mesons and new particles in high energy jets:[12),9] In Fermi's theory, in which the free emission of particles is assumed to take place from the initial high temperature equilibrium state, the number of new particles and anti-protons is of the same order of magnitude with, or more than, that of pions, as there may be many degrees of freedom for new particles. If, however, there are strong interactions between mesons and new particles, the nature of particles would change to another by their collisions and the ratio of these particles would be determined only at the critical temperature which favours pion than other heavy particles. (This result does not change even if we take into consideration the γ-charge (strangeness) conservation for meson-baryon interactions.)

Interactions between a large number of mesons may be expressed, in our first phenomenological approach, by the viscous and inertial effects, following the hydrodynamical model first introduced by Landau.[10] We assume, following Landau, that the viscosity can be neglected*** and the entropy increases mainly in the first stage which corresponds to Fermi's equilibrium state. Then the entropy increase is considered to be due to the propagation of shock waves which occur by the rapid compression of the matter. In the second stage where fluid flow out to the outside, the expansion of system proceeds practically one-dimensionally and adiabatically. From now, only the local thermal equilibrium may be established, and the energy, entropy and number densities in such an individual portion may be estimated by statistical mechanics of relativistic ideal gases.[13),14]

In the subsequent stage of three-dimensional expansion, the meson-meson interaction still plays the role. It was shown by Landau,[10] however, that the angular and energy distributions of emitted particles are essentially determined at the moment of going into

* The connection between Gev and multiple production phenomena has already been discussed by Z. Koba[9], so we shall not enter into detail here.

** The temperature is given in energy units throughout in this paper.

*** The justification of neglecting viscous effect is given by Landau himself[10),14] and by Koba[9].

three-dimensional motion from one-dimensional motion, which is different for each portion of the fluid. (The temperature at this moment is also different for different portions.) When the system expands furthermore, the temperature of the portion reaches the critical value (T_c), and this is independent of the incident energy. This temperature, which we may also call the "cooling temperature", will govern the observed number ratios of various particles and the thermal energy per one particle.

In this paper we shall treat the multiple meson production in high energy nucleon encounters with nuclei using Landau's theory. We shall show how much the distribution of emitted mesons may be distorted by a meson-meson interaction, and how the inelasticity depends on the incident energy and on the size of target nucleus, and we shall also show the asymmetry between the forward and backward directions in the center-of-mass (c. m.) system.

Here we may have to give the following excuse. Even if the idea of final interactions between produced mesons and the hydrodynamical approach to take them into account were justified, it still remains less convincing to extend the hydrodynamical picture of nuclear matter as a continuous medium also to the inter-nuclear cascade stage. (We shall give in § 2 some qualitative reasoning for it.) However, since we could not find out more consistent way to describe the nucleon-nucleus case hydrodynamically, we shall apply in this paper the relativistic hydrodynamics to the whole process of the meson production, i. e., to the inter-nuclear cascade as well as to the subsequent stages of expansion.

The nuclear mass number dependence of the multiplicity was calculated by Belenkij and Milehin[13] from the same standpoint and they obtained the result,

$$N(E_0) = (n+1)/2 \cdot N_0(E_0), \qquad (1\cdot1)^*$$

where $N_0(E_0)$ is the multiplicity in the case of a nucleon-nucleon collision with the some laboratory incident energy E_0, and n is the number of nucleons in the penetrated tunnel. They then averaged eq. (1·1) over all possible cases from the central to the peripheral collisions, and concluded that their result ($N/N_0 = A^{0.19}$, where A is the mass number) agreed well with experiments. But eq. (1·1) is not very much different from

$$N(E_0) = n^{3/4} N_0(E_0) \qquad (1\cdot2)$$

which follows from the "composite collision model"[6] of the internuclear cascade. The composite model may be regarded as the extension of Fermi's theory, and the assumptions are made there that the incident nucleon collides with the whole of the tunnel simultaneously, and that the secondary particles are emitted symmetrically between forward and backward directions in the c. m. system.

Therefore, the extended Fermi and Landau theories will be better compared with

* More precisely, eq (1·1) holds for $n \leqq 3.7$, and they got

$$N(E_0) = 0.92\,(n-\tfrac{1}{4})^{3/4} N_0(E_0) \qquad (1\cdot1')$$

for $n \geqq 3.7$. But the numerical values are practically equal if we use (1·1) instead of (1·1′).

each other in terms of the dynamical properties of emitted particles, as will be done in this paper.

We finally remark that our hydrodynamical description of inter-nuclear cascade somewhat resembles the "individual collision model"[7]. But this resemblance is of course not complete. Thus, our secondary particles show certain complicated behaviour such as turning about their direction of flow (see § 2 and § 5. 2), while it will not be the case with particles in the usual individual collision model without the final state interaction.

In the following, our discussion is divided into two parts. Part I deals with a rather mathematical problem of solving the one-dimensional hydrodynamical motion (§ 2 and § 3) and of finding out suitable approximate expressions for the energy and angular distributions (§ 4). Then in Part II these results are applied to the physical problems. In § 5, some results from the one-dimensional solution are discussed on the effective inelasticity and asymmetry in the c. m. system. In § 6, the three-dimensional motion is described and several dynamical properties of the secondary particles (angular distribution, energy spectra, transverse momenta, etc.) are considered. It will be shown that these properties for nucleon-nucleus collisions are expressed, approximately, in terms of the corresponding quantities for the nucleon-nucleon collisions with the same incident energy and a factor which depends on n and the emission angle but not on the primary energy. § 7 summarizes the results. Several figures of the observable quantities which would be important to be compared with experimental data will be given. Appendices are added to demonstrate several calculations omitted in the text.

Part I
One-dimensional solution of the fluid motion in the case of nucleon-nucleus collisions

§ 2. One-dimensional motion of nuclear fluid

Before going into mathematical details, we shall give an illustrative outline of the one-dimensional fluid motion. When a high energy nucleon collides with a nucleus, the nuclear cascade will occur only in a tunnel which penetrates into the nucleus whose radius, a, is the same as that of nucleon. ($a \approx \hbar/m_\pi c$, the Compton wavelength of pion). The nuclear cascade which occurs in this tunnel is not a usual cascade, as the decay time is much larger than the collision time of each individual collision. A large number of created mesons as well as nucleons in this tunnel are so close together that they collide with each other throughout the tunnel, and the propagation of interaction may be something like the travelling of shock waves. They then spurt out to the outside continuing these interactions. Thus, we may treat the system as if it is a fluid[13]. Then it is more convenient to take the equi-velocity system (the e.v.s.), i.e., the reference system in which the nucleon and nucleus have equal magnitude but opposite directions of velocity.

In the e. v. s., the incident nucleon has the same energy E as each nucleon in the nucleus. E is related to the laboratory incident energy E_0 by

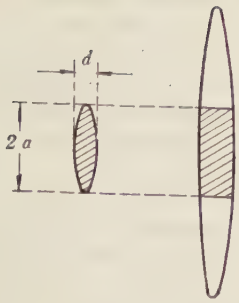

Fig. 1

$$E=Mc^2(E_0/2Mc^2)^{1/2}. \tag{2·1}$$

(M is the nucleon mass.) The Lorentz contracted diameter of the nucleon (d) is

$$d=2a(Mc^2/E), \qquad a\approx\hbar/m_\pi c. \tag{2·2}$$

Let the length of the tunnel be n times d, then n is roughly the number of nucleons in the tunnel.

In the model assumed here, the colliding particles are regarded as continuous medium. Several stages of the one-dimensional motion are sketched in Figs. 2 for $n=2$, where the temperature, measured in the local rest system of the fluid element, is chosen as one of the state variables. (Another state variable is e. g. the velocity of the fluid, v.)

The shock velocity in the tunnel, D, is determined from the continuity conditions for the energy-momentum flow through the discontinuity surface (the so-called Rankine-Hugoniot equations[15]), and is given by[13]

$$D=c_0^2. \tag{2·3}$$

To derive this, the equation of state for the high temperature matter inside the shock is assumed to be

$$p=c_0^2\mathcal{E}. \tag{2·4}$$

Here p is the pressure, and $\mathcal{E}$ is the energy per unit volume measured in the local rest system of the fluid.

c_0 is the sound velocity of the medium and its values can be evaluated if we assume, after Landau,[10, 13), 14)] that the nuclear matter can be treated as relativistic ideal gas. It is then found that $c_0^2=1/3$ at very high temperatures and decreases slowly with the cooling of the system.* Since its change is rather slow, it may be regarded as a constant, as long as we consider the stages where $T>m_\pi c^2$. Actual values of the c_0 for the pion gas are shown in Table 1; if other particles such as the K-mesons and baryons are included, the decrease of c_0 will be slightly more rapid.

Table 1. Temperature dependence of the sound velocity of relativistic pion gas (light velocity=1).

$T/m_\pi c^2$	c_0^2
∞	0.33
2.0	0.32
1.43	0.31
1.11	0.30
1.00	0.29
0.83	0.28

When the tunnel length n is shorter than the critical value[13]

$$n_c=(1+c_0)/(1-c_0)=2+\sqrt{3}, \quad \text{for} \quad c_0=1/\sqrt{3} \tag{2·5}$$

* The velocities are expressed in unit of the light velocity throughout in this paper.

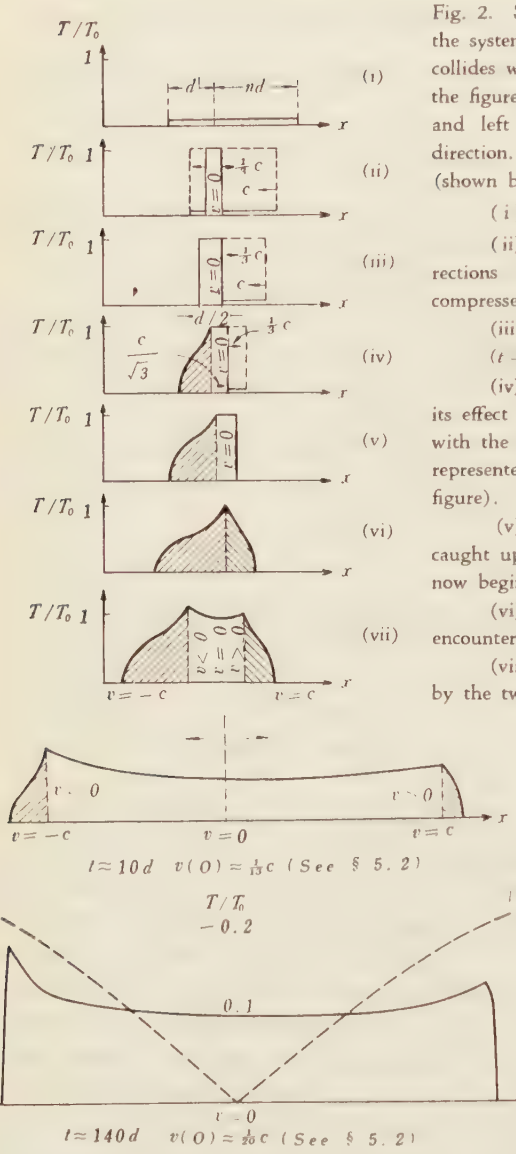

Fig. 2. Several stages of the compression and expansion of the system when a "nucleon" (thin plate) with thickness d collides with a nuclear tunnel with thickness nd ($n=2$, in the figures). The equi-velocity reference system is adopted, and left (right) corresponds to the backward (forward) direction. If $n>n_c=3.7$, we have no progressive wave region (shown by shadow) in the right direction.

(i) The instant of collision ($t=t_0$).

(ii) Shock waves start to propagate into both directions with velocity $c/3$. The matter continues to be compressed with the light velocity. $v=0$ between the shocks.

(iii) The left shock has just passed over the matter ($t-t_1$).

(iv) The outflow of matter has begun in the left, and its effect (the rarefication wave) is propagating to the right with the sound velocity, $c_0=c/\sqrt{3}$. The flowing region is represented by the progressive wave solution (shaded in the figure).

(v) The shock arrives at the right end before it is caught up by the rarefication wave ($n<n_c$). The matter now begins to flow out in the right direction ($t=t_2$).

(vi) Two progressive (rarefication) waves have just encountered. ($t=t_3$).

(vii)–(viii) The region of non-trivial solution bounded by the two progressive wave regions is formed, and expands quickly pushing the progressive wave in the narrow room very near the wave fronts.

the shock which has been travelling in the forward direction reaches the end of the tunnel sooner than the flowing out of the whole matter into the backward direction (Fig. 2 (v)).

We shall hereafter limit ourselves in the cases, $n \lesssim n_c$, and let t_1, x_1 and t_2, x_2 be the time and the position at which the outflow begins in the backward (left) and forward (right) direction, respectively (Figs. 2, (iii) and (v))*. Then we have

$$t_1-t_0=-l/c_0^2, \quad t_2-t_0=nl/c_0^2$$
$$x_1-x_0=-l, \quad x_2-x_0=nl \qquad (2\cdot6)$$

* In the case of the larger $n(n>n_c)$, the progressive wave or "rarefication" wave[13]) catches up the shock at $t=t_2$ and $x=x_2$ and is reflected by the shock. In this case the fluid motion becomes much more complicated, although we can infer some of the qualitative features as it is shown that the shock velocity becomes less than c_0^2 after that time.

where t_0 and x_0 are the instant of the time and the position of the collision, and

$$l=d[c_0^2/(1+c_0^2)]. \tag{2·7}$$

From eqs. (2·6)

$$t_2-t_1=(l/c_0^2)(n-1), \quad x_2-x_1=l(n+1). \tag{2·8}$$

The subsequent behaviour of the fluid motion is as follows. At $t=t_2$ the two progressive waves encounter (Fig. 2, (vi)), and from now we have the region of the non-trivial solution of the hydrodynamical equations (Figs. (vii)–(ix)). Here

$$t_3-t_0=\frac{1}{c_0}\left(1+\frac{1}{c_0}\right)(l/2)(n+1). \tag{2·9}$$

The energy and entropy in backward and forward progressive regions at this moment t_3 can be easily estimated as follows. As the rarefication wave propagates with velocity c_0 into the medium of temperature T_0, the amount of energy and entropy which flows into the backward (or forward) progressive wave region between t_1 and t_3 (or t_2 and t_3) is proportional to

$$c_0T^4(t_3-t_1), \quad c_0T_0^3(t_3-t_1)$$
$$c_0T^4(t_1-t_2), \quad c_0T_0^3(t_3-t_2),$$

where c_0 and T_0 is the same as in the case of $n=1$, and from eqs. (2·6) and (2·9),

$$\begin{aligned} c_0(t_3-t_1)&=l\left(\frac{n+1}{2}+\frac{n-1}{2c_0}\right)\equiv l_1 \\ c_0(t_3-t_2)&=l\left(\frac{n+1}{2}-\frac{n-1}{2c_0}\right)\equiv l_2. \end{aligned} \tag{2·10}$$

Therefore, the amount of energy or entropy contained in the backward (forward) progressive wave at $t=t_3$ is given by the corresponding amount in the case of $n=1$ times the factor $l_1/l(l_2/l)$. (See § 5. 1.)

As time goes on, the distribution of the temperature (or entropy and energy) becomes less and less uniform. The non-trivial region expands quickly pushing the progressive waves in the narrow regions very close to the wave fronts. But the state in progressive wave regions at later times is determined by its state at $t=t_3$ and does not suffer any influence from other regions (cf. Appendix A. 3). So the state of affairs in each of the progressive wave regions and at each boundary between the non-trivial region will be essentially the same as in the case of nucleon-nucleon collisions, the only difference being that the unit of length l which appears in the nucleon-nucleon solution is to be replaced by l_1 or l_2 in the nucleon-nucleus case. For the non-trivial region we cannot have such an exact and simple relation between the nucleon-nucleon and the nucleon-nucleus cases, and we must solve the hydrodynamical equations under suitable boundary conditions (§ 3). But we shall show later (§ 4) that analogous simple, approximate relation can be obtained also for the non-trivial region for sufficiently high incident

energies ($E_0 \gtrsim 10^{12}$ eV, say). That is, it is possible to express the nucleon-nucleus solution in the non-trivial region in a similar form as the nucleon-nucleon soluion.

It will be shown also that, after sufficiently long time, the distribution of the temperature and the velocity is nearly symmetric in the right and left directions, except the front regions (cf. Fig. 2, (ix)). (For $n=1$, it is of course exactly symmetric.) This is because some fraction of the matter changes the direction of flow from the left to the right, though most of the matter is initially contained in the left region ($n > 1$). On the other hand, it will be shown that most of the energy is contained in the front regions, while the entropy is largely contained in the central region[10] (see § 4). In consequence of such a behaviour of the fluid, the ratio of the energy carried in the forward direction to that carried in the backward direction soon reaches the reasonable limiting value,* $1 : n$; while the corresponding ratio for the entropy (particle number) approaches the limiting value $1 : 1$ only after infinitely long time. The actual values of the latter ratio, which depend on E_0 as well as on n, will be estimated in § 5. 2.

§ 3. The non-trivial solution

We shall now proceed to solving the one-dimensional motion and consider here the non-trivial region. (The solution of the progressive wave is given in the Appendix A. 1.) It was shown by Halatnikov[17] that in the extremely relativistic case, where the chemical potential can be neglected as compared with the temperature T, the one-dimensional problem is reduced to solving the following differential equation for the potential $\chi(y, \eta)$:

$$\chi_{\eta\eta} - c_0^2 \chi_{yy} - (1-c_0^2)\chi_y = 0, \tag{3·1}$$

and that the coordinate variables t and x are related with the state variables y and η by

$$\begin{aligned} t &= \exp(-y)\{\chi_y \mathrm{ch}\,\eta - \chi_\eta \mathrm{sh}\,\eta\}, \\ x &= \exp(-y)\{\chi_y \mathrm{sh}\,\eta - \chi_\eta \mathrm{ch}\,\eta\}. \end{aligned} \tag{3·2}$$

Here

$$T/T_0 = \exp(y);\ u_1 = \mathrm{sh}\,\eta,\ u_0 \equiv u = \mathrm{ch}\,\eta \tag{3·3}$$

and $\chi_y = \partial\chi/\partial y$, etc. $u_1 = v/\sqrt{1-v^2}$ and $u_0 = 1/\sqrt{1-v^2}$ are the components of the four velocity, and T_0 is the initial temperature of the system. Once eq. (3·1) is solved under suitable boundary condition, the solution may be substituted in (3·2) to find the functional relations between (t, x) and (y, η).

Our boundary condition (for $n \gtrsim n_c$) is that at both boundaries of the non-trivial region, the solution should be connected with the progressive wave solutions (eqs. (A. 7) and (A. 7′)). Introducing new variables,**

* The total momentum of the nuclear tunnel is $-nP$, if the momentum of the incident particle is P. Then this limiting value is expected from the conservation of the momentum and the very narrow angular distribution of the emitted energy.

** This means to introduce the characteristic curves of eq. (3·1) as variables (Cf. Appendix A).

$$\alpha=\eta-(y/c_0),\quad \beta=-\eta-(y/c_0), \tag{3·4}$$

instead of y and η, we have

$$\begin{aligned}&\alpha=0, \text{ for the backward progressive wave,}\\ &\beta=0, \text{ for the forward progressive wave.}\end{aligned} \tag{3·5}$$

(Cf. Appendix A, henceforth we put $c_0=$ constant.) If we further define the potential $\bar{\chi}(\alpha, \beta)$ by

$$\chi=\bar{\chi}\exp\{s(\alpha+\beta)\},\quad s=(1-c_0^2)/4c_0 \tag{3·6}$$

eq. (3·1) is transformed to

$$\bar{\chi}_{\alpha\beta}-s^2\bar{\chi}=0. \tag{3·7}$$

It is shown in Appendix A. 2 that the required solution of (3·7) is given by*·**

$$\begin{aligned}\bar{\chi}(\alpha, \beta)=&\int_0^{\alpha} da\, I_0(2s\sqrt{(\alpha-a)\beta})\, F(a\,; t_2, x_2)\\ &+\int_0^{\beta} db\, I_0(2s\sqrt{\alpha(\beta-b)})\, F(b\,; t_1, -x_1),\end{aligned} \tag{3·8}$$

where

$$\begin{aligned}F(a\,; t_2, x_2)=&-(1/2)\{t_2(pe^{-p\alpha}+qe^{-q\alpha}-2se^{-s\alpha})\\ &+x_2(pe^{-p\alpha}-qe^{-q\alpha})\},\end{aligned} \tag{3·9}$$

and

$$p=(1+c_0)^2/4c_0,\quad q=(1-c_0)^2/4c_0,\quad (pq=s^2). \tag{3·10}$$

The solution (3·8) is substituted in eqs. (3·2), and after some calculations we get (cf. Appendix B. 1),

$$\begin{aligned}t+x=&\exp(p\alpha+q\beta)\left\{\left[\frac{1+c_0}{2c_0}(x_2+c_0t_2)-\frac{1-c_0}{2c_0}(x_1-c_0t_1)\right]I_0(2s\sqrt{\alpha\beta})\right.\\ &-(x_2+t_2)\,p\int_0^{\alpha} da\, I_0(2s\sqrt{(\alpha-a)\beta})\, e^{-pa}-(x_1+t_1)q\int_0^{\beta} db\, I_0(2s\sqrt{\alpha(\beta-b)})\, e^{-qb}\Big\}\end{aligned} \tag{3·11}$$

$$\begin{aligned}t-x=&\exp(q\alpha+p\beta)\left\{\left[\frac{1-c_0}{2c_0}(x_2+c_0t_2)-\frac{1+c_0}{2c_0}(x_1-c_0t_1)\right]I_0(2s\sqrt{\alpha\beta})\right.\\ &-(t_1-x_1)\,p\int_0^{\beta} db\, I_0(2s\sqrt{\alpha(\beta-b)})\, e^{-pa}-(t_2-x_2)q\int_0^{\alpha} da\, I_0(2s\sqrt{(\alpha-a)\beta})\, e^{-qa}\Big\}.\end{aligned} \tag{3·12}$$

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* It is easy to show that the solution (3·8) contains, as a special case of $n=1$, the one obtained by Halatnikov[17] for a nucleon-nucleon collision.

** Though eq. (3·8) bears somewhat different form from the one given inref. 18), they are quite equivalent. The difference came from the different definition of the potential $\bar{\chi}$.

In these formulae, (t_1, t_2) and (x_1, x_2) are determined from eqs. (2·6)–(2·8) if the origin of the coordinates is fixed. In the following, we shall put $t_1=x_1=0$ when we consider the backward region ($\eta \leqq 0$, or $0 \leqq \alpha \leqq \beta$), and $t_2=x_2=0$ when we consider the forward region ($\eta \geqq 0$, or $\alpha \geqq \beta \geqq 0$). Then it is convenient to use the variables δ and ζ defined by

$$\delta=t-|x|, \qquad \zeta=t+|x|, \tag{3·13}$$

instead of t and x. (δ represents the distance from the front, which proceeds with the light velocity, to the point considered.) From (3·11) and (3·12), we get the following expressions for δ and ζ:
For the forward region ($x>0$, $\alpha \geqq \beta \geqq 0$):

$$\delta/l=\exp(q\alpha+p\beta)\, I_0(2s\sqrt{\alpha\beta})\cdot\frac{1+c_0}{2c_0}\{[(n+1)-(n-1)/c_0]$$
$$-[(n+1)-(n-1)/c_0^2]Q(\beta, \alpha; p)\} \tag{3·14}$$

$$\zeta/l=\exp(p\alpha+q\beta)\, I_0(2s\sqrt{\alpha\beta})\frac{1-c_0}{2c_0}\{[(n+1)-(n-1)/c_0]$$
$$+[(n+1)+(n-1)/c_0^2]Q(\beta, \alpha; q)\}. \tag{3·15}$$

For the backward region ($x<0$, $\beta \geqq \alpha \geqq 0$):

$$\delta/l=\exp(p\alpha+q\beta)\, I_0(2s\sqrt{\alpha\beta})\frac{1+c_0}{2c_0}\{[(n+1)+(n-1)/c_0]$$
$$-[(n+1)+(n-1)/c_0^2]Q(\alpha, \beta; p)\} \tag{3·14′}$$

$$\zeta/l=\exp(q\alpha+p\beta)\, I_0(2s\sqrt{\alpha\beta})\frac{1-c_0}{2c_0}\{[(n+1)+(n-1)/c_0]$$
$$+[(n+1)-(n-1)/c_0^2]Q(\alpha, \beta; q)$$

where

$$Q(\beta, \alpha; p)=\frac{2c_0}{1+c_0}\,(p\int_0^{\beta} db\, I_0(2s\sqrt{\alpha(\beta-b)})e^{-pb}/I_0(2s\sqrt{\alpha\beta})) \tag{3·16}$$

$$Q(\beta, \alpha; q)=\frac{2c_0}{1-c_0}\,(q\int_0^{\beta} db\, I_0(2s\sqrt{\alpha(\beta-b)})e^{-qb}/I_0(2s\sqrt{\alpha\beta})), \tag{3·17}$$

and $Q(\alpha, \beta; p)$ and $Q(\alpha, \beta; q)$ are obtained from (3·16) and (3·17) respectively, by interchanging α and β.

The integrals in (3·16) and (3·17) can be evaluated using the formulae (B·10) and (B. 10′) of the Appendix B. 1.

In the Appendix A. 3, we shall also describe an alternative way of solving one-dimensional hydrodynamical equations. The method is to follow up the characteristic curves in the (t, x) plane. Though this is less suitable for the further calculations, it will be instructive for we can follow the detailed behaviour of the flow as if we are

looking at it through a microscope.

§ 4. Approximate solution of the one-dimensional motion

In this section, we derive the approximate exprssions for the one-dimensional solution, so that we are able to discuss the physical results in a compact way (§§ 5 and 6). For this purpose, eqs. (3·14) and (3·15) are first rewritten as

$$\begin{aligned} \delta &= \Delta_n'(\beta, \alpha)\exp(\sqrt{q\alpha}+\sqrt{p\beta})^2 \\ \zeta &= \Delta_n(\beta, \alpha)\exp(\sqrt{p\alpha}+\sqrt{q\beta})^2, \end{aligned} \tag{4·1}$$

using the asymptotic formula for the imaginary Bessel function,*

$$I_0(2s\sqrt{\alpha\beta}) \approx (1/\sqrt{\pi})\left(\frac{c_0}{1-c_0^2}\right)^{1/2}(\alpha\beta)^{-1/4}\exp(2s\sqrt{\alpha\beta}). \tag{4·2}$$

In eqs. (4·1),

$$\begin{aligned} \Delta_n'(\beta, \alpha) &= (l/\sqrt{\pi})\left(\frac{c_0}{1-c_0^2}\right)^{1/2}\left(\frac{1+c_0}{2c_0}\right)(\alpha\beta)^{-1/4} \\ &\times\{[(n+1)-(n-1)/c_0]-[(n+1)-(n-1)/c_0^2]Q(\beta, \alpha\,;p)\} \end{aligned} \tag{4·3}$$

$$\begin{aligned} \Delta_n(\beta, \alpha) &= (l/\sqrt{\pi})\left(\frac{c_0}{1-c_0^2}\right)^{1/2}\left(\frac{1-c_0}{2c_0}\right)(\alpha\beta)^{-1/4} \\ &\times\{[(n+1)-(n-1)/c_0]-[(n+1)+(n-1)/c_0^2]Q(\beta, \alpha\,;q)\}. \end{aligned} \tag{4·4}$$

Eqs. (4·1)–(4·4) refer to the forward region. The corresponding expressions for the backward region will be obvious.

Noting now that the strongest dependence of δ and ζ on α and β is given by the exponential factors in eqs. (4·1), we introduce the new variables

$$\xi = \ln(\delta/\Delta_n'), \qquad \tau = \ln(\zeta/\Delta_n) \tag{4·5}$$

instead of δ and ζ. In terms of these variables, the one dimensional solution is represented in the following simple form :

$$y = -\frac{1+c_0^2}{4}(\tau+\xi)+\frac{1-c_0^2}{2}\sqrt{\tau\xi} \tag{4·6}$$

$$\eta = \pm\tfrac{1}{2}(\tau-\xi) \tag{4·7}$$

(+ for the forward and − for the backward region, respectively.)

When the temperature (y) and the velocity (η) are written as in eqs.(4·6) and (4·7), the nature of Landau's approximate solution[10),14)] for the nucleon-nucleon collision becomes clear. That is, if we first

* Cf. the formula (B·12). This asymptotic ansatz is a very good approximation for high energy cases, except the near front region.

regard $\Delta_{n}' \sim \Delta_{n}$, which is valid in the central region ($\alpha \sim \beta$), it results from (4·7) and (4·5) that

$$\eta = \pm \tfrac{1}{2} \ln \left\{ \frac{t+|x|}{t-|x|} \right\}, \quad \text{or} \quad v = x/t. \tag{4·8}$$

This velocity distribution is just what has been postulated by Landau[10] from a rather intuitive consideration. Further assuming that Δ's are constants of the order of magnitude of l, (4·6) gives ($c_0^2 = 1/3$):

$$\gamma = -\tfrac{1}{3}(\tau_0 + \tilde{\xi}_0 - \sqrt{\tau_0 \tilde{\xi}_0}), \tag{4·9}$$

with

$$\tau_0 = \ln \frac{t+|x|}{\Delta_0}, \quad \tilde{\xi}_0 = \ln \frac{t-|x|}{\Delta_0} \tag{4·10}$$

and

$$\Delta_0 \sim l \quad (\text{or} \quad \Delta_0 \sim d).$$

Again, eq. (4·9) is in agreement with Landau's approximate solution for the temperature distribution. (He put $\Delta_0 \sim (d/2)$ and $\tau_0 = \ln(t/\Delta_0)$). Thus it is seen that Landau's approximate solution represents the essential qualitative features of the one-dimensional motion rather correctly. We shall remark, however, that the correct values of the Δ's are fairly smaller than Δ_0, and this may affect the quantitative results, of the angular distribution say, to a considerable extent (see § 7 (v)).

It will become clear that it is convenient to define a parameter γ by

$$\gamma = \sqrt{\tilde{\xi}/\tau}. \tag{4·11}$$

Eqs. (4·6) and (4·7) are written as follows:

$$\gamma = \left[-\frac{1+c_0^2}{4}(1+\gamma^2) + \frac{1-c_0^2}{2}\gamma \right] \tau \tag{4·12}$$

$$\eta = \pm \tfrac{1}{2}(1-\gamma^2)\tau. \tag{4·13}$$

The range of the possible values of γ is determined from the condition $0 \leqq \beta \leqq \alpha$ ($0 \leqq \alpha \leqq \beta$) for the forward (backward) region. From eqs. (4·12), (4·13) and (3·4),

$$\left.\begin{array}{ll} \text{for the forward region:} & \sqrt{\dfrac{\beta}{\alpha}} \\ \text{for the backward region:} & \sqrt{\dfrac{\alpha}{\beta}} \end{array}\right\} = \frac{(1-c_0)-\gamma(1+c_0)}{\gamma(1-c_0)-(1+c_0)}, \tag{4·14}$$

and this gives

$$(1-c_0)/(1+c_0) \leqq \gamma \leqq 1, \tag{4·15}$$

where the left equality holds at the forward and backward boundaries of the non-trivial region and the right equality ($\gamma = 1$) corresponds to the central point ($v = 0$).

To reduce the expressions for the J_n's into a simpler form, we use the approximate integral formula (B·11) to get

$$Q(\beta, \alpha; p) \approx \frac{2c_0}{(1-c_0)\sqrt{\dfrac{\alpha}{\beta}} + (1+c_0)}$$

$$Q(\beta, \alpha\,; q) \approx \frac{2c_0}{(1+c_0)\sqrt{\dfrac{\alpha}{\beta}}+(1-c_0)}\,.$$

Then from eqs. (4·3), (4·4) and the similar eqs. for the backward region, we get

$$\Delta_n'(\gamma) \approx \Delta_n(\gamma)/\gamma, \tag{4·16}$$

and

$$\Delta_n(\gamma) \approx (l/\sqrt{\pi})\frac{(1-c_0^2)^{1/2}(1+\gamma)}{\sqrt{2(1+c_0^2)\gamma-(1-c_0^2)(1+\gamma^2)}}\cdot\frac{A_n(\gamma)}{\sqrt{\tau}} \tag{4·17}$$

where

$$A_n(\gamma) = \frac{n+1}{2}-\frac{n-1}{2c_0^2}\frac{1-\gamma}{1+\gamma}\,, \quad \text{: for forward} \tag{4·18}$$

$$= \frac{n+1}{2}+\frac{n-1}{2c_0^2}\frac{1-\gamma}{1+\gamma}\,. \quad \text{: for backward} \tag{4·18'}$$

In eq. (4·17) the variable τ still remains. In practical problems, its value will be determined from $y_c = \ln(T_c/T_0)$, where $T_c \sim m_\pi c^2$ is the critical temperature at which the interaction breaks off. If the break-off should take place during the one-dimensional motion, τ would be given by

$$\tau_c = \left[-\frac{1+c_0^2}{4}(1+\gamma^2)+\frac{1-c_0^2}{2}\gamma\right]^{-1} y_c \tag{4·19}$$

from eq. (4·12). We shall see however, in § 6, that the more consistent value of τ is given by

$$\tau \approx \frac{1}{1+\gamma^2}\ln(2E_0/Mc^2), \tag{4·20}$$

which will be obtained when the free emission of particles takes place in the three-dimensional stage of motion.*

The n-dependence of our solution is thus factorized in the form of $A_n(\gamma)$, which is given by eqs. (4·18) and (4·18'). It is here important that $A_n(\gamma)$ depends only on the parameter γ and n and does not depend on the temperature, y; that is, it does not depend on the primary energy, E_0.

We shall remark here that the non-dependence of $A_n(\gamma)$ on the variable y can be regarded as correct so long as we are considering the high energy collisions, where the effective values of $y = y_c$ have large magnitude. A superficious consideration might suggest that eqs. (4·16)–(4·18) are not so correct ones since eq. (4·3) or (4·4) contains the sum or the difference of two terms among which one contains Q and

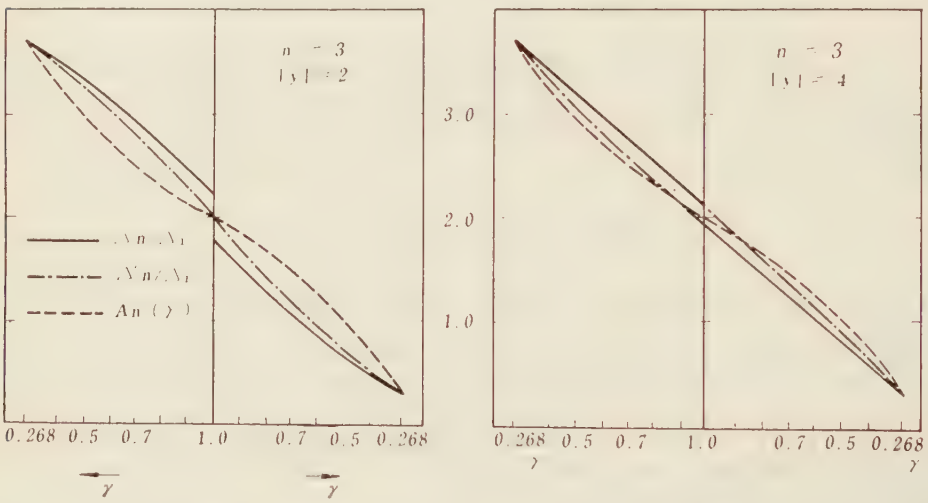
* The value of τ given by eq. (4·20) corresponds to the zeroth order approximation for τ, and contains an error of some 20–30% in itself. However, this error produces very little influences on the physical results of our actual interest.

another does not and we have used an asymptotic approximation for the Q. Indeed, we have defined $A_n(\gamma)$ by

$$A_n \fallingdotseq \Delta_n(\gamma)/\Delta_1(\gamma) \fallingdotseq \Delta_n'(\gamma)/\Delta_1'(\gamma), \tag{4·21}$$

and the correct (Δ_n/Δ_1) is different from the correct (Δ_n'/Δ_1'), and they depend not only on γ and n but also on y. However, it can be seen that our approximate expression for $A_n(\gamma)$ represents the essential features of the results rather well, in so far as we consider the events with primary energies $E_0 \geq 10^{12}$ ev, which correspond to $|y_c| \geq 2$. This will be seen in Figs. 3a, b, in which the approximate expression $A_n(\gamma)$ is compared with the correct (Δ_n/Δ_1) and (Δ_n'/Δ_1') for $n=3$. (c_0 is assumed to be $(1/3)^{1/2}$.) In these figures, the discontinuity at $\gamma=1$ for the correct functions is due to the different way of fixing the coordinate origin for the forward and backward regions. (That the approximate $A_n(\gamma)$ is continuous at $\gamma=1$ is related to the fact that the point at which $v=0$ always coincides with $x=0$ in our approximation.)

For the latter use, we give here the one-dimensional distribution of the energy and the entropy. The amount of the entropy, dS, and the energy, dE, contained in the thickness $d\hat{o}$ is, respectively,



(Fig. 3a) (Fig. 3b)

The correct $(\Delta_n(\gamma)/\Delta_1(\gamma))$, $(\Delta_n'(\gamma)/\Delta_1'(\gamma))$ and their approximate expression $A_n(\gamma)$ (for $n=3$). $|y|=2$ and 4 corresponds to $E_0 \sim 10^{12}$ ev and 10^{15} ev respectively.

$$\begin{aligned} dS &= \pi a^2 (su)\, d\hat{o}, \\ dE &= \pi a^2 |(\varepsilon + p) u^2 - p|\, d\hat{o}, \end{aligned} \tag{4·22}$$

where

$$s = s_0 (T/T_0)^{1/c_0^2}, \qquad \varepsilon = \varepsilon_0 (T/T_0)^{1+1/c_0^2} \tag{4·23}$$

are the entropy density and the energy density in the local rest system. Using the solutions (4·6) and (4·7) or (4·12) and (4·13), eqs. (4·22) are written as

$$dS \approx \pi a^2 s_0 (\Delta_n'/2) \exp\left\{-\frac{(1-c_0^2)}{4c_0^2}(\sqrt{\tau}-\sqrt{\tilde{\xi}})^2\right\} d\tilde{\xi}$$

$$= \pi a^2 s_0 (\tau \cdot \Delta_n) \exp\left\{-\frac{1-c_0^2}{4c_0^2}(1-\gamma)^2\tau\right\} d\gamma \qquad (4\cdot 24)$$

$$dE \approx (\pi a^2/4)(1+c_0^2)\varepsilon_0 \Delta_n' \exp\left\{-\frac{(1-c_0^2)^2}{4c_0^2}\left(\sqrt{\tau}-\frac{1+c_0^2}{1-c_0^2}\sqrt{\tilde{\xi}}\right)^2\right\} d\tilde{\xi}$$

$$= (\pi a^2/2)(1+c_0^2)\varepsilon_0 (\tau \cdot \Delta_n) \exp\left\{-\frac{(1-c_0^2)^2}{4c_0^2}\left(1-\frac{1+c_0^2}{1-c_0^2}\gamma\right)^2 \tau\right\} d\gamma \qquad (4\cdot 25)$$

where we have neglected the smaller terms and used the approximate relation

$$d\delta = \Delta_n' \exp(\tilde{\xi}) d\tilde{\xi} = 2\tau \Delta_n \cdot \exp(\gamma^2 \tau) d\gamma. \qquad (4\cdot 26)$$

Eqs. (4·24) and (4·25) show respectively, that, as pointed out by Landau[10], the entropy is largely concentrated in the region $\gamma = 1$, while the energy is concentrated in the narrow region $\gamma = (1-c_0^2)/(1+c_0^2)$, in the $(\tau, \tilde{\xi})$ scale. (In the ordinary (t, x) scale, the maxima of the entropy and of the energy distributions are both very close to the fronts, but the entropy distribution has much larger tail towards the central region.)*

Having derived several formulae for the (one-dimensional) motion of nuclear matter, we shall now proceed to apply them to get the physical results of the hydrodynamical theory.

Part II
Applications

§ 5. Some results from the one-dimensional solution

Before considering the three-dimensional motion, we shall discuss in this section some important results which can be inferred from the one-dimensional solution.

5.1 Energy and particle number contained in the progressive wave

It has been pointed out by Gerasimova and Černavskij[16] that the estimation of energy and particle number contained in the progressive wave region has some significance for discussing the dynamical properties of secondary particles. Since the calculation is essentially the same as theirs, we shall mention here only the results leaving detailed calculations to be made in Appendix C.

Denoting by ΔN_1 and ΔE_1 the number of particles and the energy (in the e. v. s.) which are contained in the backward progressive wave, and by N_0 and $2E$ the total number of secondary particles and the total energy (in the e. v. s.) in the case of a

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* Landau's statement[10),14] on this point is not very accurate. He has perhaps overlooked the logarithmic dependence of $\tilde{\xi}$ and τ on $\delta = t - |x|$ and $\zeta = t + |x|$.

nucleon-nucleon collision, we define by

$$\alpha_1 = \Delta N_1 (N_0/2), \qquad \beta_1 = \Delta E_1 / E \tag{5·1}$$

the fraction of particle number and energy, respectively, contained in the backward progressive wave. Then using the results in § 3 and Appendix A. 1, we get (Appendix C),

$$\alpha_1 = (l_1/l)\alpha, \qquad \alpha_2 = (l_2/l)\alpha. \tag{5·2}$$

$${}_l\beta_1 = (l_1/l)\,{}_l\beta, \qquad {}_l\beta_2 = (l_2/l)\,{}_l\beta, \tag{5·3}$$

where the suffix 2 refers to the corresponding quantities in the forward progressive wave, and

$$\begin{aligned} \alpha &\approx (1/2)\,(E_0/2Mc^2)^{-(1-2c_0^2)/4c_0^2} \approx (1/2)\,(E_0/2Mc^2)^{-1/4} \\ \beta &\approx \frac{1+c_0}{2}(E_0/2Mc^2)^{-(1-c_0-c_0^2)/4c_0^2} \approx \frac{1+\sqrt{3}}{2\sqrt{3}}(E_0/2Mc^2)^{-(2-\sqrt{3})/4} \end{aligned} \tag{5·4}$$

refer to the nucleon-nucleon collision.* E_0 is the laboratory incident energy and

$$\begin{aligned} l_1 &= l\left\{\frac{n+1}{2} + \frac{n-1}{2c_0}\right\} \\ l_2 &= l\left\{\frac{n+1}{2} - \frac{n-1}{2c_0}\right\} \end{aligned} \qquad l = (c_0^2/(1+c_0^2))d \tag{5·5}$$

are the appropriate units of length in the backward and the forward progressive wave regions, respectively.** (Cf. Table 2, in which l_1 and l_2 are shown for $c_0 = (1/3)^{1/2}$.) We note that although eqs. (5·4) are approximate ones, the relations in eqs. (5·2) and (5·3) are exact.** The values of α and β are given in Table 5. 2 assuming $c_0 = (1/3)^{1/2}$.

As pointed out by Gerasimova and Černavskij[16], the actual number of particles contained in each progressive wave is nearly equal to one-half in the case of a nucleon-nucleon collision and this is independent of the incident energy. For a collision with a tunnel of the length n, eqs. (5·2), (5·3) and Table 2 indicate that the number of particles contained in the backward progressive wave is roughly equal to n, while the corresponding number in the forward progressive wave is practically zero for $n \gtrsim 2$. As for the energy carried by a single particle, we see from eqs. (5·2), (5·3) and (5·4) that it is $(\Delta E/\Delta N) = {}_l\beta E/(\alpha N_0/2) \approx {}_l\beta E$ for either of the backward and the forward region. (Note that ${}_l\beta_1/\alpha_1 = {}_l\beta_2/\alpha_2 = {}_l\beta/\alpha$.) It should be remembered, however, that there is practically no particle in the forward progressive wave (for $n \gtrsim 2$), so that the energy is difficult to find its carrier in this region. We may therefore expect that a particle, which will find itself largely in the non-trivial region close to the boundary, will carry out about such an amount of energy. (Cf. the discussions in § 7).

* There are some mistakes in the equations of reference[16].

** See the argument in § 2.

Table 2.

n	l_1/l	l_2/l	l_1/l_2
1	1	1	1
2	2.73	0.634	3.73
3	3.73	0.268	13.9
3.73	4.73	0	∞

Table 3.

E_0(ev)	α	β	E_0(ev)	α	β
10^{12}	0.104	0.435	10^{16}	0.010	0.235
10^{13}	0.059	0.373	10^{17}	0.0059	0.201
10^{14}	0.033	0.320	10^{18}	0.0033	0.176
10^{15}	0.019	0.274			

5.2 *Partition rate of secondary particles in wide and narrow cones*

In the study of jet showers, it has sometimes been the subject of discussions whether the angular distribution of secondary particles is symmetric with respect to the forward and backward directions in the center-of mass systm. In our model there is no such symmetry and it will be of some interest to consider how the energy and the particles are partitioned among the forward and the backward directions, in the e. v. system. Although more detailed behaviour of the angular distribution is discussed in § 6 and § 7, our calculation in this sub-section needs only the one-dimensional solution, and can be performed with much better accuracy.

We first consider the approximate formulae (4·24) and (4·25) for the entropy and energy distributions, and note that the variables τ and $\tilde{\xi}=\gamma^2\tau$ introduced by eqs. (4·5) and (4·11) are defined in each direction separately. Therefore, eq. (4·24) (or (4·25)) gives, when integrated with respect to γ from 1 to $(1-c_0)/(1+c_0)$, the amount of entropy (or energy) contained in one side of the non-trivial region. Our task is to get the ratio of the integral for the forward direction to that for the backward direction.

If we consider the distributions at sufficiently long after the collision, the variable τ is large enough and the exponential factors in eqs. (4·24) and (4·25) have strong peaks at $\gamma=1$ and $\gamma=(1-c_0^2)/(1+c_0^2)$, respectively. Moreover, the progressive wave regions will have no important contributions for very large τ. Therefore, the ratio of the integrals for both directions can be obtained by substituting for the parameter γ in the coefficient Δ_n the values at which the contributions are largest ($\gamma=1$ for the entropy and $\gamma=(1-c_0^2)/(1+c_0^2)$ for the energy), and this can be regarded to include both the non-trivial and the progressive wave regions. Now, from eqs. (4·12), (4·13) (4·18) and (4·18′), we get,

for $\gamma=1$, $(\eta=0, y=-c_0^2\tau)$

$$A_n(\gamma)=(n+1)/2, \quad \text{for both forward and backward,} \tag{5·6}$$

and for $\gamma=(1-c_0^2)/(1+c_0^2)$, $\left(\zeta=\pm\frac{2c_0^2}{(1+c_0^2)^2}\tau,\ y=-\frac{c_0^2}{1+c_0^2}\tau\right)$

$$A_n(\gamma)=\begin{cases}1: \text{for forward,}\\ n: \text{for backward.}\end{cases} \tag{5·7}$$

Thus we can infer that after an infinitely long time, the partition rate of particles and the partition ratio of energy are given by (note that $J_n(\gamma)\propto A_n(\gamma)$, and that entropy is proportional to particle number):

$$N_{\text{forward}}:N_{\text{backward}}=1:1, \qquad (t=\infty) \tag{5·8}$$

$$E_{\text{forward}}:E_{\text{backward}}=1:n. \tag{5·9}$$

The result (5·9) is just what to be expected from the consideration of the momentum conservation (see the footnote on p. 248), and we may expect that it will be also true for larger $n(n>n_c)$. However, we must note that the results in eqs. (5·8) and (5·9) are correct only at the limit of infinitely long time. As will be shown just below (eqs. (5·11) and (5·12)), the energy ratio approaches the limiting value (5·9) very rapidly but the number ratio is slow in approaching (5·8)*. Therefore (5·8) is roughly correct only for slow particles in the e. v. s. ($\gamma\sim1$), but not for the whole particles. Now it is needless to say that those particles which are moving backwards in the e. v. s. with large velocities also become slow when transformed into the laboratory system, and can be observed in the wide cone. Hence we need a little more detailed calculation to get the required partition rate.

In the approximation used in § 4, the point at which the velocity is zero remains to be at the origin. Actually, this point moves slowly with the velocity, $v(0)$, which becomes smaller with time. Then, there is a flow of energy and entropy from the backward region ($v<0$) into the forward region ($v>0$), and their amount per unit time is given by

$$\frac{dE}{dt}=\pi a^2\mathcal{E}_0(T/T_0)^{1+1/c_0^2}|v(0)|, \quad dS/dt=\pi a^2 s_0(T/T_0)^{1/c_0^2}|v(0)|, \tag{5·10}$$

where T/T_0 is the temperature at the point where $v=0$ divided by the initial temperature, and πa^2 is the area of the cross section. Using the expression for $v(0)$ given in Appendix D (eq. (D·7)), and $T/T_0=\exp(y)$, eqs. (5·10) are written as

* It must be noted also that the assumption of the constant sound velocity used in § 4 can be regarded as correct only for high temperature region ($T\gtrsim T_c$ $m_\pi c^2$, say), and breaks down for lower temperatures. ($T>0$ everywhere as $t\to\infty$). In this sub-section, we are dealing with a hypothetical problem in which c_0 is supposed to be constant ($=1/\sqrt{3}$) for all the values of the temperature. But our method of calculation is such that the final result eq. (5·14) is correct, apart from some approximations made in the course of the calculation, so long as we can regard $c_0=$const. for $T\geq T_c$.

$$\left(\frac{dE}{dt}\right)dt=(E/2)\,(n-1)\,g(-y)\exp(y)\,d(-y) \tag{5·11}$$

$$\left(\frac{dS}{dt}\right)dt=(S_0/4)\,(n-1)\,g(-y)\,d(-y) \tag{5·12}$$

with

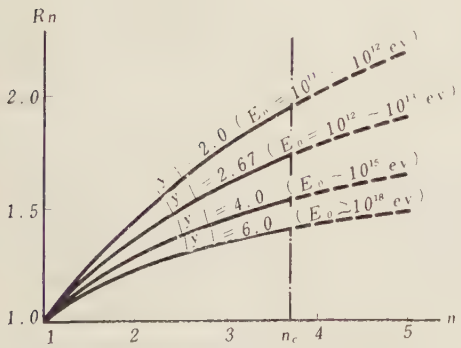
$$g(-y)=(1/2\sqrt{\pi(1-c_0^2)})\,(-y)^{-3/2}\left(1+\frac{2+3c_0^2}{2(1-c_0^2)}\,\frac{c_0^2}{(-y)}\right).$$

Here E and S_0 are, as before, the incident energy in the e. v. s. and the total entropy in the case of the nucleon-nucleon collision with the same incident energy. The total entropy in the nucleon-nucleus case is $S=S_0(n+1)/2$.

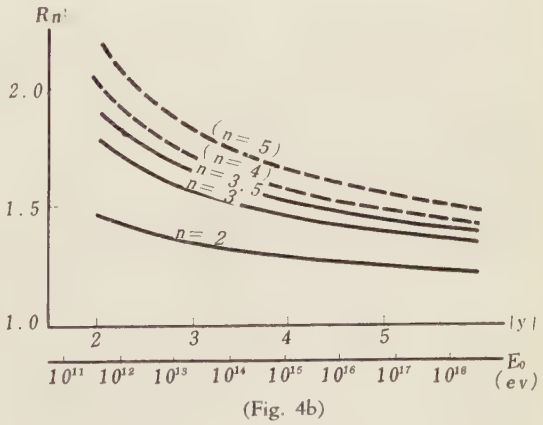
Eq. (5·11) confirms our previous statement that the limiting ratio (5·9) is attained quickly. For the entropy, on the other hand, we now integrate eq. (5·12) from a given time t to $t=\infty$, and get, for the amount of entropy which moves from the backward to the forward direction after the time t,

$$\Delta S(-y)=(S_0/2)\frac{n-1}{2}\,\frac{1}{\sqrt{\pi(1-c_0^2)}}\,\frac{1}{\sqrt{-y}}\left(1+\frac{2+3c_0^2}{6(1-c_0^2)}\,\frac{c_0^2}{(-y)}\right). \tag{5·13}$$

Noting that the total entropy $S=S_0(n+1)/2$ is equally divided in both directions at $-y=\infty$, we get for the entropy in each direction at $y=y_c$,



(Fig. 4a)



(Fig. 4b)

$R_n=N_{\text{back}}/N_{\text{for}}$, where N_{back} and N_{for} stand for the number of the particles emitted backward and forward respectively, in the equi-velocity system. E_0 is the incident energy in the laboratory system, and n is the number of nucleons in the tunnel.

$$S_f(-y_c)=S_f(\infty)-\Delta S(-y_c)=\frac{S}{2}-\Delta S(-y_c),$$

$$S_b(-y_c)=S_b(\infty)+\Delta S(-y_c)=\frac{S}{2}+\Delta S(-y_c),$$

where the suffix f stands for "forward" and b for "backward". Using eq. (5·13), and again noting that entropy is proportional to particle number, we obtain the following result for the required number ratio

$$R_n(-y_c)=\frac{N_b(-y_c)}{N_f(-y_c)}=\frac{(n+1)+(n-1)f(-y_c)}{(n+1)-(n-1)f(-y_c)} \tag{5·14}$$

with

$$f(-y)=(1/\sqrt{\pi(1-c_0^2)(-y)})\left(1+\frac{2+3c_0^2}{6(1-c_0^2)}\frac{c_0^2}{(-y)}\right) \tag{5·15}$$

$$=\sqrt{\frac{3}{2\pi}}\frac{1}{\sqrt{-y}}\left(1+\tfrac{1}{4}\frac{1}{(-y)}\right), \quad \text{for} \quad c_0^2=1/3. \tag{5·15′}$$

The ratio R_n is nothing but the ratio of particles contained in the wide cone to those contained in the narrow cone, provided that these are distinguishable experimentally. The values of R_n calculated by eqs. (5·14) and (5·15) are shown in Figs. 4a and b for several values of the incident energy.* (For the values of y_c in eq. (5·15′), we use eq. (D·8) and Table D. 2 in the Appendix D.) In these figures, our original one-dimensional solution cannot be used for $n\gtrsim 3.73$, and the dotted curves, which are also calculated from eqs. (5·14) and (5·15′), will probably represent the lower limit of R_n for $n>3.73$.

§ 6. Energy spectrum and angular distribution of secondary particles.**

In the initial stage of the expansion, the motion of the nuclear matter can well be considered as one-dimensional, since the matter is initially concentrated in a very thin circular plate. The distributions of energy and entropy in this one-dimensional stage have been given in § 4.

The subsequent three-dimensional motion of the fluid can be solved only approximately and we shall closely follow Landau's treatment.[10]

Consider a particular element of the fluid which lies at distance δ from the front, when the transverse spread of the system is b. The element is always under the action of the transverse force which will spread out the system sidewards, as well as the longitudinal force which will accelerate the element. The fomer force will be roughly proportional to $1/b$ while the latter to $1/\delta$. Then, as long as the scattering angle is small,

* The accuracy of our calculation was checked by the numerical calculation (graphical method, cf. Appendix A. 3) in the case of $n=2$. The following figures show that our calculation in this sub-section is sufficiently accurate for practical use:

Numerical calculation: $R_2(2.35)=1.4$, $R_2(4.85)=1.2$,

Eqs. (5·14) and (5·15′): $R_2(2.35)=1.41$, $R_2(4.85)=1.25$.

The accuracy can also be estimated from Table D. 1 in Appendix D.

** In this section we put $c_0=1/\sqrt{3}$ for simplifying the expressions.

the ratio of the velocities perpendicular and parallel to the x-axis will be given by $\Delta v_\perp / \Delta v_\parallel \sim \delta/b$.

Now, the free emission of particles will take place during the three-dimensional stage since the cooling of the system is most appreciable in this stage. But the action of the transverse force will become rapidly small after the moment at which the three-dimensional trend of motion predominates over the one-dimensional. Then the emission angle θ will be mainly determined during the one-demensional motion, and the final angle θ will be nearly equal to the angle at the end of one-dimensional motion, which differs for each portion of fluid.

On the other hand, it can be shown[10] that the quantity δ for a particular element approaches rapidly to a constant in the three-dimensional stage. From these considerations, it follows the angle θ of the particular element is given, roughly speaking, by

$$\theta \sim \delta/a \quad (\text{for } \theta \ll 1) \tag{6·1}$$

and this is kept approximately constant during the three dimensional (conical) expansion.

We shall further rewrite eq. (6·1), after Landau[10], so that it includes also those particles which will be emitted with $\theta \sim 1$. That is, hereafter we use the "ansatz,"

$$\tan(\theta/2) = \delta/a. \tag{6·2}*$$

We shall now consider how our parameter γ is related to θ.

i) Connection between θ and γ.

From the equation $db/dt = v\theta = \theta$, which determines the transverse spread of the element, and from (6·2), we have ($\xi = t + |x| \approx 2t$)

$$b \approx a + (\delta\xi/a), \tag{6·3}$$

where a is the value of b at $t=0$. As mentioned above, the three-dimensional motion becomes appreciable when the second term of eq. (6·3) becomes of the same order of a. Thus we connect the one-dimensional solution with the three-dimensional one at this moment ζ_1, where

$$\zeta_1 = a^2/\delta. \tag{6·4}$$

(Hereafter we use the suffix 1 for every quantity to indicate its value at this moment.) On the other hand, it follows by definition that (cf. §4),

$$\delta = (\zeta_1/\gamma_1)\exp[\tau_1(\gamma_1^2 - 1)]. \tag{6·5}$$

From eqs. (6·4) and (6·5) and $\zeta = \Delta e^\tau$, we get

$$\zeta_1/\Delta(\gamma_1) = \exp(\tau_1)$$

* In the following, θ should be replaced by $\pi - \theta$ for the backward region.

$$= \left[\left(\frac{a}{J(\gamma_1)} \right)^2 \gamma_1 \right]^{1/(1+\gamma_1^2)} |L \cdot B|^{1/(1+\gamma_1^2)} \qquad (6 \cdot 6)$$

where

Fig. 5, a

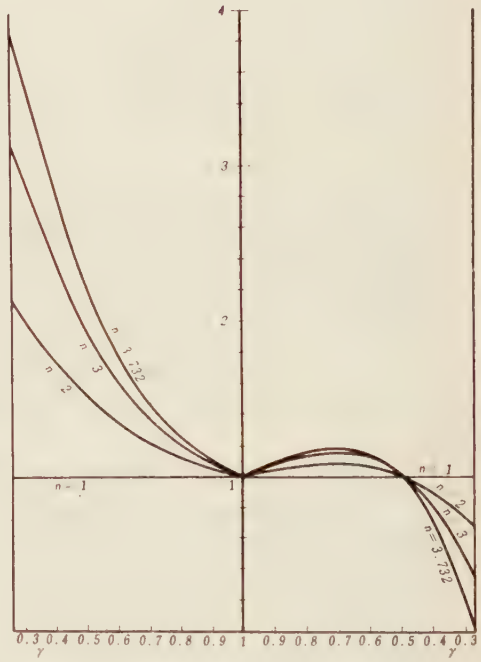


Fig. 5, b

γ is the parameter introduced to simplify the description and connected with $\tan\theta/2$ by the fig. 5 a in the case of nucleon-nucleon collision. In the case of nucleon-nucleus collision we must multiply the factor in Fig. 5 b to the curves in Fig. 5 a. E_0 is the incident energy in L. S. n is the length of penetrating tunnel measured in unit of the nucleon diameter.

$$L(E_0, \gamma) = (2E_0/Mc^2)\{\pi(4\gamma - 1 - \gamma^2)\tau\gamma/(1+\gamma)^2\}$$
$$\approx (2E_0/Mc^2)\ln(2E_0/Mc^2)$$
$$\times \{\pi\gamma(4\gamma - 1 - \gamma^2)/(1+\gamma^2)(1+\gamma)^2\} \qquad (6 \cdot 7)^*$$

and

$$B_n(\gamma) = (1/A_n^2(\gamma))(\tau^{(n)}(\gamma)/\tau^{(1)}(\gamma)) \approx (A_n(\gamma))^{-2}. \qquad (6 \cdot 7')$$

* The variable τ in eq. (6·7) is approximated by $\tau = 1/(1+\gamma^2) \cdot \ln(L \cdot B_n) \approx 1/(1+\gamma^2)\ln \cdot (2E_0/Mc^2)$ (see the statement in the footnote of page. 253).

Eq. (6·2) is then written as follows,

$$\tan\theta/2=(1/\sqrt{\gamma_1})\cdot[L(E_0,\gamma_1)\cdot B_n(\gamma_1)]^{(\gamma_1^2-1)/2(1+\gamma_1^2)} \tag{6·8}$$

Thus, the angle θ is expressed in terms of the parameter γ and n (Figs. 5a and 5b). The factor $B_n(\gamma)$, defined by eq. (6·7′), is such a function of n and γ that becomes identically unity for $n=1$. Eq. (6·8) then means that the relation between θ and γ for $n>1$ is obtained from that for $n=1$ by multiplying the factor $[B_n(\gamma)]^{(\gamma^2-1)/2(1+\gamma^2)}$, which is shown in Fig. 5b.

ii) *Energy carried by a single particle and the transverse component of momentum.*

We shall now try to get several formulae for various dynamical properties of secondary particles. First, we consider the energy E_π, carried away by single particle in the e. v. s. .

Noting that the thermal energy participated for a single particle in the local rest system of the fluid element is ε/ρ, and that the momentum distribution is isotropic in this rest system, we get in the e.v.s. in which the element is moving with the time component u of the four-velocity,

$$E_\pi=(\varepsilon/\rho)u=(\varepsilon_0/\rho_0)(T/T_0)u. \tag{6·9}$$

Since the energy density ($\sim\varepsilon u^2$) and the entropy density ($s\,u$) in the e.v.s. are kept approximately constant at the stage of conical expansion*, and since ρ is proportional to s, we can infer that E_π is determined at the time t_1 and is kept on constant. Therefore

$$E_\pi\approx(\varepsilon_0/\rho_0)(T_1/T_0)u_1, \tag{6·10}$$

or (see Appendix E, we shall hereafter omit the suffix 1 on the parameter γ),

$$\frac{E_\pi}{m_\pi c^2}\approx(1/k)(E_0/2Mc^2)^{1/4}(5.4\sim6.0)[L\cdot B_n]^{(1+2\gamma-5\gamma^2)/(1+\gamma^2)} \tag{6·11}$$

where the lower and upper limits of the numerical values in the right-hand of (6·11) correspond to $T_C=(0.7-1.4)m_\pi c^2$. (For the numerical calculations below, we used 6.0.) k is a numerical constant (we take $k\sim2$) which will be determined from experimental data on the total multiplicity. (Figs. 6a and 6b).

The transverse momentum is obtained as $cP_\perp\approx E_\pi\sin\theta$ and the result is

$$p_\perp/m_\pi c\approx(2/k)(E_0/2Mc^2)^{1/4}(5.4\sim6.0)\gamma^{-1/2}\times[L\cdot B_n]^{(\gamma-\gamma^2-1)/3(1+\gamma^2)}. \tag{6·12}$$

* This is inferred from the conservation of energy and entropy flow through the cone. See reference[10].

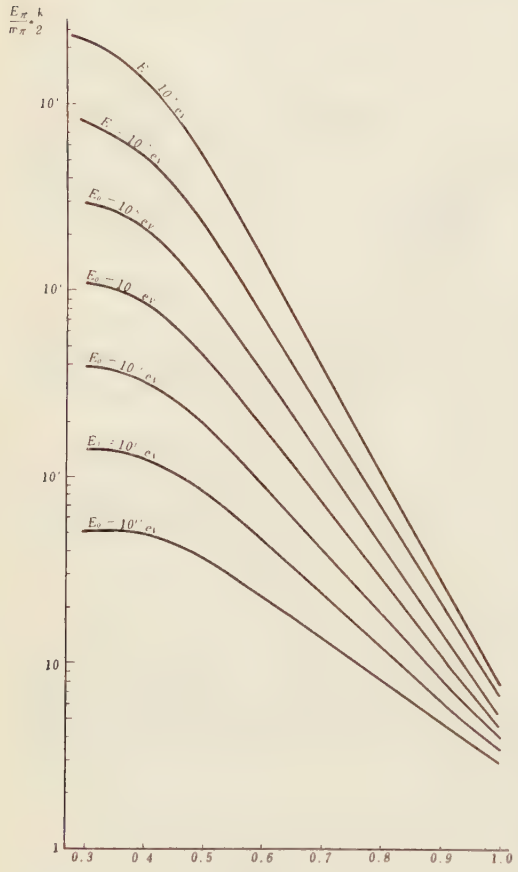


Fig. 6, a

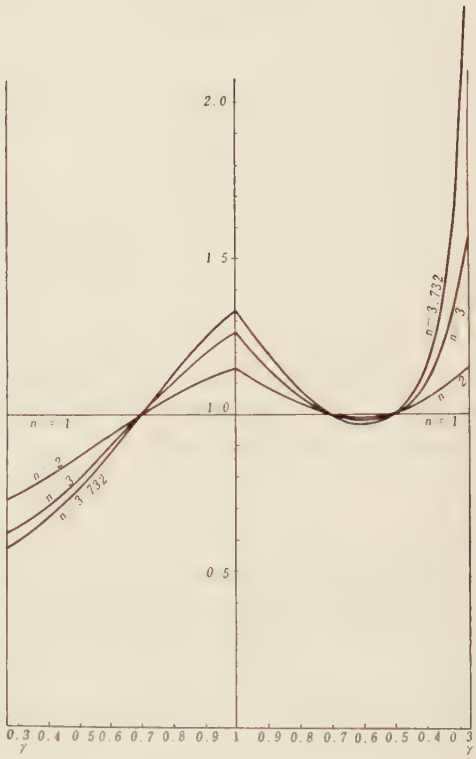


Fig. 6, b

This curve shows E_π *versus* γ for $n=1$, where E_π is the energy of secondary particles in the e. v. s. By making use of the curve in Fig. 5, we get the relation between E_π and emitted angle θ in the e. v. s. In the case of nucleus collision we must multiply the factor in Fig. 6b to the curves in Fig. 6a.

iii) *Angular distributions of particles and energy; Energy spectrum.*

The distributions of particles and energy at the moment of free emission can be estimated by substituting the value τ_1 of eq. (6·6) and the value γ_1 for the variables τ and γ in (4·24) and (4·25). Here we have again assumed that the essential feature of the distributions is determined at t_1, which corresponds to the beginning of three-dimensional motion. Thus we get,

$$\frac{dN}{d\gamma} \approx C \frac{\gamma^{1/2}}{(1+\gamma^2)} [L \cdot B_n]^{(-1+\Upsilon-\Upsilon^2)/1+\Upsilon^2} \tag{6·13}$$

$$\frac{dE}{d\gamma} \approx D \frac{\gamma^{1/2}}{(1+\gamma^2)} [L \cdot B_n]^{-(11\Upsilon^2-8\Upsilon+5)/6(1+\Upsilon^2)}, \tag{6·14}$$

where C and D are constants depending on the incident energy:

$$C \approx 2\pi b_1^2 a \rho_0 \ln(2E_0/Mc^2), \quad (b_1 \approx 2a)$$
$$= 8N_0(E_0/2Mc^2)^{1/2}\ln(2E_0/Mc^2) = 8k(E_0/2Mc^2)^{3/4}\ln(2E_0/Mc^2) \qquad (6\cdot15)$$
$$D \approx (2/3)\pi b_1^2 a \varepsilon_0 \ln(2E_0/Mc^2)$$
$$= (8/3)E_0 \ln(2E_0/Mc^2). \qquad (6\cdot16)$$

Numerical results for the γ-distribution of particles are shown in Figs. 7a and 7b.

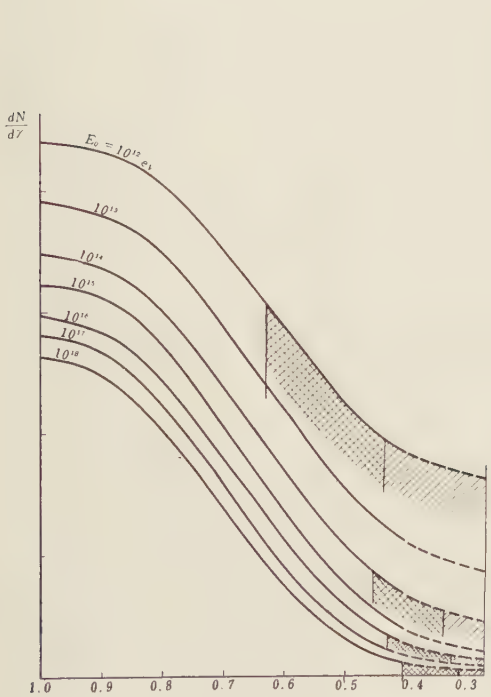


Fig. 7, a

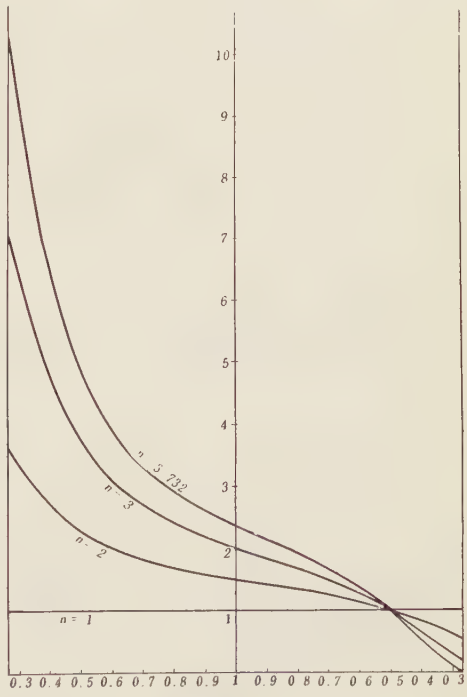


Fig. 7, b

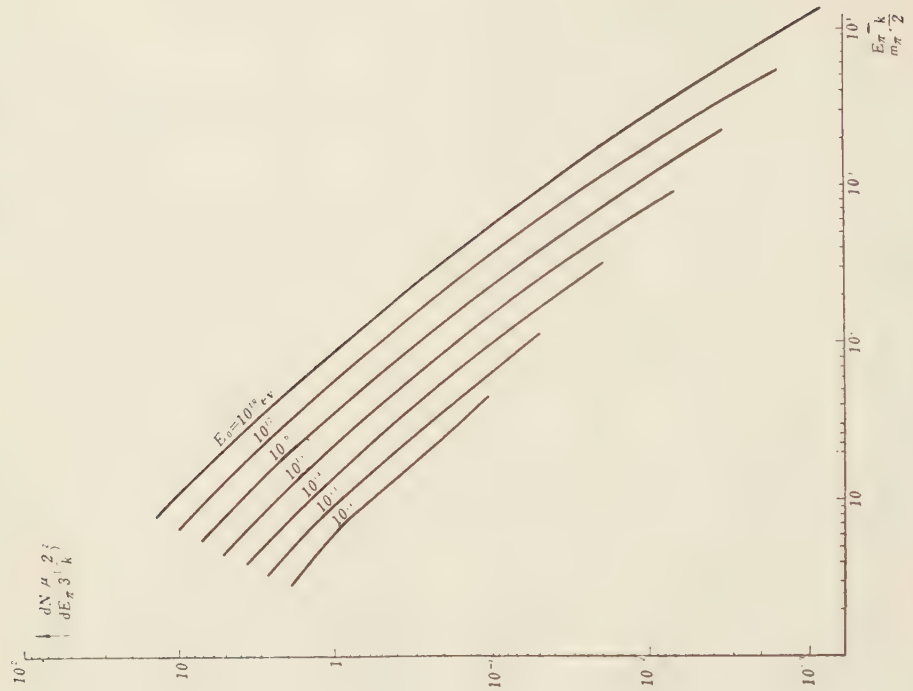
Fig. 7a $dN/d\gamma$ *versus* γ curve for $n=1$. The regions shaded and cross-shaded are non-trivial regions in which the 1st and 2nd particle of greatest energy are contained, respectively. In the case of nucleon-nucleus collision we must multiply the factor in Fig. 7b to the curves in Fig. 7a. We get, for example, the curves in Fig. 9 at the energy $E_0=10^{12}$ ev.

To get the energy spectrum of secondary particles (in the e.v.s), we use $E_\pi \propto \exp[(1-5\gamma^2+2\gamma)\tau_1/6]$, and this gives,

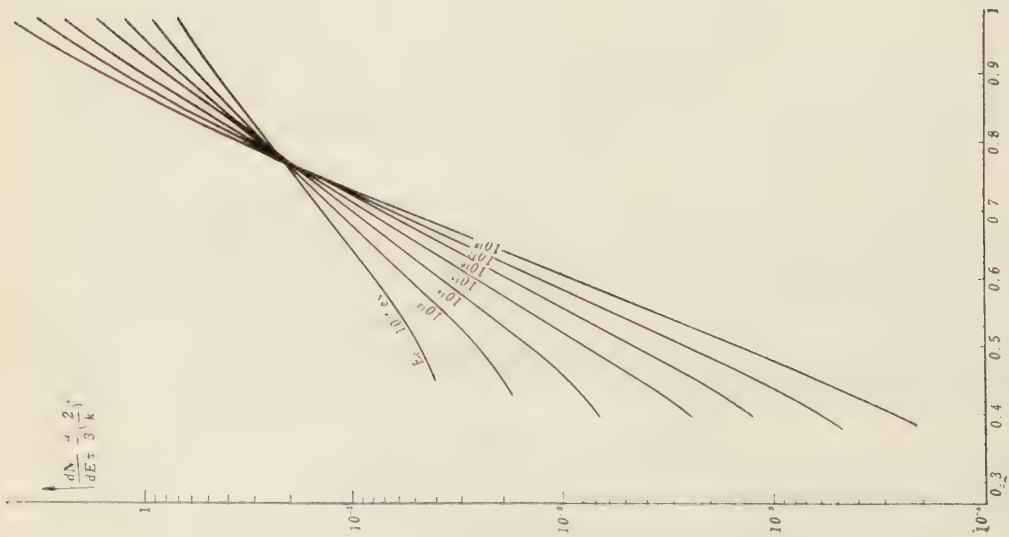
$$dE_\pi/d\gamma = [(1-5\gamma)/3]\cdot\tau_1\cdot E_\pi.$$

Therefore, from eqs. (6·13), (6·15), (6·17) and (4·20), (6·17)

$$m_\pi c^2(dN/dE_\pi) = \frac{3(1+\gamma^2)}{1-5\gamma}\left(\ln\frac{2E_0}{Mc^2}\right)^{-1}(m_\pi c^2/E)(dN/d\gamma)$$


$F_\pi = 10^{16}$ eV
$10^{15.5}$
10^{15}
$10^{14.5}$
10^{14}
$10^{13.5}$
10^{13}
$\frac{dN}{dE_\pi} \propto E_\pi^{-\frac{\gamma+2}{k}}$
$E_\pi - \frac{k}{m_\pi c^2}$

(Fig. 8b)


$\gamma = 1.0$
$\gamma = 1.2$
$\gamma = 1.4$
$\gamma = 1.6$
$\gamma = 1.8$
$\gamma = 2.0$
$\gamma = 2.2$
$\frac{dN}{dE_\pi} \propto E_\pi^{-\frac{\gamma+2}{k}}$
$\frac{E_\pi}{E_0}$

(Fig. 8a)

$$\approx 3(k/2)^2 (E_0/2Mc^2)^{3/4} \frac{8(1+\gamma^2)}{(1-5\gamma)} \left(\frac{m_\pi c^2}{E_\pi} \frac{2}{k} \right)$$

$$\times \left[\frac{\sqrt{\gamma}}{1+\gamma^2} (L \cdot B_n)^{(-1+\gamma-\gamma^2)/(1+\gamma^2)} \right]. \qquad (6 \cdot 18)$$

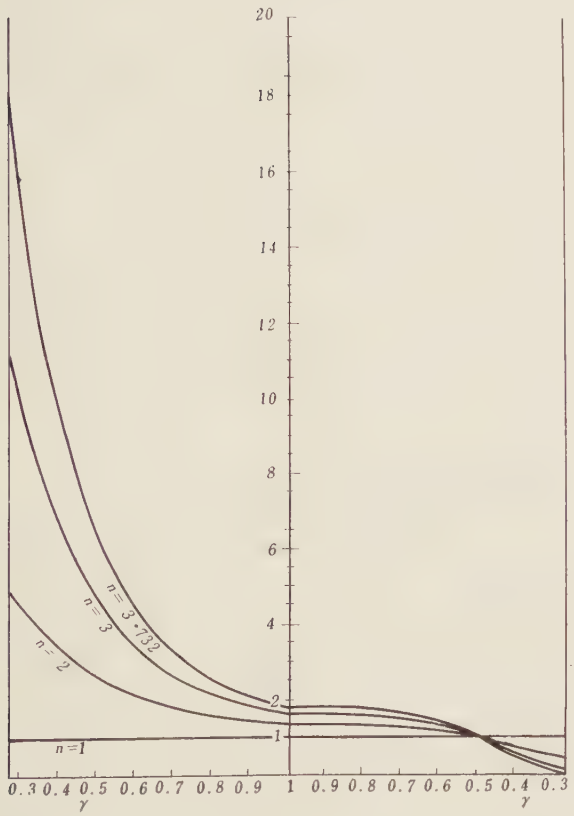


Fig. 8, c

The energy spectum in e. v. s. of secondary particles
(a) dN/dE_π *versus* γ curves, for $n=1$.
(b) dN/dE_π *versus* E curves, for $n=1$.
(c) In the case of nucleon-nucleus collision we must multiply the factor in Fig. 8c to the curves in Fig. 8a.

This is shown in Fig. 8a. The curves in Fig. 8b are obtained by eliminating γ from eqs. (6·18) and (6·11). The n-dependence of the energy spectrum is given by

$$\left(\frac{dN}{dE_\pi} \right)_n \Big/ \left(\frac{dN}{dE_\pi} \right)_{n-1} = B_n(\gamma)^{-(\gamma^2-4\gamma+7)/6(1+\gamma^2)} \qquad (6 \cdot 19)$$

and this is shown in Fig. 8c.

The angular distributions of particles and energy are obtained from eqs. (6·13) and (6·14) if we connect γ with θ by eq. (6·8). Here we should like to say once again that the three-dimensional motion has been treated only approximately, and the results for the angular distributions indicate only the qualitative features, particularly because the relation (6·8) between γ and θ is a rough one. (The same should be stated for the transverse momentum, eq. (6·12).) However, we can expect that the results for the energy spectrum (Figs. 8a-8c) will be more accurate, since we need not use eq. (6·8) to calculate it.

§ 7. Summary of results and discussion

We shall now discuss some physical consequences which can be inferred from our calculations in §§5-6, thereby comparing some of them with the experimental knowledge.

i) *Effective inelasticity*

In the Fermi-Landau theory of multiple production, it is assumed that all the incident energy is used for the production, and therefore the inelasticity parameter is just unity. It is however one of the most significant consequences of the hydrodynamical theory that very few particles which are emitted in a narrow angle in the laboratory system carry away most of the available energy. Then such a fraction of the energy may be interpreted as giving the effective elasticity, since each of such high energy particles will produce again the secondary shower of almost the same size as the primary event, and one could not distinguish this high energy secondary from the primary particle after the collision which suffers some energy loss.*

We have already estimated in §5·1 how much portion of the total available energy is carried away by the progressive wave, which corresponds to about one-half of particle in the case of a nucleon-nucleon collision. Therefore, the non-trivial region close to the front must share the remaining one-half to form a particle. The range of the parameter γ is occupied by this fraction of particle and the range for the next one particle is shown in Fig. 7a.** Then from this and Fig. 6a for E_π, we can estimate how much of the energy is carried away by these two particles. The results are summarized in Table. 4.

Table 4. Energy carried away by the first two particles of greatest energy, in the case of the nucleon-nucleon collision ($n=1$) and nucleon-nucleus ($n=2$) collisions.

Incident energy in the L. S. (ev.)		10^{12}	10^{14}	10^{16}	10^{18}
$n=1$	The fraction of energy carried by the first particle of the greatest energy. (%)	58	45	33	25
	The fraction of energy carried by the next particle of the greatest energy. (%)	22	19	15	11
$n=2$	The fraction of energy carried by the first particle of the greatest energy. (%)	49	37	28	22
	The fraction of energy carried by the next particle of the greatest energy. (%)	16	14	12	10

The situation will be somewhat different in the case of a nucleon-nucleus collision, where there are fewer particles in the forward front region than in the case of $n=1$ (see §5·1 and Fig. 7b), while the values of E_π in the forward non-trivial region is not very different from those of $n=1$ (Fig. 6b). Therefore, there will be the tendency that the average energy of the fastest particle in the case of nucleon-nucleus collisions becomes smaller than that of the nucleon-nucleon collisions. Such a possibility may be fairly large for lower energies ($E_0 \sim 10^{12}$ eV, say), and this will result in the increase of the effective

* In emulsion data, we may estimate the inelasticity from other low energy, and rather large angle particles which contain almost all of the secondary particles. This definition of effective inelasticity is convenient for the experimental analysis.

** γ is a theoretical parameter which is approximately proportional to $\ln \tan \theta/2$, where θ is the angle of emission in the equi-velocity reference system (see § 6 i) and Figs. 5).

(or apparent) inelasticity.* This is also shown in Table 4 for the case of $n=2$.

ii) *The transverse momentum.*

Recently, J. Nishimura[18] pointed out that the analysis of the transverse component of momenta carried by the secondary particles emitted from the jet showers is a useful mean to judge the theory of multiple meson production. According to his analysis, various events so far published yield rather constant values of about 100 to 2,300Mev/c for this quantity, for fairly wide range of the incident energy ($10^{11}\sim10^{15}$ev). We shall therefore consider here what values of the transverse momenta will be obtained by the hydrodynamical theory.

From (6·12)

$$\begin{aligned}\frac{p_\perp}{m_\pi c} &\approx 6(2/k)(E_0/2Mc^2)^{1/12}\left(\frac{n+1}{2}\right)^{-1/3}, \quad \text{for} \quad \gamma=1 \\ &\approx 6(2/k)(E_0/2Mc^2)^{1/20}\left\{\begin{array}{c}1\\ n^{-2/5}\end{array}\right\}, \quad \text{for} \quad \gamma=\tfrac{1}{2},\end{aligned} \tag{7·1}$$

or numerically (we put $k=2$)

$$\begin{aligned}p_\perp &\approx 12\, m_\pi c \approx 2 \text{ Gev}/c, \quad \text{for} \quad E_0=10^{13}\sim10^{15} \text{ ev} \\ &\approx 4 \text{ Gev}/c, \quad \text{for} \quad E_0=10^{18} \text{ ev},\end{aligned} \tag{7·2}$$

and we see that the transverse momenta depend only slightly on the length of tunnel and on the angle (γ), as well as on the incident energy.

The above value of the transverse momenta is due to the hydrodynamical motion, and this will be overlapped by the fluctuation due to the thermal motion. The latter is estimated to be about $2m_\pi c\approx300$ Mev/c (cf. Appendix E).

Thus our theory yields about 1 to 3 Gev/c for the transverse momenta for $E_0=10^{12}\sim10^{15}$ev. These values are somewhat larger than the experimental values of Nishimura's analysis, but may not be said to contradict with the experiment, since our treatment of the emission angle was a rather crude one. It is certain, on the other hand, that the high-temperature equilibrium hypothesis of Fermi will give much larger values.**

iii) *Energy spectrum of secondary particles.*

As mentioned already, the results for energy spectrum is more reliable than the re-

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* This consequence may be interpreted as follows: At the beginning of the process the individual collision between nucleons will occur successively, but after the moderate time the decay or emission of mesons begins in each portion, so the smaller the decay time and the larger the length of the tunnel, the larger is the inelasticity. The ratio between the collision time and the decay time will be roughly proportional to the length of the tunnel $n\left(\frac{\text{collision time for nucleus}}{\text{decay time for nucleus}}\sim\frac{n(M/\mu)}{E_0^{1/4}(\text{Lab. Bev})}\right)$, and this figure will give us some critical measure for the inelasticity.

** The smallness of the (hydrodynamical) transverse momenta is due to the fact that the longitudinal change of pressure is more predominant than the transversal one. The mesons are so close together and each particle has much larger longitudinal momentum than the transverse momentum, and the collision occurs mostly in the longitudinal direction.

sults for angular distributions. In § 6, we have given the result only in the e.v.s. (Fig 8), but they may be easily transformed to the laboratory system. The remarkable consequence of our theory is that the energy spectra for large n show marked asymmetry between forward and backward directions in the e.v.s. (Fig. 8c), and this results in the relative increase of low energy secondary particles in the laboratory system as compared with the case of nucleon-nucleon collisions.

iv) *Angular distribution of particles; symmetry in the c. m. system.*

From the usual individual collision model of inter-nuclear jet cascade, in which the final interactions between produced particles are neglected, we may expect that both the energy and the particles are divided with the common ratio $1:n$ between forward and backward regions in the e.v.s. In our hydrodynamical model, however, we have seen in §5·2 that some particles change the direction of flow during the expansion of system, and that the above ratio for particles between $1:n$ and $1:1$ (see Fig. 7). Although the hydrodynamical theory is still lacking a firm foundation, such a complex behaviour of secondary particles may be considered as an effect of the final interactions.

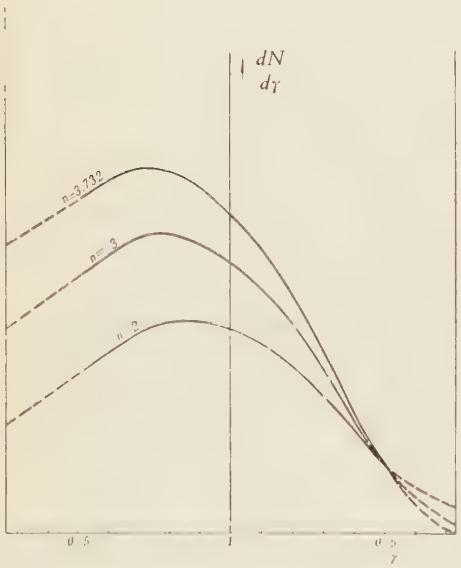


Fig. 9.
$dN/d\gamma$ *versus* γ curves, for $E_0=10^{12}$ ev and several n values. The scale of ordinate axis is arbitrary.

The above mentioned result also means that the wide cone contains more particles than the narrow cone, and that our model does not predict the forward-backward symmetry of number distribution in the c.m. system. Therefore, the usually employed "half-angle method" of estimating the incident energy will become questionable. We shall then examine how much of the error will become questionable. We shall then examine how much of the error will be committed for the incident energy by the half-angle method, *when* the actual nucleon-nucleus collisions should obey our model.

Denoting by ϑ_F and ϑ_{1-F} the laboratory angles which contain the fraction F and $1-F$ of particles respectively, we get for the "apparent incident energy" in the laboratory system.

$$\text{"}E_0\text{"}=2Mc^2/(\vartheta_F\cdot\vartheta_{1-F}). \qquad (7\cdot3)$$

This "E_0" coincides with true E_0 if the angular distribution has the forward-backward symmetry in the c.m. system. In (7·3), ϑ_F and ϑ_{1-F} are obtained by the transformation formula

$$\tan\vartheta=(2Mc^2/E_0)^{1/2}\tan(\theta/2), \qquad (7\cdot4)$$

if E_0 is fixed, and θ_F and θ_{1-F} (angles in the e.v.s.) are determined.

We shall take here the case of $E_0=10^{12}$ ev, as it lies in the most familiar energy region for emulsion experiments. θ_F and θ_{1-F} can be obtained from Fig. 9, where the $(dN/d\gamma)$ *vs.* γ relation is shown for various n, and from Figs. 5a and 5b. The results of calculation are shown in Table 5, where we have defined E_0' and E_0'' by

$$E_0'=2Mc^2/(\vartheta_{1/2})^2, \quad E_0''=2Mc^2/(\vartheta_{1/4}\cdot\vartheta_{3/4}). \tag{7·5}$$

It is seen from this table that the formula (7·3) gives somewhat smaller values for incident energy than the true one. The discrepancy is larger for larger n, as is to be expected.

Table 5. Discrepancy between the true and apparent incident energies in the laboratory system, for true $E_0=10^{12}$ ev. E_0' and E_0'' are given by (7·5).

n	E_0' (ev)	E_0'' (ev)
1	10^{12}	10^{12}
2	4.0×10^{11}	4.1×10^{11}
3.73	2.1×10^{11}	2.4×10^{11}

Next we shall consider Kaplon and Ritson's method[2] of examining the symmetry in the c.m. system, which is to plot the normalized integral angular distribution in the laboratory system

$$F(\vartheta)=\int_0^{\vartheta}(dN(\vartheta')/d\vartheta')\,(d\vartheta'/N) \tag{7·6}$$

against $\log\vartheta$. (N is the total number of shower particles.) It is easy to see that, if the distribution is symmetrical between forward and and backward directions in the c.m. system, then the plot is symmetrical under the reflection through the point $P(F=1/2,\ \log\vartheta_{1/2})$. (cf. Fig. 10). As their data showed this symmetry, Kaplon and Ritson[2] regarded it as an (not unique) evidence in favour of the symmetrical distribution in the c.m. system.

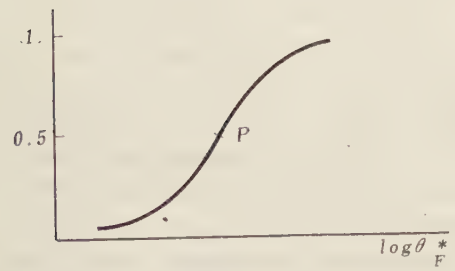


Fig. 10

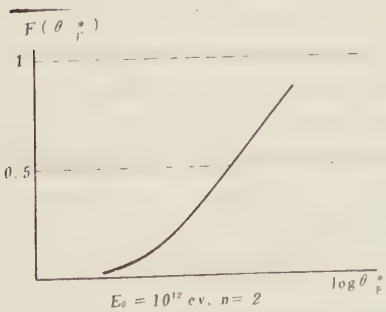


Fig. 11a

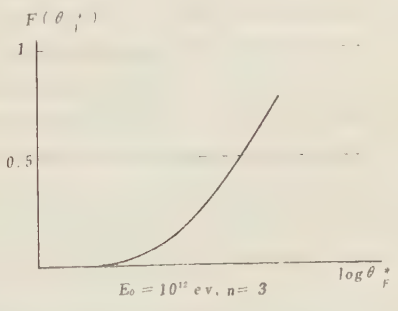


Fig. 11b

Figs. 11a and 11b show the same plot from our theory for $n=2$ and 3, where again $E_0=10^{12}$ eV was chosen. It may be seen that each curve also shows nearly symmetrical behaviour through a point. (The edge parts cannot be given because our approximation for the γ-$\tan(\theta/2)$ relation is bad for $\theta\sim 0$ and $\theta\sim\pi$.) That is, our model, which does *not* predict the symmetrical distribution in the c.m system, is rather consistent with Kaplon and Ritson's experimental result.

These circumstances may be understood as follows. From (7·4), we get ($\tan\vartheta$ can be always approximated by ϑ),

$$\log\vartheta=\log\,(2Mc^2/E_0)^{1/2}+\log\tan(\theta/2)\,. \tag{7·7}$$

It is, however, seen from Figs. 5 that $\log\tan(\theta/2)$ has a practically linear relation with the parameter γ. We may therefore replace the abscissa of Figs. 11 by γ, and we may as well consider the $F'(\gamma)=\int (dN\,d\gamma')\,(d\gamma'/N)$ *vs.* γ plot. On the other hand, we see from Fig. 9 that the point of $F'(\gamma)=\frac{1}{2}\,(\gamma=\gamma_{1/2})$ lies close to the maximum point of $dN/d\gamma-\gamma$ curve, and that the main part of the curve is approximately symmetric about this maximum point. Even if the actual curve deviates from the symmetry, it produces only small influences when we consider the integral $(\int(dN/d\gamma)d\gamma)$. Thus we may get a $F'(\gamma)-\gamma$ plot which is very nearly symmetrical with respect to the reflection through the point $(F'=\frac{1}{2},\ \gamma_{1/2})$, and a similar $F(\vartheta)-\log\vartheta$ plot.

v) *Angular distribution of particles; the anisotropy parameter of Kaplon and Ritson*

In order to measure the deviation from isotropic angular distribution in the c.m. system, Kaplon and Ritson[2] further introdued the quantity X, which is defined by

$$X=x/x_1, \tag{7·8}$$

where $x=(\vartheta_{3/4}/\vartheta_{1/4})$ and x_1 is the value of x for the isotropic distribution in the c.m. system. (Larger deviation from isotropy corresponds to larger X.)

Our theory gives X as a function of E_0 and n, and the results of calculation are shown in Figs. 12a and 12b. Fig. 12a shows for $n=1$ how X increases with increasing the incident energy, and Fig. 12b shows for a given incident energy ($E_0=10^{12}$ev) how X decreases towards unity with the increase of n.

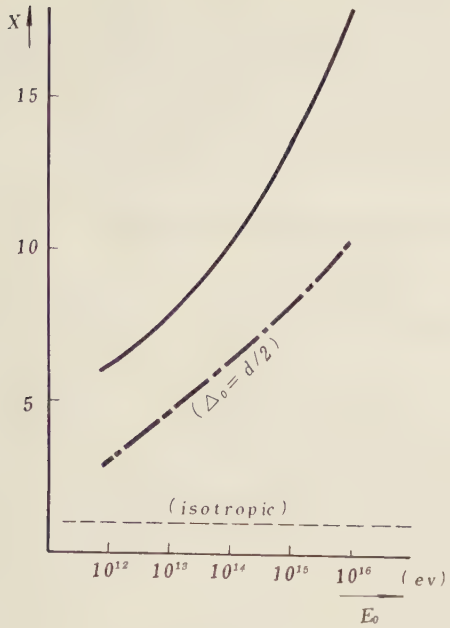
We will not enter into detailed comparison with experimental data (Kaplon and Ritson[2]), but it seems to be important to remark that the calculated values of X are rather sensitive to the precise determination of Δ, which has been defined in § 4. As mentioned there, the exact values of Δ are fairly smaller than $\Delta_0\approx d/2$ used by Landau[10], and this difference can give large influences on the quantitative results of some dynamical properties of secondary particles, in particular on the angular distribution.*

These circumstances may be seen clearly in Fig. 12a, where X calculated from Landau's

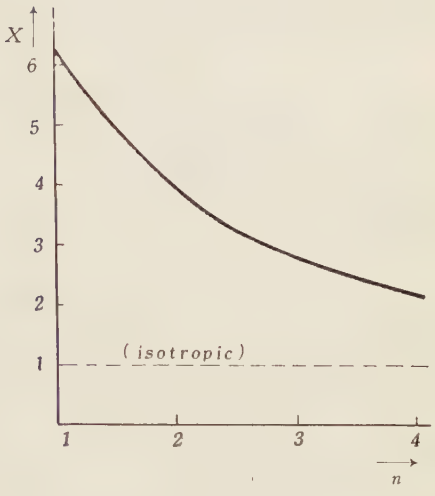
* In Landau's formula for the angular distribution (cf. reference[10])

$$dN/d\lambda\sim\exp(\sqrt{\ }L^2-\lambda^2),\qquad \lambda=-\ln\tan(\theta/2), \tag{7·9}$$

L becomes larger than $L=\frac{1}{2}\ln(E_0/2Mc^2)$ used by Landau, if we use correct values for Δ. Thus, using the correct Δ requires a smaller value of E_0 (smaller by about two orders of magnitude) to get the same angular distribution as given by (7·9).



(Fig. 12a)



(Fig. 12b)

Fig. 12a. *X versus* E_0 curve, for $n=1$. x/x_1, where $X=(\vartheta_{3/4}/\vartheta_{1/4})$ and ϑ_F is the angle in laboratory system which contains the fraction F of particles. E_0 is the incident energy in L. S.

Fig. 12b. *X versus n*curve, for $E_0=10^{12}$ ev n is the length of penetrated tunnel measured in the unit of the radius of nucl eon.

approximate formula (7·9) is drawn by a broken line. We can see that in order to get a given valuse of X, the value of about one hundred times the correct value is required for E_0 if we use (7·9). (For instance, $X=8$ corresponds to $E_0\approx10^{13}$ ev by the full line and to $E_0\approx10^{15}$ ev by the broken line.)

Therefore, we should like to say that the analysis of Kaplon-Ritson's data by Rozental' and Černavskij[1], in which Landau's approximate formula for the angular distribution was used, should be considered as a rather crude one**.

However, there will be large fluctuations in the cosmic ray data, which would disturb the more detailed discussions on these points. We hope, inconcluding this paper, that future experiments would furnish us with more abundant and reliable data.

We would like to express our sincere gratitude to Prof. Z. Koba for his many valuable and stimulating discussions. We would also like to express our sincere thanks to Prof. J. Nishimura for informing us of his results on transverse momenta. We are also indebted to any other theorists and experimentalists with whom we had useful conversations during the "Symposium on Extremely High Energy Phenomena" held at the Research Institute for Fundamental Physics, Kyoto University, Feb 1956, as well as to the

* What was actually analysed by Rozental' and Černavskij[1] includes the data on nucleon-nucleus collisions. But it is easy to show that their procedure to extend Landau's formula (7·9) to the nucleon-nucleus case gives also smaller X than the curve in Fig. 12b.

hospitality of the Institute staff. We are also indebted to Prof. S. Hayakawa, Prof. S. Takagi, Dr. D. Ito, Mr. K. Mori, and Mr. Y. Nishida for their continual discussions and encouragements. Prof. Z. Koba and Mr. T. Yoshimura kindly took the trouble of informing of and translating the Russian papers for us. Two of the authors (C. I. and M. S.) are indebted to the Yomiuri Yukawa Fellowship for the financial aid.

Appendix A. Solution of the one-dimensional problem

A. 1. Progressive waves

The one dimensional equations of the relativistic hydro-dynamics can be written in the following form.[15]

$$\frac{\partial \alpha}{\partial t}+\frac{v-c_0}{1-vc_0}\frac{\partial \alpha}{\partial x}=0, \tag{A·1}$$

$$\frac{\partial \beta}{\partial t}+\frac{v+c_0}{1+vc_0}\frac{\partial \beta}{\partial x}=0, \tag{A·1′}$$

when we introduce varibles α and β by

$$\alpha=\gamma-\lambda(\gamma) \tag{A·2}$$

$$\beta=-\gamma-\lambda(\gamma). \tag{A·2′}$$

Here the quantities c_0 (the sound velocity) and $\lambda(\gamma)$ are given by

$$c_0^2=\left(\frac{s}{T}\frac{dT}{ds}\right), \qquad \lambda(\gamma)=\int(d\gamma/c_0), \tag{A·3}$$

in our extremely relativistic case ($T \gg \mu$, where μ is the chemical potential).* It is easy to see that (A·1) and (A·1′) reduce to $d\alpha/ds=0$ and $d\beta/ds=0$, respectively, along the *characteristic curves*, whose directions in (t, x)-plane are specified by,

$$dx/dt=(v-c_0)/(1-vc_0) \tag{A·4}$$

and

$$dx/dt=(v+c_0)/(1+vc_0) \tag{A·4′}$$

respectively.

* The general definition of λ is

$$\lambda(\varepsilon)=\int d\varepsilon \sqrt{f'(\varepsilon)}/(\varepsilon+f(\varepsilon))$$

where $p=f(\varepsilon)$ is the equation of state.[20] In the opposite extreme case ($T \ll \mu$), we have, instead of (A·3),

$$\lambda=\int c_0(d\rho/\rho), \qquad c_0^2=(\rho/\mu)(d\mu/d\rho),$$

where ρ is the particle density.

The disturbance is said to propagate as a *progressive wave* when either the α or the β is kept constant over a region of the fluid. In our problem, it is intuitively expressed as such a region that the characteristic relation in it at the instant of the outflow does not suffer any influence from the behaviour of fluid in other regions.

Let us assume, for instance, that $\alpha=\alpha_0=$const. Then from (A·2) and (A·2′) we have $\beta=\alpha_0-2\eta$, and eq. (A·1′) is written as

$$\partial\eta/\partial t+l'(\eta)\partial\eta/\partial x=0, \qquad l'(\eta)=(v+c_0)/(1+vc_0), \tag{A·5}$$

where c_0 and v are regarded as functions of η. The general solution of eq. (A·5) is

$$x=l'(\eta)t+f(\eta), \tag{A·6}$$

where $f(\eta)$ is an arbitrary function of η.

The value of α_0 is determined from the initial condition. As will be described in detail in. A. 3, we can choose $\alpha=0$ for the backward progressive wave. On the other hand, $f(\eta)$ is determined from the choice of the reference system so that the progressive wave is at $x=x_1$ for $t=t_1$. Thus we have, for the backward progressive wave, the solution

$$x-x_1=\frac{v+c_0}{1+vc_0}(t-t_1), \quad \text{for} \quad \alpha=0. \tag{A·7}$$

By a similar consideration, we have

$$x-x_2=\frac{v-c_0}{1-vc_0}(t-t_2), \quad \text{for} \quad \beta=0 \tag{A·7′}$$

for the forward progressive wave which will be present in the case of small $n(n<n_c)$.

A. 2. Derivation of eq. (3·8)

We shall now consider to solve the differential eq. (3·7). Hereafter c_0 is assumed to be a constant, and α and β are defined by eqs. (3·4).

Substituting the relations (3·2) in eqs. (A·7) and (A·7′), the *boundary conditions* for the potential χ are obtained as follows:

$$\frac{\partial\chi}{\partial\beta}=\tfrac{1}{4}\left[(1-c_0)(t_1+x_1)e^{\beta(1-c_0)/2}-(1+c_0)(t_1-x_1)e^{-\beta(1+c_0)/2}\right], \quad \text{for} \quad \alpha=0, \tag{A·8}$$

$$\frac{\partial\chi}{\partial\alpha}=\tfrac{1}{4}\left[(1-c_0)(t_2-x_2)e^{\alpha(1-c_0)/2}-(1+c_0)(t_2+x_2)e^{-\alpha(1+c_0)/2}\right], \quad \text{for} \quad \beta=0. \tag{A·8′}$$

Since each of these equations has a simple form, containing only one variable, they can be integrated along the respective axis ($\alpha=0$ or $\beta=0$) ;* and after some calculations we get

* The main reason by which we had to restrict ourselves in the cases $n\leq n_c$ is that the boundary condition can not be expressed in such a simple form for $n>n_c$.

$$\bar{\chi}(0, \beta) = \tfrac{1}{2}\{t_1(e^{-p\beta} + e^{-q\beta} - 2e^{-s\beta}) - x_1(e^{-p\beta} - e^{-q\beta})\}, \tag{A·9}$$

$$\bar{\chi}(\alpha, 0) = \tfrac{1}{2}\{t_2(e^{-p\alpha} + e^{-q\alpha} - 2e^{-s\alpha}) + x_2(e^{-p\alpha} - e^{-q\alpha})\}, \tag{A·9'}$$

where $\bar{\chi}$ is related to χ by eq. (3·6), and the arbitrary constant for the potential is chosen as $\bar{\chi}(0, 0) = 0$.

After such procedures, the solution of eq. (3·7) is straightforward. We shall use the Riemann's integral method for the characteristic initial value problem for the hyperbolic partial differential eqations[21]. The Riemann's function belonging to eq. (3·7) is

$$v(a, b\,;\, \alpha, \beta) = I_0(2s\sqrt{(\alpha - a)(\beta - b)}), \tag{A·10}$$

where $I_0(x)$ is the zeroth order Bessel function for imaginary argument, and the solution is

$$\begin{aligned}\bar{\chi}(\alpha, \beta) = {} & \bar{\chi}(0, 0)v(0, 0\,;\, \alpha, \beta) \\ & + \int_0^{\alpha} da\, v(a, 0 : \alpha, \beta)\bar{\chi}_a(a, 0) \\ & + \int_0^{\beta} db\, v(0, b\,;\, \alpha, \beta)\bar{\chi}_b(0, b).\end{aligned} \tag{A·11}$$

In our particular case $\bar{\chi}(0, 0) = 0$ and $\bar{\chi}_a(a, 0)$ and $\bar{\chi}_b(0, b)$ are obtained from (A·9) and (A·9'). Thus eq. (3·8) is obtained with

$$F(a\,;\, t_2, x_2) \equiv \bar{\chi}_a(a, 0) \quad \text{and} \quad F(b\,;\, t_1, -x_1) \equiv \bar{\chi}_b(0, b). \tag{A·12}$$

A. 3. Graphical method of solving one-dimensional problem

We describe here a method of solving the equations (A·1) and (A·1') graphically by following up the characteristic curves (C.C.'s hereafter)*. The directions of C.C.'s at each point on the (t, x)-plane are determined with the help of equations (A·4) and (A·4') when v(or η) at the point is given and α or β is kept constant along a C.C. as was described in ***A. 1.***

Fig. A. 1 shows the schematic behaviour of C.C.'s on the (t, x)—plane, in the case of $n = 2$. Each of the moments t_0, t_1, etc., corresponds to the same one in Fig. 2. The line of shock waves are represented by a broad line, and the C.C.'s along which $\alpha =$ const. and $\beta =$ const. are represented by full and dotted lines, respectively.

We investigate first the behavior of C.C.'s at the moment just after the arrival of the shock wave at the left end of the nuclear matter (x_1), when the nuclear matter is just going to flow into the vacuum to the left ; and shall find a suitable initial condition imposed on the state of the nuclear matter at this moment. (See fig. A·2)

Though $v = 0$ and $T = T_0$ everywhere in the nuclear matter except at x_1, at this point x_1, where the nuclear matter is in contact with the vacuum, the nuclear matter is going to flow with the light velocity and the energy density is zero, i.e. $v = -1$ and $T = 0$. So we shall connect the region where $v = 0$ and $T = T_0$ with the point x_1 with the help of an infinitesimal region and impose on it the initial condition which assures

* The application of this method to the problem was first suggested by Prof. Z. Koba.

a progressive wave to appear in the region where the matter flow to the left is not yet affected by the matter flow to the right, which takes place after the arrival of the shock wave at the right end x_2.

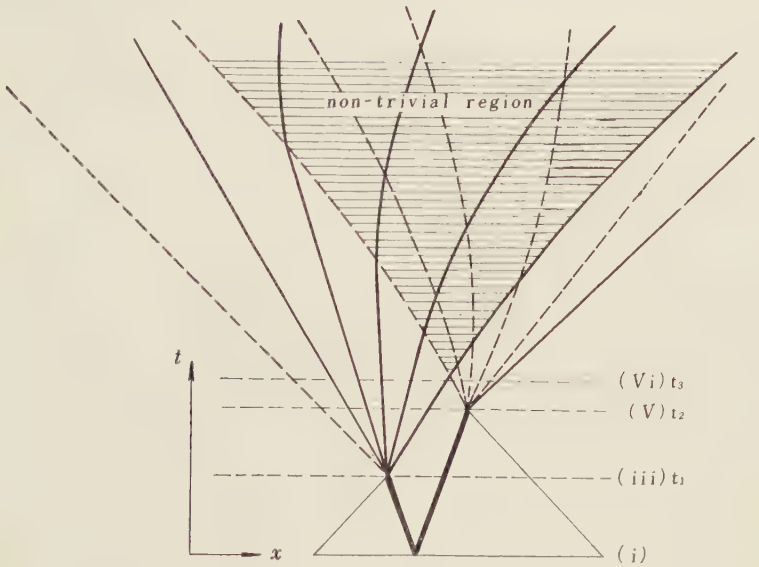


Fig. A 1.

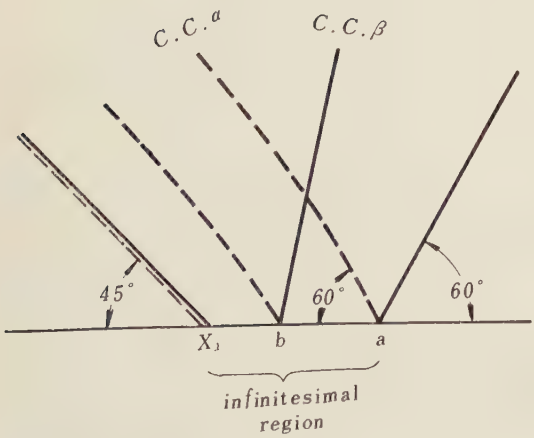


Fig. A 2.

Up to the point a, $v=0$, $T=T_0$, i.e. $y=0$, $\eta=0$; and at the point x_1 $v=-1$, $T=0$, i.e., $y=-\infty$, $\eta=-\infty$; so the C.C.'s start from the points a and x_1 as is illustrated in Fig. A. 2. The initial condition (the distributions of v and T, i.e. of y and η) imposed on the infinitesimal region between a and x_1 determines the directions in which each of the C.C.'s starts from each point between a and x_1.

In setting up the initial condition, we notice the fact that both at the point x_1 and in the region up to the point a*

$$\sqrt{3}\,y-\eta=-\alpha=0.$$

Then we find that the initial condition we need is to require the above relation to hold also in the infinitesimal region between a and x_1. To see this we take two C.C.'s, one

* For definiteness, we put $c_0=1/\sqrt{3}$ in this sub-section.

(CC, β) starting from a point, say b, in the region between a and x_1, and along which

$$-\beta \equiv \sqrt{3}\,y+\eta=\text{const}=-\beta_b$$

and another (CC, α) starting from a point right to b, for example a, and along which

$$-\alpha \equiv \sqrt{3}\,y-\eta=\text{const}=0.$$

Then at the crossing point of the two C.C.'s say c, hold the simultaneous equations,

$$\begin{cases} \sqrt{3}\,y+\eta=-\beta_b \\ \sqrt{3}\,y-\eta=0, \end{cases}$$

which determine the values of y and η at the crossing point. It is easy to see that these simultaneous equations are the equations which determine the values of y and η at the point b, and y as well as η has the same value at b and c. Hence the CC, β does not change its direction and the region of $\sqrt{3}\,y-\eta=0$ (progressive wave) appears (only CC, β's and not CC, α's are straight lines in this case).

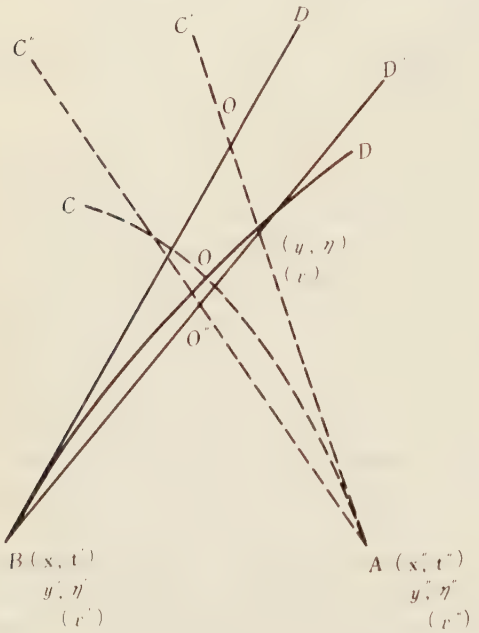


Fig. A, 3

C''(or D'') represents the line whose gradient is the mean of the gradients of the CC's at O and A(or B).

Many full lines starting from the point x_1 in Fig. A·1 are those which start from such infinitesimal region as mentioned above. Dotted curves are discarded since we do not need them in getting the actual solution. At the point x_2 the same procedure is introduced, the only difference being the reversion of the role of α and β.

After the time $t=t_3$, CCβ's strating from x_1 meet with CCα's starting from x_2 along which $\alpha \fallingdotseq 0$, and both CCα's and CCβ's begin to curve. Hence the non-trivial region appears where both CC's are curves.

In the non-trivial region we can follow the CC's only approximately. (See Fig. A·3) Suppose we know the values (y', η'), (y'', η'') of y and η at the points $B(x', t')$, $A(x'', t'')$, so we can get the values of y and η at the crossing point O of the two CC's, one (CC' α) starting from the point A and another (CC' β) starting from the point B, by solving the smultaneous equations. Next, to find the crossing point $O(x, t)$, we approximate the small part of the CC's OA and OB with two straight lines

(C′ and D′) whose gradients coincide with those of CC's at A and B respectively and find the crossing point of the two straight lines graphically or numerically. If better approximation is needed, we can use as the gradient of the straight line not that of the CC at A (or B) itself but the mean of the gradients of the CC at O and A(or B). This procedure corresponds to approximating the curve OA (or OB) by a symmetrical curve (e.g. a part of a circle) between O and A (or B).

Appendix B. Some formulae about the integrals involving $I_0(z)$

B. 1. In the course of deriving eqs. (3·11) and (3·12) from the solution (3·8) for the potential $\bar{\chi}(\alpha, \beta)$ and the relations (3.2), which are now rewritten as

$$\begin{aligned} t+x &= -(1/c_0)\exp(p\alpha+q\beta)\{(1+c_0)\bar{\chi}_\alpha+(1-c_0)\bar{\chi}_\beta+2s\bar{\chi}\}, \\ t-x &= -(1/c_0)\exp(q\alpha+p\beta)\{(1-c_0)\bar{\chi}_\alpha+(1+c_0)\bar{\chi}_\beta+2s\bar{\chi}\}, \end{aligned} \tag{B·1}$$

we need to calculate the derivatives $(\partial J/\partial\alpha)$ and $(\partial J/\partial\beta)$ of the integrals

$$J(\alpha, \beta; r) \equiv \int_0^\alpha da\, I_0(2s\sqrt{(\alpha-a)\beta})\exp(-ra). \tag{B·2}$$

$$(r=p,\ q,\ s)$$

Using the power series expansions for the functions I_0 and I_1

$$I_0(2s\sqrt{(\alpha-a)\beta}) = \sum_{m=0}^{\infty} \frac{s^{2m}}{(m!)^2}(\alpha-a)^m\beta^m, \tag{B·3}$$

$$\frac{I_1(2s\sqrt{(\alpha-a)\beta})}{\sqrt{(\alpha-a)\beta}} = \sum_{m=0}^{\infty} \frac{s^{2m+1}}{m!\,(m+1)!}(\alpha-a)^m\beta^m, \tag{B·4}$$

and making use of suitable integrations by parts, we obtain easily the following results:

$$\frac{\partial J}{\partial\alpha} = I_0(2s\sqrt{\alpha\beta}) - r\int_0^\alpha da\, I_0(2s\sqrt{(\alpha-a)\beta})e^{-ra}, \tag{B·5}$$

$$\frac{\partial J}{\partial\beta} = \frac{s}{r}\left\{\alpha\frac{I_1(2s\sqrt{\alpha\beta})}{\sqrt{\alpha\beta}} - s\int_0^\alpha da\, I_0(2s\sqrt{(\alpha-a)\beta})^{-ra}\right\}, \tag{B·6}$$

and similar ones for $J(\beta, \alpha; r)$, which is obtained by interchanging α and β in eq. (B·2). The first term in eq. (B·6) does not contribute to eqs. (3·11) and (3·12). We shall also note that there are following relations among the p, q, and s from their definition in eqs. (3·10):

$$\begin{aligned} &(1-c_0)p^2+(1+c_0)s^2-2sp = (1+c_0)q^2+(1-c_0)s^2-2sq = 0, \\ &(1+c_0)p^2+(1-c_0)s^2-2sp = 2c_0p \\ &(1-c_0)q^2+(1+c_0)s^2-2sq = 2c_0q, \quad \text{etc.}. \end{aligned}$$

B. 2. When p and q are arbitrary constants, the following formula is obtained:

$$-p\int_0^{\alpha} da\, I_0(2\sqrt{pq(\alpha-a)\beta})e^{-pt}-q\int_0^{\beta} db\, I_0(2\sqrt{pq\alpha(\beta-b)})e^{-qh}$$

$$=e^{-(p\alpha+q\beta)}-I_0(2\sqrt{pq\alpha\beta}), \tag{B·7}$$

and this can be used to reduce two integrals in eqs. (3·11) and (3·12) into one (note that $pq=s^2$). In doing so, the first term in eq. (B·7) yields a constant term which can be put to zero by the suitable choice of the origin.

The proof of eq. (B·7) is as follows: Using the similar formula as eq. (B·3), we get

$$-p\int_0^{\alpha} da\, I_0(2\sqrt{pq(\alpha-a)\beta})e^{-pt}$$

$$=\sum_{m=0}^{\infty}{}'\sum_{n=0}^{\infty}{}'\frac{(-q\beta)^m(-p\alpha)^{m+1}}{(m!)^2 n!}\int_0^{\alpha} da(\alpha-a)^m(-pa)^n \tag{B·8}$$

$$=\sum_{m=0}^{\infty}{}'\sum_{n=0}^{\infty}{}'\frac{(-p\alpha)^{m+n}(-q\beta)^m}{m!(m+n)!}$$

where we have used the relation,

$$\int_0^1 x^n(1-x)^m dx=\frac{m!\,n!}{(m+n+1)!}.$$

Next, using the power series expansion for the n-th order Bessel functions:

$$I_n(2\sqrt{(p\alpha)(q\beta)})=(-1)^n\sum_{m=0}^{\infty}{}'\frac{(-p\alpha)^{m+n/2}(-q\beta)^{m+n/2}}{m!(m+n)!} \tag{B·9}$$

eq. (B·8) becomes,

$$-p\int_0^{\alpha} da\, I_0(2\sqrt{pq(\alpha-a)\beta})e^{-pt}=\sum_{n=1}^{\infty}{}'(-1)^n(p\alpha/q\beta)^{n/2}I_n(2\sqrt{(p\alpha)(q\beta)}). \tag{B·10}$$

Similarly, exchanging p with q and α with β,

$$-q\int_0^{\beta} db\, I_0(2\sqrt{pq\alpha(\beta-b)})e^{-qh}=\sum_{n=1}^{\infty}{}'(-1)^n(q\beta/p\alpha)^{n/2}I_n(2\sqrt{(p\alpha)(q\beta)}). \tag{B·10′}$$

Then eq. (B·7) follows immediately if we use the relation,

$$e^{-1/2x(t+1/t)}=I_0(x)+\sum_{n=1}^{\infty}{}'(-1)^n(t^n+t^{-n})I_n(x).$$

Since the convergence of the series in eqs. (B·10) and (B·10′) (and those obtained by interchanging α and β there) are rather rapid, we can use them to evaluate numerically the integrals in Q's eqs. (3·16) and (3·17).

B. 3. The following approximate formula is obtained when $2s\sqrt{\alpha\beta}\gg 1$,

$$\int_0^\alpha da\, I_0(2s\sqrt{(\alpha-a)\beta})e^{-ra} \approx \frac{1}{s\sqrt{\beta/\alpha}+r}\cdot\frac{1}{\sqrt{4\pi s(\alpha\beta)^{1/2}}}\exp(2s\sqrt{\alpha\beta})$$

$$\approx \frac{1}{s\sqrt{\beta/\alpha}+r} I_0(2s\sqrt{\alpha\beta}). \qquad (B\cdot 11)$$

To prove it, we retain only the first term in the asymptotic expansion of the function $I_0(z)$:

$$I_0(z) \approx (2\pi z)^{-1/2}\exp(z)\left(1+\frac{1}{8z}+\frac{9}{128}\frac{1}{z^2}+\cdots\right) \qquad (B\cdot 12)$$

$$(z \gg 1)$$

and put $a=0$ everywhere except in the exponential factor since the integrand is large only at $a\sim 0$. The approximation is rather good for $2s\sqrt{\alpha\beta}\sim 1$, too.

Appendix C. Derivation of equations in § 5. 1

From the equation of state, $p=c_0^2\varepsilon$, and the thermodynamic relations, $p+\varepsilon=Ts$, $dp=sdT$, we have

$$(\varepsilon/\varepsilon_0)=(T/T_0)^{1+1/c_0^2}, \quad (s/s_0)=(T/T_0)^{1/c_0^2}, \qquad (C\cdot 1)$$

where ε_0 and s_0 are densities at $T=T_0$.

The entropy and the energy contained in the backward progressive wave are given by

$$\Delta S_1=\pi a^2\int_0^{\delta_1} su\,d\delta=\pi a^2 l s_0(l_1/l)\int_0^{\delta_1/l_1}(s/s_0)u\,d(\delta/l_1) \qquad (C\cdot 3)$$

$$\Delta E_1=\pi a^2 l\varepsilon_0(l_1/l)\int_0^{\delta_1/l_1}(1+c_0^2)(\varepsilon/\varepsilon_0)u^2 d(\delta/l_1) \qquad (C\cdot 2)$$

in which (s/s_0) and $(\varepsilon/\varepsilon_0)$ can be expressed as functions of t and $\delta=t-|x|$ by substituting the progressive wave solution (cf. eq. (A·7)),

$$\frac{T}{T_0}=\left[\frac{\delta}{(2t-\delta)}\frac{1-c_0}{1+c_0}\right]^{c_0/2}, \qquad (C\cdot 4)$$

in eq. (C·1). Similarly, from $u=\mathrm{ch}\,\eta$ and $\alpha=\eta-(\gamma/c_0)=0$,

$$u=\left[1-\left(\frac{c_0t+|x|}{t+c_0|x|}\right)^2\right]^{-1/2}$$

$$\approx\frac{1}{2}\left[\frac{2t-\delta}{\delta}\frac{1-c_0}{1-c_0}\right]^{1/2}. \qquad (C\cdot 5)$$

In eqs. (C·2) and (C·3), δ_1 is the value of δ at the boundary of the non-trivial and the (backward) progressive wave regions, and this is obtained from eq. (3·14′) by putting there $\alpha=0$.

The result is (use also eq. (C·4) to eliminate β),

$$\frac{\delta_1}{l_1}=\left(\frac{2t-\delta_1}{l_1}\right)^{(2c_0-1)/(2c_0+1)}\left(\frac{1+c_0}{c_0}\right)\left(\frac{1-c_0}{c_0}\right)^{-(2c_0-1)/(2c_0+1)} \tag{C·6}$$

and l_1 is given in eqs. (5·5).

Denoting by V_0 the initial volume of the system in the case of a nucleon-nucleon collision, we have $V_0/2=\pi a^2 l$, and

$$S_0=s_0V_0, \qquad 2E=\varepsilon_0V_0, \tag{C·7}$$

for the total entropy and the total energy in the e.v.s. for this case. Therefore the fractions of entropy and energy contained, in the progressive wave are

$$\alpha_1\equiv\frac{\Delta S_1}{(S_0/2)}=\frac{l_1}{l}\int_0^{\delta_1/l_1}(s/s_0)\,u\,d(\delta/l_1), \tag{C·8}$$

$$\beta_1\equiv\frac{\Delta E_1}{E}=\frac{l_1}{l}\int_0^{\delta_1/l_1}(1+c_0^2)\,(\varepsilon/\varepsilon_0)\,u^2\,d(\delta/l_1). \tag{C·9}$$

These integrals can performed easily for $\delta\ll t$ and the results are

$$\alpha_1=\frac{l_1}{l}\left(\frac{1-c_0}{c_0}\frac{l_1}{2t}\right)^{(1-c_0^2)/c_0(1+2c_0)}=\frac{l_1}{l}\,(T_1/T_0)^{1/c_0^2-2} \tag{C·10}$$

$$\beta_1=\frac{l_1}{l}\left(\frac{1+c_0}{2}\right)\left(\frac{1-c_0}{c_0}\frac{l_1}{2t}\right)^{(1-c_0-c_0^2)/c_0(2c_0+1)}=\frac{l_1}{l}\left(\frac{1+c_0}{2}\right)(T_1/T_0)^{(1-c_0-c_0^2)/c_0^2} \tag{C·11}$$

where T_1 is the temperature at the boundary. For the actual estimation of α and β, we shall use for T_1 the value $T_c\approx m_\pi c^2$ at which the free emission of particles takes place. Thus (cf. Appendix E),

$$T_0/T_1\approx T_0/m_\pi c^2\approx 2(E_0/2Mc^2)^{1/4}. \tag{C·12}$$

Substituting this in eqs. (C·11) and (C·12) we get eqs. (5·2), (5·3) and (5·4).

Appendix D. The motion of the point at which the velocity is zero

The coordinates $t=t(0)$ and $x=x(0)$ at which the velocity is zero are obtained from eqs. (3·11) and (3·12) by putting there $\alpha=\beta$. With the help of eqs. (B·7) and (B·10′), we get

$$t(0)=l(n+1)e^{(p+\eta)\alpha}\left\{\tfrac{1}{2}\left(\frac{1}{c_0}-1\right)I_0(2s\alpha)+\sum_{m=1}^{\infty}(-1)^{m-1}\times\left(\frac{1-c_0}{1+c_0}\right)^m I_m(2s\alpha)\right\}+(l/2)\,(n+1)$$

$$-x(0)=(l/c_0^2)\,(n-1)e^{(p+\eta)\alpha}\left\{\tfrac{1}{2}(1-c_0)\,I_0(2s\alpha)-\sum_{m=1}^{\infty}(-1)^{m-1}\right. \tag{D·1}$$

$$\times\left(\frac{1-c_0}{1+c_0}\right)^m I_m(2s\alpha)\Big\}-(l/2c_0^2)(n-1),$$

where the origins of t and x are so chosen that $x_1+x_2=t_1+t_2=0$. In particular, $(p+q)\alpha=-2y$ and $2s\alpha=-y$ for $c_0=1/\sqrt{3}$. The left half of Table D. 1 shows the results by eqs. (D·1) for several values of the parameter y.

For the estimation of the forward to backward ratio of energy and entropy, we need the approximate expression for the velocity, $v(0)$, of the motion of $x(0)$:

$$v(0)=(dx(0)/dt(0))=(dx(0)/dy)/(dt(0)/dy). \tag{D·2}$$

It can be seen that we get better approximation by starting from the relations which relate $t(0)$ and $x(0)$ with the potential χ(cf. eqs. (3·2)),

$$\begin{aligned} t(0)&=\exp(-y)\,(\chi_y)_\eta=0 \\ -x(0)&=\exp(-y)\,(\chi_\eta)_\eta=0 \end{aligned} \tag{D·3}$$

than by approximating eqs. (D·1). Using eqs. (3·8) and (3·6) for χ, and neglecting smaller terms, we get

$$\begin{aligned} t(0)&\approx(l/2c_0)(n+1)e^{(2s+c_0)\alpha}(2s+\partial/\partial\alpha)J(\alpha) \\ -x(0)&\approx(l/2c_0^2)(n-1)e^{(2s+c_0)\alpha}K(\alpha), \\ &\qquad\qquad (\alpha=-y/c_0) \end{aligned} \tag{D·4}$$

where

$$\begin{aligned} J(\alpha)&\equiv\int_0^\alpha da\,I_0(2s\sqrt{\alpha(\alpha-a)})\,(pe^{-p\alpha}-qe^{-q\alpha}) \\ &\approx(4\pi s)^{-1/2}(c_0/\sqrt{\alpha})\left(1+\frac{1}{16s\alpha}\right)e^{2s\alpha} \\ K(\alpha)&\equiv\int_0^\alpha da\,(sa/\sqrt{\alpha(\alpha-a)})\,I_1(2s\sqrt{\alpha(\alpha-a)})\,(2se^{-s\alpha}-pe^{-p\alpha}-qe^{-q\alpha}) \\ &\approx(c_0^2/4\sqrt{\pi s})\,(1/\alpha^{3/2})\left(1+\frac{9}{16s\alpha}\right)e^{2s\alpha}, \end{aligned} \tag{D·5}$$

In eqs. (D·5), the approximate expressions are derived by using the asymptotic expansion for Bessel functions :

$$I_0(z)\approx e^z(2\pi z)^{-1/2}\left(1+\frac{1}{8z}\right),\quad I_1(z)\approx e^z(2\pi z)^{-1/2}\left(1-\frac{3}{8z}\right),$$

and by retaining only terms up to the first order in expanding in (a/α). From eqs. (D·4) and (D·5),

$$t(0)\approx(l/2\sqrt{\pi})\,(n+1)\,(1-c_0^2)^{1/2}(-y)^{-1/2}\left[1-\frac{c_0^2}{4(1-c_0^2)}\,\frac{1}{(-y)}\right]e^{-y/c_0^2}$$

$$-x(0)\approx(l/4\sqrt{\pi})\,(n-1)\,(c_0^2/\sqrt{1-c_0^2})\,(-y)^{-3/2}\left[1+\frac{9c_0^2}{4(1-c_0^2)}\frac{1}{(-y)}\right]e^{-y/c_0^2}. \tag{D·6}$$

The results by approximate formulae (D·6) are shown in the right half of Table D. 1. It is seen that the difference from the exact values (eqs. (D·1)) is only about ten percents.

Table D. 1. Values of $t(0)$ and $x(0)$ for several values of the parameter $-y$. Approximate values (eqs. (D. 6)) are compared with the exact ones (eqs. (D. 1)). For the sound velocity, $c=1/\sqrt{3}$ is assumed.

$-y$	Exact values (eqs. (D. 1))		Approximate values (eqs. (D. 6))	
	$t(0)/l(n+1)$	$-x(0)/l(n-1)$	$t(0)/l(n+1)$	$-x(0)/l(n-1)$
2	66.8	15.06	62	12.8
3	1090	137.7	1030	124
4	18924	1575	18150	1501
5	339400	20950	328300	20630
6	6215000	305650	6044000	305700

From eqs. (D·2) and (D·6), the velocity $v(0)$ is obtained in terms of the parameter $-y$. The result is

$$|v(0)|=\frac{n-1}{n+1}\frac{c_0^2}{2(1-c_0^2)}\frac{1}{(-y)}\left[1+\frac{3+2c_0^2}{2(1-c_0^2)}\frac{c_0^2}{(-y)}\right] \tag{D·7}$$

where we have again expanded in $(1/y)$. Eqs. (5·11) and (5·12) in the text are obtained by substituting eq. (D·7) in eqs. (5.10), and using the first of Eqs. (D·6) to express $dt(0)$ in terms of $-y$.

For the practical calculations, the values of $-y$ are chosen to correspond to the critical temperature $T_c\approx m_\pi c^2$,

$$y_c=\ln(T_c/T_0), \quad \text{with} \quad T_0/T_c\approx 2(E_0/2Mc^2)^{1/4}. \tag{D·8}$$

Here E_0 is the incident energy in the laboratory system. (See Appendix E for eq. (D·8).

Table D. 2. y_c and $(E/Mc^2)=(E_0/2Mc^2)^{1/2}$ for several values of the laboratory incident energy.

E_0 (ev)	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}	10^{17}	10^{18}
E/Mc^2	22.4	70.7	224	707	2240	7070	22400
$-y_c$	2.25	2.82	3.40	3.97	4.55	5.12	5.69

The values of y_c(eq. (D·8)) are shown in Table D. 2 for several values of the incident energy.

Appendix E. Some thermodynamical relations used in the text

Assuming that the local element of nuclear fluid can be treated as a relativistic ideal gas, we get for π-mesons the following expressions for the energy, number, and entropy densities in the local rest system,

$$\begin{aligned} \varepsilon_\pi &= (g_\pi/2\pi^2)\, T\,(T/\hbar c)^3 \Phi(Z_\pi) \\ \rho_\pi &= (g_\pi/2\pi^2)\,(T/\hbar c)^3 F(Z_\pi) \\ s_\pi &= (g_\pi/2\pi^2)\,(T/\hbar c)^3 G(Z_\pi). \end{aligned} \qquad \text{(E·1)}$$

Here $g_\pi=3$ is the statistical weight for π-mesons, and Φ, F, and G are functions which decrease slowly with increasing $z_\pi=m_\pi c^2/T$. Their numerical values are given in Ref. 13) and 14). From (E·1), the average thermal energy (including rest mass) per particle (ε/ρ) is given by

$$E_{\mathrm{th}}=T(\Phi/F), \qquad \text{(E·2)}$$

as a function of the temperature. Table E·1 shows E_{th} for π-mesons, K-mesons and nucleons, calculated assuming that there were initially no K-mesons and baryons (nucleons and hyperons).

Now consider the initial state of the system in which various particles may be present. Let $2E$, S_0 and V_0 be the total energy, total entropy and the initial volume of the system in the c.m. system of the two colliding nucleons. Then

Table E. 1. Average thermal energies per one particle for π-mesons, K-mesons ($m_K=3.5m_\pi$) and nucleons.
At high temperatures, $E_{\mathrm{th}}\approx 2.7\ T$ for Bosons
$\approx 3.2\ T$ for Fermions,
At low temperatures, $E_{\mathrm{th}}\approx m_0c^2+(3/2)T$.

$T/m_\pi c^2$	$E_{\mathrm{th}}{}^\pi/m_\pi c^2$	$E_{\mathrm{th}}{}^k/m_\pi c^2$	$E_{\mathrm{th}}{}^N/m_\pi c^2$
10	27.2	28.2	33.0
5	13.8	15.2	18.4
3.33	9.3	11.0	13.8
2.5	7.2	9.0	11.8
2.0	5.8	7.7	10.7
1.67	5.0	6.9	9.8
1.43	4.3	6.4	9.3
1.25	3.9	6.0	8.8
1.11	3.5	5.6	8.5
1.00	3.3	5.3	8.3
0.83	2.8	4.9	7.9
0.67	2.4	4.5	7.7

$$2E=\mathcal{E}_{\rm tot}(T_0)V_0\equiv B(T_0)T_0^4V_0,\quad \mathcal{E}_{\rm tot}=\sum_i\mathcal{E}_i$$
$$S_0=S_{\rm tot}(T_0)V_0\equiv A(T_0)T_0^3V_0,\quad s_{\rm tot}=\sum_i s_i \qquad (E\cdot 3)$$

where the suffix i refers to the various kinds of particles. Since the expansion of the system is assumed to be adiabatic, S_0 remains constant and is related to the total multiplicity, N_0, by

$$N_0=\alpha S_0,\quad \alpha\equiv(\rho_{\rm tot}(T_c)/s_{\rm tot}(T_c)), \qquad (E\cdot 4)$$

where T_c is the "cooling temperature". On the other hand, we have

$$N_0=k(E_0/2Mc^2)^{1/4}=k(E/Mc^2)^{1/2}. \qquad (E\cdot 5)$$

where k is a parameter which should be determined from the experiment.

The determination of T_0 does not depend so much on the possible kinds of particles existing in the system and on their statistical weights. From (E·3), (E·4) and (E·5) we have

$$T_0=\frac{A(T_0)}{B(T_0)}\frac{2E}{N}\alpha=\frac{A(T_0)}{B(T_0)}\frac{2\alpha}{k}Mc^2(E_0/2Mc^2)^{1/4}, \qquad (E\cdot 6)$$

where $A(T_0)/B(T_0)$ can be put equal to a constant $(\approx(G_0/\Psi_0)=1.33)$ so long as $T_0\gg m_\pi c^2$, and α may be approximated by $\alpha=(\rho_\pi(T_c)/s_\pi(T_c))$. Then from (E·2) and (E·6), we get for the initial thermal energy per one pion,

$$E_{\rm th}^\pi(T_0)=(\mathcal{E}_0/\rho_0)=(2Mc^2/k)(F_c/G_c)(G_0/F_0)(E_0/2Mc^2)^{1/4}$$
$$=m_\pi c^2(2/k)(6.0\sim 5.4)(E_0/2Mc)^{1/4} \qquad (E\cdot 7)$$

where the upper and lower limits of numerical values in the right-hand side correspond to $T_c=(1.4-0.7)m_\pi c^2$.

Eq. (6·11) is §6 in obtained from (E·7) and

$$T_1/T_0=\exp(\gamma_1)=\exp\left[-\tfrac{1}{3}(1+\gamma_1^2-\gamma_1)\tau_1\right]$$
$$=[L\cdot B_n]^{(\gamma_1-1-\gamma_1^2)/3(1+\gamma_1^2)} \qquad (E\cdot 8)$$
$$u_1\approx(1/2)|L\cdot B_n|^{(1-\gamma_1^2)/2(1+\gamma_1^2)}. \qquad (E\cdot 9)$$

The thermal fluctuation of the transverse momenta, $\Delta P_\perp$, will be estimated by the mean square root of the component (to some particular direction) of the momentum which correspond to the thermal energy ($E_{\rm th}$). That is

$$\Delta p_\perp=\sqrt{\frac{\overline{E_{\rm th}^2}-m^2c^4}{3c^2}}\approx\frac{E_{\rm th}}{\sqrt{3}c}. \qquad (E\cdot 10)$$

From Table E. 1, we see that $P_\perp/m_\pi c\approx 2$ for π-mesons, ≈ 3 for K-mesons and ≈ 5 for nucleons.

Eq. (D·8), which has been sometimes used in the text, is obtained from (E·6) by substituting the numerical values. The results for E_π and $P_\perp$ are sensitive to the

value of k (we used $k=2$), but other quantities such as α and β in §5·1 or the relative magnitudes of angular distributions, etc., are not.

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Note on Models of Multiple Meson Production at Extremely High Energy*

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Takagi's model and Heisenberg's model (modified with regard to the total cross section) for multiple meson production are examined, and they are found to conform to two sources of experimental information on the properties of secondary particles in jet showers: their composition and the distribution of their transverse momenta. The main aspects of these models, as well as those of the Landau model, are discussed.

§ 1. Introduction and summary

1·1. In a previous note[1] it has been remarked that two sources of experimental information concerning the properties of secondary particles can be regarded as important criteria for the theory of multiple production. The first is the ratio of heavy to light mesons,[2] and the second is the distribution of their transverse momenta.[3] Both of them appear to indicate the necessity of modifying Fermi's original treatment[4] in favour of Landau's hydrodynamical theory,[5][6] and this theory implies the existence of a strong pion-pion interaction at a lower energy, say, near 1 Gev.

This way of reasoning, however, will not be the only one possible. The main purpose of the present note is to examine two plausible models of multiple production, different from both Fermi's and Landau's and to see whether they satisfy the criteria previously mentioned.

1·2. One of the promising modifications of the Fermi theory of multiple production was proposed by Takagi[7] and his idea was further developed by Kraushaar and Marks.[8] According to their model, the two colliding nucleons exchange a certain amount of momentum during the collision and become excited, and then each of them independently decays and emits secondary particles. The statistical treatment similar to Fermi's is applied to this second stage of particle formation; the equilibrium volume consists now of two separate spheres, each being of the radius of the order of $1/\mu$, the Compton wave length of the pion**, in the respective center of mass systems of these two excited

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** Units in which $\hbar=c=k=1$ are used throughout this note. (k is the Boltzmann constant.)

nucleon-meson clouds.

Recent experimental data[9)-12)] suggest that most of the high energy collisions are not completely inelastic ; that is to say, an appreciable part of the energy available does not go over to the secondary particles, but remains in the original particle. One advantage of the above mentioned model is that it can take this circumstance into account in a rather natural way.

This model we shall investigate in § 2. The results can be reconciled with the two criteria referred to above, if we introduce *a priori* a certain inelasticity factor, which does not depend much on the incident energy.

1·3. In § 3 we shall examine the predictions of the Heisenberg theory. Even before the discovery of mesons, Heisenberg has been discussing in a variety of ways the multiple formation of secondary particles by nuclear collisions at very high energy.[13)14)] His fundamental point of view is that the nucleon-meson system is governed by a wave equation involving strong non-linearity,* and as a characteristic feature of the latter a conspicuous energy dissipation, which corresponds to the formation of a large number of low energy mesons, takes place during the expansion process of the concentrated nucleon-meson cloud.

Many events have been recorded in nuclear emulsions with a large number of secondary particles, though most of them belong to a relatively lower energy. This fact may be regarded as supporting Heisenberg's point of view.[15)-19)] At extremely high energy, on the other hand, his theory seems to give either too large a number of secondary particles or too large a cross section.[20)21)]

The latest version of his theory has been worked out semi-quantitatively and can be compared with experimental data. In the present note I shall study this particular version of his theory, excepting, however, his arguments concerning the impact parameter, inelasticity and total cross section.** Indeed this part of his work has been criticized[6)21)] because it appears to involve an extrapolation of the classical mechanics concept beyond its limit of applicability.

It appears possible to separate logically this particular part of Heisenberg's work (for instance, III c) of the reference 14) from the rest of it. I shall assume in this note therefore that the total cross section of nucleon-nucleon collisions remains at a constant value π/μ^2, instead of increasing logarithmically with energy as in the original work.***

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* The definition of the strong non-linearity in Heisenberg's works appears somewhat unclear. It would be necessary to assume not only the presence of a parameter of the dimension of a positive power of a length, but also the presence of a derivative of the wave function in the non-linear term.

** Heisenberg extends namely the integration with respect to the impact parameter r to such a value as to correspond to the minimum inelasticity : $r_{\max}=-1/\mu \log \eta_{\min}$ with $\eta_{\min}=2\bar{p}_0/W$. Here $\bar{p}_0$ is the average energy of the secondary mesons in the center of mass system, which, in his theory, depends on the incident energy W in the center of mass system only logarithmically. Thus the total cross section turns out to be

$$\sigma \approx \frac{\pi}{\mu^2}\left[\log\left(\frac{W}{2\bar{p}_0}\right)\right]^2. \qquad (1\cdot1)$$

*** See the expression in the above footnote.

This alteration will make it possible to apply Heisenberg's theory to the case of nucleon-nucleus collisions at extremely high energy.*

The results can again be regarded as consistent with the two criteria referred to.

1·4. In the final section, the main aspects of the three models for Landau, Takagi and Heisenberg are discussed. First of all, they differ in the prediction of the total number of secondary particles as a function of incident energy. If we assume an almost constant average inelasticity over a wide region of energy, we should be able to discriminate between them. Another difference is that Heisenberg's and Landau's models lead to the presence of pion-pion interaction, while Takagi's model does not.

Concerning the mechanism of partially elastic collisions, there are at least two points of view possible : spacial localization of the self-energy of a nucleon and concentration of a large amount of energy on a small number of secondary particles.

§ 2. Predictions of Takagi's model

2·1. First let us give a somewhat simplified formulation of multiple production in Takagi's model, so as to see the qualitative aspects of this picture. The incident energy in the laboratory system will be denoted by E_0, and the total energy in the center of mass system by W. The momentum carried away by each of the clouds in the center of mass system will be denoted by P. We also introduce a parameter,** M^*, the fictitious mass of the excited nucleon-meson system,

$$M^{*2}=(W/2)^2-P^2. \tag{2·1}$$

In the case of completely inelastic collisions, when the inelasticity $\eta=1$, the final momentum P vanishes and

$$M^*=W/2. \tag{2·2}$$

When $\eta<0.9$, on the other hand, η and M^* are connected by an approximate relation

$$\eta\approx 1-M/M^*, \quad \text{for} \quad E_0>10^3 M, \tag{2·3}$$

M being the nucleon mass. (See the appendix.)

The energy available for the emission of secondary particles in each cloud in its own center of mass system is equal to

* It is to be remarked that also in this modified Heisenberg theory, the inelasticity has to be introduced without reference to the impact parameter, otherwise it would be unreasonable to assume as small a value for inelasticity as required from experiments in the case of nucleon-nucleus collision, because in the "tunnel" inside the nuclear matter (jet model) no extremely peripheral nucleon-nucleon collision can take place except the collision with a peripheral nucleon in the nucleus.

An attempt has been made to apply Heisenberg's theory without any modification to the individual collision treatment of the jet model.[22] This procedure does not seem to be quite consistent, however, because the concept of tunnel penetration (of the radius $1/\mu$) and the small inelasticity (which implies a very large impact parameter in the original form of the theory) are used at the same time.

** In the original article of Takagi there is still another parameter available. Also our definition of "fictitious mass" is different from that given by Takagi.

$$M^*-M=\begin{cases}(W/2)-M & \text{for} \quad \eta=1,\\ \{\eta/(1-\eta)\}M & \text{for} \quad \eta\leq 0.9,\end{cases} \tag{2·4}$$

so that the energy density $\mathcal{E}$ is given by*

$$\mathcal{E}=\begin{cases}((W/2)-M)\mu^3, & \text{for} \quad \eta=1,\\ \{\eta/(1-\eta)\}M\mu^3, & \text{for} \quad \eta\leq 0.9.\end{cases} \tag{2·5}$$

Notice that in the case of inelasticity not very close to 1, the energy or energy density available for the secondary particle emission does not depend on the initial energy explicitly.** The energy density determines the corresponding temperature, which, in turn, gives the number of secondary particles.

Table 2·1. Equilibrium temperatures (in μ) in Fermi's and Takagi's models for nucleon-nucleon collision.*

Incident energy (in M)	Temperature in Fermi model**		Temperature in Takagi model		
	$r=0$	$r=0.9/\mu$	$\eta=1$	$\eta=0.5$	$\eta=0.1$
10^3	6	4	2.5	1.3	0.9
10^4	11	8	3.3	1.3	0.9
10^5	19	14	4.4	1.3	0.9

* The temperatures have been estimated by using the formulae (2·5), (2·9) and (2·10).

** In Fermi's case the results for head-on collision (impact parameter $r=0$) and peripheral collision ($r=0.9/\mu$) are given.

As is seen in table 2·1 the equilibrium temperature in the Takagi model is much reduced compared with the original Fermi model. This is due to two reasons. i) A part of the available energy is used as mass motion and not as thermal motion. ii) The equilibrium volume is not Lorentz-contracted. The conspicuous feature of this model is, as was pointed out above, that one has a definite temperature if one specifies an inelasticity not very close to one.

2·2. The distribution of the transverse momenta of secondary particles in this model depends on the following two factors. First, the transverse component of the momentum exchange between the two nucleons will give the mean value of transverse momenta of the particles in the diffuse and narrow cones. Second, the thermal motion corresponding to the equilibrium temperature will be superposed on this average value. The first term

* For simplicity the volume is taken to be equal to $1/\mu^3$.

** This can be understood as follows. Let us assume, for the moment, that a metastable excited state of the nucleon with energy kM exists, and further, that the production of secondary particles takes place exclusively through this intermediary stage. The total energy of the system is expressed by $2\Gamma kM$, Γ being the Lorentz factor of the outgoing excited cloud in the center of mass system; the energy of the original nucleons in the final state is $2\Gamma M$. The inelasticity in this case is obviously $(k-1)/k$, if we identify the total energy with the total available energy. (This is permissible if k is not too large.) In this approximation, therefore, the inelasticity is a function of k only, and vice versa.

does not seem, however, to play an essential role. If it should be of any importance, one could find an asymmetry of narrow and diffuse cones with respect to the direction of primary incidence.

Further, the ratio of heavy to light mesons is determined by the equilibrium temperature just before particles fly out, as was explained previously.[1]

It is expected, therefore, that the low temperature equilibrium predicted by this model for $\eta \sim 1$ will imply reasonable values with regard to these two criteria, and it will be worthwhile to study this model in more detail.

2·3. Since most of the experimental data are concerned with nucleon-nucleus collisions, we shall now extend the model to this more complicated case. At extremely high energy ($\gtrsim 10^4$ Gev), the nucleon-nucleus collision can be fairly well approximated by the so-called jet model:[23)24)] the incident nucleon interacts only with the nuclear matter in the "tunnel" which it penetrates into the nucleus, and the remaining part of the nucleus suffers little excitation. In this approximation we can still employ two opposite points of view: composite collision[20] and individual collision,[22] but the former appears to be the more adequate description when the energy of the incident particle is very large.

2·4. We shall treat, therefore, the composite collision model. As regards the final division of the nucleons and the total energy among the two clouds, there is no *a priori* criterion that would come out of a consideration of the nucleon-nucleus collision, but, for simplicity, we shall treat the following two cases.

A) Case of $n:1$ division.

We assume that, after exchange of momentum and energy, two excited clouds containing n and 1 nucleons (or, more precisely, "nucleons minus antinucleons") respectively, will appear, the ratio of fictitious masses M_1^* and M_2^* of these clouds also being $n:1$. This means that, when referred to their own center of mass systems, both the energy and the equilibrium volumes of these clouds are in the ratio $n:1$.

If the collision is completely inelastic, $\eta=1$,

$$\begin{aligned} M_1^* &= (n/(n+1))W, \\ M_2^* &= (1/(n+1))W, \end{aligned} \tag{2·6}$$

W being the total energy in the center of mass system.

If the inelasticity is not close to 1, it can be shown that the approximate relation

$$\eta = 1-(M/M_2^*), \quad \eta < 0.8 \tag{2·7}$$

$$\text{for} \quad (E_0/M) \geq 10^3 \quad \text{and} \quad n \leq 5$$

holds in this case, too. Thus the density of available energy, which turns out common to both clouds, is

$$\varepsilon = \begin{cases} (W/(n+1)-M)\mu^3, & \text{for} \quad \eta=1, \\ (\eta M/(1-\eta))\mu^3, & \text{for} \quad \eta<0.8. \end{cases} \tag{2·8}$$

The temperature corresponding to this energy density can be determined by the formulae :[25)1)]

$$\varepsilon=\varepsilon(\pi)+\varepsilon(\theta)+\varepsilon(N)$$

$$\varepsilon(\pi)=\frac{3T^4}{2\pi^2}G\left(\frac{\mu}{T}\right),\qquad \varepsilon(\theta)=\frac{4T^4}{2\pi^2}G\left(\frac{m_\theta}{T}\right), \tag{2·9}$$

$$\varepsilon(N)=\frac{8T^4}{2\pi^2}G^*\left(\frac{M}{T}\right),$$

with

$$G(x)=x^2\sum_{\nu=0}^{\infty}\frac{1}{1+\nu}\left\{\frac{3K_2((1+\nu)x)}{1+\nu}+xK_1((1+\nu)x)\right\},$$

$$G^*(x)=x^2\sum_{\nu=0}^{\infty}(-1)^\nu\frac{1}{1+\nu}\left\{\frac{3K_2((1+\nu)x)}{1+\nu}+xK_1((1+\nu)x)\right\}. \tag{2·10}$$

m_θ means the mass of the heavy meson. Here we have assumed the statistical weight of the heavy mesons to be equal to 4.*

Once the temperature is determined, we can derive the number density for each kind of particles and multiplying it by the volume we get the total number.

The results of the computation are shown in the following table.

Table 2·2. Temperature, total number, and composition. Case of $n:1$ division.†

(i) $\eta=1$

n	E_0/M	T/μ	Total number of secondary particles	Composition ($\pi:K:N$)
1	10^2	1.9	1.0×10	1 : 0.7 : 0.5
	10^3	2.6	3×10	1 : 0.9 : 0.8
	10^4	3.3	8×10	1 : 1 : 1.1
	10^5	4.4	2×10^2	1 : 1.1 : 1.4
	10^6	5.7	5×10^2	1 : 1.2 : 1.6
3	10^2	1.9	1.9×10	1 : 0.7 : 0.5
	10^3	2.5	6×10	1 : 0.9 : 0.8
	10^4	3.2	1.4×10^2	1 : 1 : 1.1
	10^5	4.2	4×10^2	1 : 1.1 : 1.4
	10^6	5.5	9×10^2	1 : 1.2 : 1.6
5	10^2	1.8	2×10	1 : 0.7 : 0.4
	10^3	2.4	7×10	1 : 0.9 : 0.8
	10^4	3.1	1.8×10^2	1 : 1 : 1
	10^5	4.1	5×10^2	1 : 1.1 : 1.3
	10^6	5.4	1.3×10^3	1 : 1.2 : 1.6

* That is to say, we have assumed only one kind of heavy meson with isospin 1/2 and spin 0. This is the least possible value for the statistical weight of the heavy meson.

(ii) $\eta<1$ (independent on incident energy)

η	T/μ	Total number $n=1$	$n=3$	$n=5$	Composition $(\pi : K : N)$
0.75	1.7	6	13	19	1 : 0.6 : 0.3
0.5	1.3	2	4	6	1 : 0.5 : 0.1
0.3	1.1	1	2	3	1 : 0.3 : 0.08
0.1	0.9	0.3	0.6	0.9	1 : 0.2 : 0.03

† In the case of $E_0/M=10^2$, the jet model is probably not valid, and in the cases, where the number of secondary particles turns out very small, the thermodynamical approximation obviously fails. Such cases are listed only for rough orientation.

B) Case of symmetrical (1 : 1) division

We assume that after collision two excited clouds fly out in perfect symmetry. This may seem less plausible than the case A), if one adheres to the original picture of momentum exchange. It could be regarded, however, as the opposite extreme case to A); and so it would give a rough idea of how much the results depend on the assumption concerning the division of energy and momentum between two clouds.

In this case we have

$$W_1=W_2=W/2, \tag{2·11}$$

$$M_1^*=M_2^*=\{(W/2)^2-P^2\}^{1/2}. \tag{2·12}$$

For a completely inelastic collision

$$M_1^*=W/2, \tag{2·13}$$

$$\varepsilon=\left(\frac{W}{2}-\frac{n+1}{2}M\right)\frac{2\mu^3}{n+1}, \tag{2·14}$$

while in the case of $\eta<0.8$ (See appendix.)

$$\eta=1-(n+1)M/2M^*, \tag{2·15}$$

so that

$$M^*=(n+1)M/2(1-\eta), \tag{2·16}$$

$$\varepsilon=[\eta/(1-\eta)]M\mu^3. \tag{2·17}$$

The case of completely inelastic collision is of course, identical with the case A), because two clouds have zero momentum and two ways of division do not yield any difference. The case of $\eta\cdot 1$ is, as can be expected, the same as the case of nucleon-nucleon collision with mass and volume multiplied by a factor of $(n+1)/2$.

The results are given in the following table.

Table 2·3. Temperature, total number, and composition. Case of 1 : 1 division.*

(i) $\eta=1$. The results are the same as A) (i).
(ii) $\eta<1$. (Independent on incident energy)

η	T/μ	Total number $n=1$	$n=3$	$n=5$	Composition ($\pi : K : N$)
0.75	1.7	6	13	19	1 :0.6 : 0.3
0.5	1.3	2	4	6	1 : 0.5 : 0.1
0.3	1.1	0.8	1.6	1.4	1 : 0.2 : 0.08
0.1	0.9	0.4	0.8	1.2	1 : 0.2 : 0.03

* Cf. the footnote to table 2·2.

2·6. This model may be regarded as an extreme idealization. It is seen, nevertheless, from the above numerical values that it reproduces some of the general tendencies of jet showers. It predicts a very small average number of secondary particles, if one assumes a small inelasticity. This point may be fatal. Stars with quite a large number of prongs observed in nuclear emulsions could not be explained except by considering the fluctuations in the inelasticity, or by taking into account some additional mechanism which would result in the formation of a large number of low energy secondary particles.

§ 3. Predictions of Heisenberg's theory

3·1. In the Heisenberg theory the transverse component of the secondary particles is essentially determined by the Fourier component of the transverse dimension of the initial excited state (Lorentz-contracted disk), and this dimension is of the order of $1/\mu$. Thus the transverse momenta will be of the order of pion rest mass. The energy dissipation characteristic of the Heisenberg theory is concerned only with the longitudinal component. In both Heisenberg and Landau theories the essential feature of the angular distribution is determined by its one-dimensional character in the expanding process. This feature will not be changed in a nucleon-nucleus collision, either.

3·2. The problem of the composition of the secondary particles will be considered next. Since most of the experimental data are concerned with nucleon-nucleus collision at extremely high energy ($>10^3$ Gev), we must slightly modify the original formulae, so that they can be applied to the general picture of the jet model.

Instead of two flat disks of the same size colliding, we have now a contracted nucleon and a contracted tunnel-like part of the nucleus containing n nucleons, n being less than 6 for the nuclear emulsions. This picture is compatible with our modified assumption of the geometrical cross section at extremely high energy.

Thus we should put

$$p_0^{\max}=\mu I'/n, \tag{3·1}$$

where $p_0^{\max}$ is the maximum energy of a secondary particle and I' is a certain average of the Lorentz factors in the center of mass system. This expression corresponds to the

so-called composite collision model.

As is explained in detail in Heisenberg's works, if the strong non-linearity* of the meson wave field is assumed, the energy spectrum of the emitted particles will be of the form,

$$dN \propto dp_0/p_0^2, \tag{3·2}$$

p_0 being the energy of the secondary particle; the total number of a certain kind, j, of particles will be given to a rough approximation by

$$N_j \propto g_j \int_{m_j}^{p_0^{\max}} dp_0/p_0^2, \tag{3·3}$$

g_j and m_j being respectively the statistical weight** and the mass of that particle.

From (3·2) and (3·3) one has

$$N_j \propto g_j\left(\frac{1}{m_j} - \frac{n}{\mu l'}\right), \tag{3·4}$$

whence one gets easily the number ratio for a given incident energy and a given value of n. The results are shown in the Table 3·1. These values are found, considering the rough approximation we have used to be consistent with the experimental informations.

Moreover, minor discrepancies, even if they should be found, would never be fatal for the Heisenberg theory, because the detailed behaviour of the low energy part of the energy spectrum of secondary particles and consequently the number ratio of various particles is rather sensitive to the particular properties of the non-linear term involved in the wave equation. This situation might be regarded as corresponding to the sharp dependence on the critical temperature in the Landau theory. A rough idea of this fact will be given by the last column of the Table 3·1.*** These values have been computed using a slightly more plausible spectrum given by Heisenberg:[10]

$$N_j \propto \frac{g_j}{m_j}\left\{1+2\left(\frac{m_j}{p_0^{\max}}\right)^2 - 2\frac{m_j}{p_0^{\max}}\sqrt{1+\left(\frac{m_j}{p_0^{\max}}\right)^2}\right\}. \tag{3·5}$$

One can conclude, therefore, that the composition of the secondary particles cannot discriminate between Landau's, and Heisenberg's theories.

3·3. A somewhat subtler distinction may be possible, however, if one can detect distribution of heavier particles. In the Landau picture the critical temperature T_c is determined by the pion-pion interaction and is common to all the elements of the expanding nucleon-meson cloud, so that one can expect the same number ratio of heavy to light mesons in all directions. In the Takagi picture, too, the situation is similar. In

* See the footnote on page 289.

** Heisenberg does not necessarily identify g_j with the statistical factor. Here I have assumed this for simplicity.

*** The results are much more sensitive to a change in the value of g_j.

Table 3·1. Number ratio of various particles predicted by the Heisenberg theory

Incident energy	Number of nucleons in the "tunnel"	Maximum momentum of secondary particles	Number ratio by (3·4)	Number ratio by (3·5)
E_0/M	n	$p_0^{\max}/\mu$	$N_\pi : N_K : N_N$	$N_\pi : N_K : N_N$
10^2	1	7.1	1 : 0.22 : 0.025	1 : 0.20 : 0.098
	3	1.4	1 : 0 : 0	1 : 0 : 0
10^3	1	22	1 : 0.34 : 0.29	1 : 0.31 : 0.24
	3	4.3	1 : 0.094 : 0	1 : 0.14 : 0
	5	2.0	1 : 0 : 0	1 : 0 : 0
10^4	1	71	1 : 0.37 : 0.37	1 : 0.35 : 0.34
	3	14	1 : 0.30 : 0.22	1 : 0.27 : 0.18
	5	6.3	1 : 0.20 : 0	1 : 0.18 : 0
10^5	1	2.2×10^2	1 : 0.38 : 0.39	1 : 0.37 : 0.38
	3	43	1 : 0.36 : 0.34	1 : 0.34 : 0.31
	5	20	1 : 0.33 : 0.28	1 : 0.30 : 0.23
10^6	1	7.1×10^2	1 : 0.38 : 0.39	1 : 0.38 : 0.39
	3	1.4×10^2	1 : 0.37 : 0.38	1 : 0.37 : 0.37
	5	63	1 : 0.37 : 0.36	1 : 0.35 : 0.33
∞	1, 3, 5	∞	1 : 0.38 : 0.40	1 : 0.38 : 0.40

the Heisenberg picture, on the other hand, the transverse energy is so low from the outset that few heavy mesons can appear in the perpendicular direction in the center of mass system.

This difference is due to the fact that Heisenberg deals with a well-defined wave motion, in which, so to speak, all the particles are found in a definite phase relation, while Takagi and Landau regard the nucleon-meson system as a thermodynamical system without any definite phase relation between constituent particles. The thermal motion in the latter pictures can furnish enough energy to create heavy mesons independently of the direction of the "mass" motion.

§ 4. Discussion

4·1. As we have examined in the previous[1] and the present notes, three models for multiple meson production, namely those of Landau, Takagi and Heisenberg, give similar results, compatible with experimental information now available, concerning the composition[2] and the distribution of transverse momenta[3] of secondary particles.

The main difference between the predictions of these theories can be found in the dependence of the total number of created particles on the incident energy. The distinctions are obscured, however, by the intervention of another parameter, inelasticity. Indeed the results are expressed by

$$N \propto \eta E_0^{1/2} \qquad \text{Heisenberg,}$$

$$N \propto \eta E_0^{1/4} \qquad \text{Landau,*}$$

$$N \propto \begin{cases} E_0^{3/8} & \text{for} \quad \eta = 1 \\ (\eta/(1-\eta))^{3/4} & \text{for} \quad \eta \lesssim 0.8 \end{cases} \qquad \text{Takagi.}$$

If we assume for the present that the average inelasticity is practically constant and much less than 1 over a wide range of incident energy, as is suggested by the analysis of latitude dependence of various components and extensive air showers, then we could discriminate between these theories by examining a large number of events statistically. Such a treatment is necessary because the present theoretical predictions are concerned with the average behaviour only and fluctuations from the latter are expected to be quite large.** In analyzing emulsion data we should further notice that events with small numbers of secondary particles are likely to escape detection. Thus the arguments put forward so far[15] which infer the relation of total number to incident energy from a few events of conspicuous stars, do not appear very convincing.

4·2. One difference between these three models, which may have some bearing on the analysis of the phenomena at lower energy, is that both Landau's and Heisenberg's pictures lead naturally to the presence of a pretty strong pion-pion interaction at lower energy, near 1 Gev, while in Takagi's picture it is not necessary to introduce such an interaction.

The role of the pion-pion interaction in the nucleon-nucleon and nucleon-meson reactions is reported to be not yet quite clear. It has also been suggested that this interaction, if it exists, will be strongly dependent on energy and isotopic spin. Such a circumstance might have slight influence on the results of Landau and Heisenberg.

Takagi's model, too, has its analogue in the Gev energy region : one can assume the existence of a similar mechanism in the one-or two pion production[27] or strange particle production by nucleon-nucleon collision.

4·3. The point which is left open is how to incorporate the small value of inelasticity into the theory.

Bhabha[28] has once proposed to take into account the possible localization of a certain part of nucleon self-energy in a small central region of the extended nucleon. Although this idea, if taken too literally, is subject to criticism in the same way as Heisenberg's argument, it may have something to do with the real situation. Indeed the concept of pion-pion interaction seems to be in a sense consistent with this picture of energy localization.

* This relation has been derived assuming that the equilibrium volume in the center of mass system is also proportional to η. If we presume this volume to be independent of η, we have : $N \propto \eta^{3/4} E_0^{1/4}$.

** Podgoreckij, Rozental, and Černavskij[26] have estimated the fluctuations based on statistical considerations and found them fairly large. Since the statistical treatment itself is sometimes to be regarded as a rough approximation, the deviations could be still larger.

It may be worth noting that the concentration of an appreciable amount of the available energy on one or two secondary particles can be regarded as producing inelasticity. In Landau's theory, in particular, this kind of concentration can really take place, as has been pointed out by Gerasimova and Černavskij.[29] If this mechanism for inelasticity is true, one might expect an anomalous behaviour corresponding to the case that this particular secondary particle happens to be a π^0-meson or some other particle which decays into γ's immediately.

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Appendix
Definition and evaluation of inelasticity in Takagi's model

When a nucleon with energy E_0 collides with an n-nucleon nuclear aggregate at rest, the velocity of the center of mass v is given by

$$v=\sqrt{\mathcal{E}^2-1}/(n+\mathcal{E}), \tag{A·1}$$

with

$$\mathcal{E}=\frac{E_0}{M}. \tag{A·2}$$

The total energy, the energies of the first and the second clouds in the center of mass system will be denoted by W, W_1, and W_2 respectively. Then we have

$$W=(n^2+2n\mathcal{E}+1)^{1/2}M, \tag{A·3}$$

$$W_1+W_2=W, \tag{A·4}$$

and define M_1^* and M_2^* by

$$\begin{aligned} W_1^2-P^2&=M_1^{*2},\\ W_2^2-P^2&=M_2^{2*}. \end{aligned} \tag{A·5}$$

In the case A) we have

$$M_1^*=nM_2^*, \tag{A·6}$$

and so

$$W_1=(W/2)[1+(n^2-1)\,(M_2^*/W)^2]$$
$$W_2=(W/2)[1-(n^2-1)\,(M_2^*/W)^2] \tag{A·7}$$

and in the case B),

$$W_1=W_2=W/2, \tag{A·8}$$

$$M_1^{*2}=M_2^{*2}=(W/2)^2-P^2. \tag{A·9}$$

In the theoretical treatment we generally define the inelasticity by

$$\eta_{\rm th}=\frac{\text{energy of all the secondary particles in the center of mass system}}{\text{kinetic energy of incident } (n+1) \text{ particles in the center of mass system}}. \tag{A·10}$$

Experimentally it is usual to put

$$\eta_{\rm exp}=\frac{\text{energy of all the secondary particles in the laboratory system}}{\text{kinetic energy of the incident nucleon in the laboratory system}}. \tag{A·11}$$

In this case the final energy of the target nucleon (or the nuclear matter containing n nucleons) is sometimes included in the secondary energy, because it may not be distinguished from created particles.

In the case A) we have

$$\eta_{\rm exp}=\frac{E_0-\dfrac{M}{M_2^*\sqrt{1-v^2}}\,(W_2+vP)}{E_0-M}, \tag{A·12}$$

because the final energy and momentum of the incident nucleon in the center of mass system are given by $W_2(M/M_2^*)$ and $P(M/M_2^*)$ respectively. Thus

$$\eta_{\rm exp}=\frac{1-(M/M_2^*)\,\alpha}{1-1/\mathcal{E}} \tag{A·13}$$

with

$$\alpha=\left(1+\frac{n}{\mathcal{E}}\right)\left[\frac{1}{2}\left\{1-\frac{n^2-1}{n^2+2n\mathcal{E}+1}\left(\frac{M_2^*}{M}\right)\right\}\right.$$
$$+\frac{1}{2}\left(1-\frac{1}{\mathcal{E}^2}\right)^{1/2}\left(1+\frac{n}{\mathcal{E}}\right)^{-1}\left\{1-\frac{2(n^2+1)}{n^2+2n\mathcal{E}+1}\left(\frac{M_2^*}{M}\right)^2\right.$$
$$\left.\left.+\frac{(n^2-1)^2}{(n^2+2n\mathcal{E}+1)^2}\left(\frac{M_2^*}{M}\right)^4\right\}^{1/2}\right]. \tag{A·14}$$

$\mathcal{E}$ is very large and $n\gtrsim 5$, so that for $M_2^*/M\lesssim 5$ we get $0.9<\alpha<1$. We have then approximately (within 2% error)

$$\eta=1-M/M_2^* \quad \text{for} \quad \eta\leq 0.8. \tag{A·15}$$

In the case of nucleon-nucleon collision this relation is valid for $\eta \leq 0.9$.

On the other hand

$$\eta_{\text{th}} = \frac{W - ((nM/M_1^*)W_1 + (M/M_2^*)W_2)}{W-(n+1)M}$$

$$= \frac{1 - M/M_2^*}{1 - \dfrac{n+1}{\sqrt{n^2+2n\varepsilon+1}}}.$$

Again this can be approximated within 6% error by

$$\eta_{\text{th}} \approx 1 - M/M_2^* \quad \text{for } \varepsilon \geq 10^3,\ n \leq 5$$

in agreement with η_{exp} for $\eta \leq 0.8$. So we have used in the text a conventional combination* of η_{th} and η_{exp} : $\eta = \eta_{\text{th}}$ for $\eta = 1$, and $\eta = \eta_{\text{exp}} \approx 1 - M/M_2^*$ for $\eta \leq 0.8$.

In the case B) it would be reasonable to define the elastic part of the energy as that carried away by $(n+1)/2$ nucleons in the incident direction.

$$\eta_{\text{th}} = \frac{W - 2\left(\dfrac{W}{2}\dfrac{(n+1)M}{2M^*}\right)}{W-(n+1)M}$$

$$= \frac{1 - \dfrac{n+1}{2}\dfrac{M}{M^*}}{1 - \dfrac{n+1}{(n^2+2n\varepsilon+1)^{1/2}}}$$

$$\eta_{\text{exp}} = \frac{E_0 - \dfrac{1}{\sqrt{1-v^2}}\left(\dfrac{W}{2}\dfrac{M}{M^*}\dfrac{n+1}{2} + v\sqrt{\left(\dfrac{W}{2}\right)^2 - M^{*2}}\dfrac{(n+1)M}{2M^*}\right)}{E_0 - M}$$

$$= \frac{1 - \dfrac{n+1}{2}\dfrac{M}{M^*}\left(1+\dfrac{n}{\varepsilon}\right)\left[\dfrac{1}{2} + \dfrac{1}{2}\left(1 - \dfrac{1}{\varepsilon^2}\right)^{1/2}\left(1+\dfrac{n}{\varepsilon}\right)^{-1}\left\{1 - \dfrac{4}{n^2+2n\varepsilon+1}\left(\dfrac{M^*}{M}\right)^2\right\}^{1/2}\right]}{1 - 1/\varepsilon}.$$

When $\varepsilon \geq 10^3$, $n \leq 5$, η_{exp} can be expressed within 5% error by

$$\eta = 1 - \frac{n+1}{2}\frac{M}{M^*} \quad \text{for} \quad \eta \leq 0.8.$$

η_{th} can also be approximated by

---

* The maximum value of η_{exp} is not exactly one:

$$(\eta_{\text{exp}})_{\max} = \frac{1 - \dfrac{1}{(2n\varepsilon)^{1/2}}\left(1+\dfrac{n}{\varepsilon}\right)\left(1+\dfrac{n^2+1}{2n\varepsilon}\right)^{-1/2}}{1 - \dfrac{1}{\varepsilon}}.$$

$$\eta = 1 - \frac{(n+1)M}{2M^*}$$

within 6% error for $n \leq 5$, $\varepsilon > 10^3$.

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Letters to the Editor

The Surface Tension of Liquid He^3

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The extent to which liquid He^3 behaves as an ideal F.D. gas has been the object of considerable study in recent years. Experiments to date have failed to give a unique answer to this problem. It is the purpose of the present note to point out that the recent measurements of surface tension of liquid He^3 by Lovejoy[1] and Zinoveva[2] allow a quantitative check on this question.

Kothari and Auluck[3], in a recent communication, arrived at the following expression for surface tension of a completely degenerate ideal F.D. gas.

$$\sigma = g\pi/16 \cdot (h^2/m)\,(p_0/h)^4, \qquad (1)$$

where p_0 is given by the cubic equation

$$(p_0/h)^3 - (3S/16V)\,(p_0/h)^2 - 3n/4\pi g = 0; \qquad (1a)$$

p_0 is the maximum momentum characterizing the Fermi distribution, n the number of particles per unit volume, m the mass and g the weight-factor of a particle : σ is the surface tension. Here V is the volume while S is the surface area.

Assuming liquid He^3 to be a completely degenerate ideal F.D. gas and ignoring (as in the case of metals) the second term in eq. (1a), eq. (1) reduced (taking $m=$ mass of a He^3 atom, $g=2$) to

$$\sigma = 2 \cdot 35 \cdot \rho^{4/3} \text{ dyne. cm}^{-1}, \qquad (2)$$

where ρ(gm. cm^{-3}) is the density of liquid He^3.

On the basis of this theory, the value of surface tension of liquid He^3 at 0°K (by making use of the extrapolated density at 0°K, as given by Kerr[4]) is found out to be 0.084 dyne/cm, as compared to the corresponding value of 0.154 dyne/cm, determined by Lovejoy[1] on the basis of Atkin's theory[5]. We have calculated the surface tension of liquid He^3 as a function of temperature by making use of eq. (2) and the data for density of liquid He^3 as determined recently by Kerr[4].

The results so obtained are compared in Fig. 1 with the experimental results of Lovejoy[1] and Zinoveva[2]. For the sake of comparison, we have also plotted in Fig. 1 our theoretically calculated results by using Pollara's theory[6],[7]. As is evident from Fig. 1, eq. (1) expresses the correct order of magnitude but the predicted dependence of the surface tension of liquids He^3 on temperature is not satisfactory. The agreement is much better in the case of Pollara's thermo-dynamical theory.

It leads us to the conclusion that although there is some 'a priori' basis for believing that liquid He^3 should approximate an ideal F.D. gas assembly of particles, it now seems apparent that the actual interaction forces between the particles invalidate the applicability of this simple

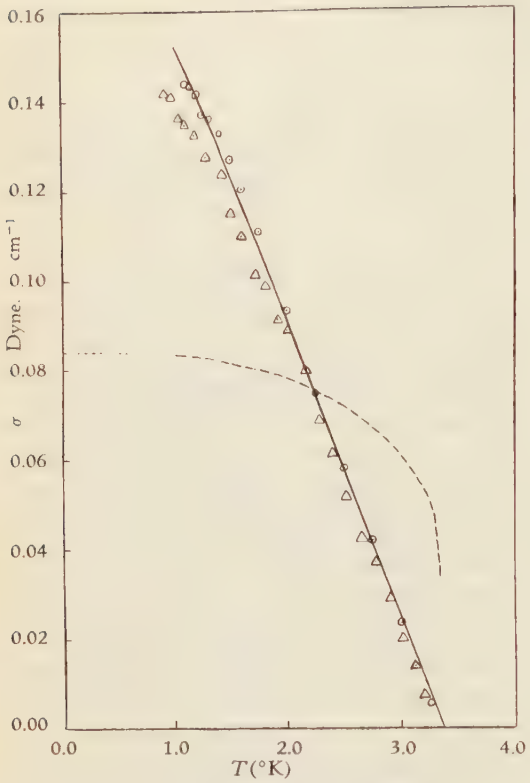

Fig. 1. Surface Tension of Liquid He^3 versus temperature T°K.
——: Pollara's theory[6),7)].
- - - -: Ideal F.D. gas theory, eq. (2).
⊙: Smoothed experimental results of Lovejoy[1)].
△: Experimental results of Zinoveva[2)].

limiting model. We have extended the model of liquid He^3 recently proposed by Nanda and Trikha[8)], for the two-dimensional case. It is found that, if the contribution to the free energy on account of the lattice vibrations is ignored for the two dimensional case, the agreement between the predicted values and the experimental data of surface tension of liquid He^3 is fairly satisfactory[9)].

Our thanks are due to Prof. D. S. Kothari for taking interest in the above investigations.

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Note on Bloch-Nordsieck Transformation and Electron-Lattice Interaction

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Using the Bloch-Nordsieck transformation Bardeen[1)] obtained an expression for the interaction energy between electrons and lattice vibrations. An interaction between electrons can take the place of the interaction between electrons and lattice vibrations. The expression for the interaction between electrons which was given by Bardeen is

$$U=$$
$$-\sum_{sij}{}'|m_s|^2\cos\boldsymbol{s}\cdot(\boldsymbol{x}_i-\boldsymbol{x}_j)/(\omega_s^2-(\boldsymbol{V}\cdot\boldsymbol{s})^2)$$
$$+1/2\sum_{sij}{}'(3(\boldsymbol{V}\cdot\boldsymbol{s})^2-\omega_s^2)\{m_s^2e^{i\boldsymbol{s}(\boldsymbol{x}_i+\boldsymbol{x}_j)}$$
$$+m_s^{*2}e^{-i\boldsymbol{s}(\boldsymbol{x}_i+\boldsymbol{x}_j)}\}/\{\omega_s^2-(\boldsymbol{V}\cdot\boldsymbol{s})^2\}^2. \quad (1)$$

The first of these interaction terms contributes to the energy in the first order, and the second only in higher orders.

The aim of the present note is to point out that the second term in (I) may be

eliminated by a suitable choice of the generating function of the B–N transformation.

Following Bardeen, we write the Hamiltonian as follows,

$$H=\sum_i \boldsymbol{V}\cdot\boldsymbol{P}_i+\sum_{is} q_s f_{si} + 1/2\sum_s (p_s^2+\omega_s^2 q_s^2), \qquad (2)$$

where $f_{si}=m_s e^{i\boldsymbol{s}\boldsymbol{x}_i}+m_s^* e^{-i\boldsymbol{s}\boldsymbol{x}_i}$. Let us introduce the new coordinates,

$$q_s'=q_s+\sum_i \alpha_s f_{si}\,;\ \boldsymbol{x}_i'=\boldsymbol{x}_i \qquad (3)$$

and take the wave function for the system in the form,

$$\Phi(\boldsymbol{x},q)=\exp\left[iS(\boldsymbol{x}',q')/\hbar\right]\Phi'(\boldsymbol{x}',q'). \qquad (4)$$

The operators p_s and $\boldsymbol{P}_i$ acting on Φ give

$$p_s\Phi=\exp(iS/\hbar)\left[(\partial S/\partial q_s')+p_s'\right]\Phi';$$
$$\boldsymbol{P}_i\Phi=\exp(iS/\hbar)\left[\boldsymbol{P}_i'+\nabla_i S + \sum_s (\partial S/\partial q_s')+p_s'\nabla_i q_s'\right]\Phi'. \qquad (5)$$

We take S in the form,

$$S=-\sum_{is} q_s'\alpha_s(\boldsymbol{V}\cdot\boldsymbol{\Gamma}_i)f_{si} + i\sum_{ijs}\alpha_s^2(3(\boldsymbol{V}\cdot\boldsymbol{s})^2-\omega_s^2)/4(\boldsymbol{V}\cdot\boldsymbol{s}) \cdot\left[m_s^2 e^{i\boldsymbol{s}(\boldsymbol{x}_i+\boldsymbol{x}_j)}-m_s^{2*}e^{-i\boldsymbol{s}(\boldsymbol{x}_i+\boldsymbol{x}_j)}\right], \qquad (6)$$

where $\alpha_s=1/(\omega_s^2-(\boldsymbol{V}\cdot\boldsymbol{s})^2)$. (Bardeen took S in the form, $S=-\sum_{i,s} q_s'\alpha_s(\boldsymbol{V}\cdot\boldsymbol{\Gamma}_i)f_{si}$). Then the transformed Hamiltonian operating Φ' is

$$H'=\sum_i \boldsymbol{V}\cdot\boldsymbol{P}_i'+1/2\sum_s (p_s'^2+\omega_s^2 q_s'^2) -\sum_{ijs}|m_s|^2\cos\boldsymbol{s}\cdot(\boldsymbol{x}_i-\boldsymbol{x}_j)/(\omega_s^2-(\boldsymbol{V}\cdot\boldsymbol{s})^2). \qquad (7)$$

Thus the second term in (I) is eliminated.

Next we consider the following Hamiltonian, which is based on the Bloch model,

$$H=\sum_i \boldsymbol{V}\cdot\boldsymbol{P}_i+\sum_q(\hbar qs/2)(P_q^2+Q_q^2) +\sum_{iq} M_q(P_q\cos\boldsymbol{q}\boldsymbol{x}_i+Q_q\sin\boldsymbol{q}\boldsymbol{x}_i), \qquad (8)$$

where $M_q=(4C^2\hbar q/9nvMs)^{1/2}$ and $[P_q, Q_{q'}]=-i\delta_{qq'}$.

Introduce the new coordinates,

$$Q_q'=Q_q-\sum_i \sigma_q\sin\boldsymbol{q}\boldsymbol{x}_i\,;\ \boldsymbol{x}_i'=\boldsymbol{x}_i, \qquad (9)$$

and take the generating function S in the form,

$$S=\sum_{iq}\hbar\sigma_q Q_q'\cos\boldsymbol{q}\boldsymbol{x}_i +\sum_{ijq}(\hbar\sigma_q^2/2)\sin\boldsymbol{q}\boldsymbol{x}_i\cos\boldsymbol{q}\boldsymbol{x}_j. \qquad (10)$$

Then we get

$$P_q\Phi=\exp(iS/\hbar)(P_q'+\sum_i\sigma_q\cos\boldsymbol{q}\boldsymbol{x}_i)\Phi';$$
$$\boldsymbol{P}_i\Phi=\exp(iS/\hbar)\left[\boldsymbol{P}_i'-\sum_q\hbar\sigma_q\boldsymbol{q}\cdot(P_q'\cos\boldsymbol{q}\boldsymbol{x}_i+Q_q'\sin\boldsymbol{q}\boldsymbol{x}_i) -\sum_{jq}(\hbar\sigma_q^2\boldsymbol{q}/2)(\cos\boldsymbol{q}\boldsymbol{x}_i\cos\boldsymbol{q}\boldsymbol{x}_j +\sin\boldsymbol{q}\boldsymbol{x}_i\sin\boldsymbol{q}\boldsymbol{x}_j)\right]\Phi'. \qquad (11)$$

Putting $\sigma_q=M_q/(\hbar\boldsymbol{V}\cdot\boldsymbol{q}-\hbar qs)$, we obtain the following expression for the transformed Hamiltonian

$$H'=\sum_i \boldsymbol{V}\cdot\boldsymbol{P}_i'+\sum_q(\hbar qs/2)(P_q'^2+Q_q'^2) -\sum_{ijq}(M_q^2/2\hbar)\cos\boldsymbol{q}(\boldsymbol{x}_i-\boldsymbol{x}_j)/(qs-\boldsymbol{V}\cdot\boldsymbol{q}). \qquad (12)$$

It is easily seen that the last term in (12) is equivalent to the last term in (7).

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On the Exchange Effect in the Stripping Reactions

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It is well known that the characteristic forward peak appearing in the angular distribution from certain (d, p) and (d, n) reactions can be explained by means of the stripping process of deuterons. By the usual stripping theory (Butler's theory), however, it is impossible to explain the rise of the differential cross section at the large angles which appears in some experiments. To explain this backward component, Madansky and Owen[1] proposes the "heavy particle stripping mechanism" and emphasized that the reaction products are also ejected from the target nucleus as much as from the deuteron stripping. On the heavy particle stripping model, the differential cross section is described as the sum of the contributions from deuteron and heavy particle stripping, but both must be normalized to the experimental data separately since there is no relation between these contributions.

In this note we examine the possibility of the heavy particle stripping mechanism taking into account the effect of nucleon exchange. We take the (d, n) reactions for illustration, and consider the exchange phenomena between the neutron which initially composes the deuteron and those which are in the outer orbits of the target nucleus by properly symmetrizing the wave function of the total system.

Assuming that there are n neutrons in the outermost j-orbit in the target nucleus the wave functions for these n neutrons in the target nucleus can be expressed in terms of those for $n-1$ neutrons by means of the fractional parentage coefficients, and we consider the exchange of the neutron, say 3, which is taken out in this way, and that of the incident deuteron, say 1. Then the reaction matrix element is given by

$$((1-\sum_{j=3}^{n+2} \Lambda_{1j})/\sqrt{n+1}\cdot \Psi_f,$$

$$V(1-\sum_{j=3}^{n+2} \Lambda_{1j})/\sqrt{n+1}\cdot \Psi_i).$$

For the case of $n=1$, neglecting the Coulomb interaction, this becomes

$$(\Psi_f, (V(1,2)+V(1,I))\Psi_i)$$
$$-(\Lambda_{13}\Psi_f, (V(3,I)+V(3,1)$$
$$+V(3,2))\Psi_i),$$

where 2 stands for the proton in the deuteron, Λ_{1j} is the exchange operator of the space and spin coordinates of particle 1 and j, Ψ_i and Ψ_f are the initial and final state wave functions in the plane-wave Born approximation respectively.

For the calculation of the matrix element we take the following approximations. In the first term the interaction between the neutron 1 which is originally in the initial deuteron and the initial target nucleus $V(1, I)$ is neglected. This procedure corresponds to the usual Butler theory. In the second term the interaction between the incident deuteron and the neutron 3, $V(3, 1)+V(3, 2)$, is dropped. This means the neglect of the knock out process of the Austern-Butler-McManus type.[2] The remaining interaction $V(3, I)$ corresponds to the interaction which causes the heavy

particle stripping process. In the evaluation of this term we assume that the wave function of the deuteron can be replaced by the zero-range wave function. We take the single particle wave function moving in a suitable square well potential as that of the neutron 3. There are also two parameters which enter in the first term as the interaction radius a, and in the 2nd term as the radius of the target nucleus r_0. The relative contributions of the first and the second term to the matrix element depend on a, r_0 and the angular momenta of the initial and final nucleus in a complicated manner.

We calculate the case of $C^{13}(d, n)N^{14}$ ($E_d=0.865$ Mev). In Fig. 1 is shown the comparison between the theory and the experiments of Green et al.[3]

Recent experimental evidences show the effects of compound nucleus formation in (d, p) or (d, n) reactions. Since in the case of C^{13} which we have treated the nucleus is composed of a loosely bound extra neutron and the comparatively rigid core, the excitation of the core through the compound nucleus formation may not make a large contribution. Therefore, it may be said that our mechanism mainly represents the compound effects. The rather good agreement between our calculation and the experiments shows that the contribution from our mechanism is considerable for this sort of nucleus in these energy region.

Thus we may conclude that the backward components of the angular distribution in (d, p) or (d, n) reactions should be explained by the compound nucleus formation. Further, it may be concluded that the single particle-like treatment as ours is a good approximation to the compound nucleus process in the reaction $C^{13}(d, n)N^{14}$.

The analyses of (d, p) or (d, n) reactions involving the process of compound nucleus formation are now in progress and will be published in near future.

The authors wish to express their sincere thanks to Mr. M. Sano and Mr. S. Yamasaki for their valuable suggestions and discussions.

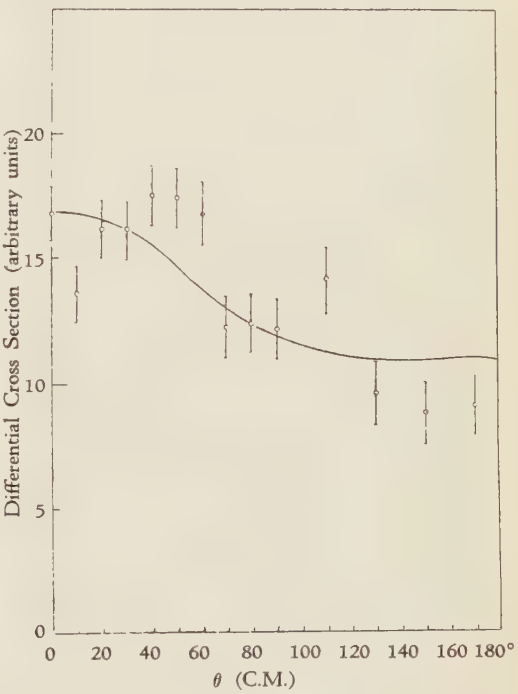

Fig. 1. The angular distribution of the ground state neutrons from the $C^{13}(d, n)N^{14}$ reaction for $E_d=0.865$ Mev. Parameters used are: $a=4.9\times10^{-13}$ cm, $r_0=3.4\times10^{-13}$ cm.

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A Model for the Inelastic Scatteriug of Nucleons

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According to the statistical theory of nuclear reaction, the relative level density of target nucleus can be calculated from the observed energy distribution of inelastically scattered protons. However, the curve of the predicted level density plotted against excitation energy of residual nucleus is concave downward in disagreement with experiments. Furthermore, the experimental results[1] indicate that the observed energy variation of the level densities may not depend on the excitation energy of residual nucleus primarily, but, instead, there is some evidence that it is a function of the energy of emitted protons. These facts can never be explained by the statistical theory of nuclear reaction. In this note it is pointed out that these conflicts are due to the *single-particle aspect of nuclear behavior in the compound system.* A model is proposed for the description of the energy exchange between the incoming nucleon and the target nucleus.

The energy distributions[2] of inelastically scattered protons from Ag, observed at various scattering angles, are shown in Fig. 1. There are two prominent peaks : the one peak is broader than the other. What is concerned in this note is the broad peak (appeared at about 8-Mev excitation energy) in Fig. 1. According to the usual theory of nuclear reaction, the two peaks may be

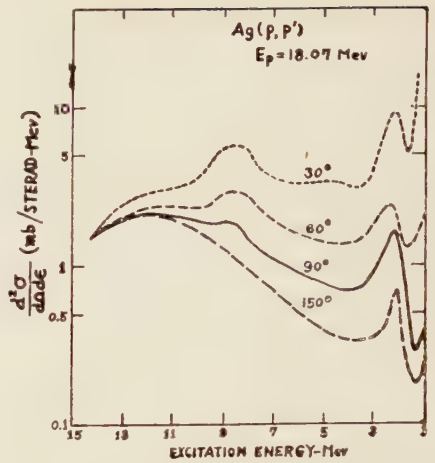

Fig. 1. The energy distributions of Ag observed at different angles.

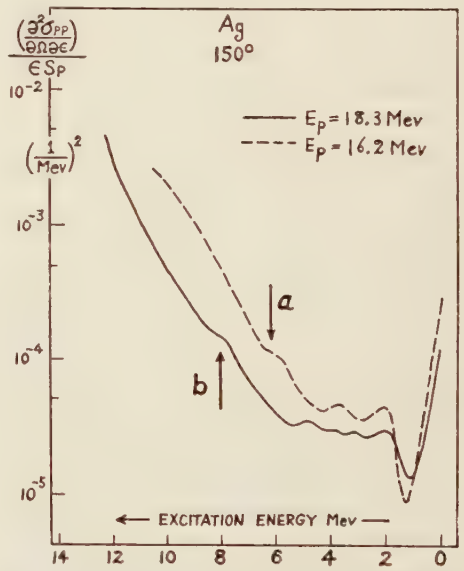

Fig. 2. The relative level density of Ag obtained from the scattering of 18.3-Mev and of 16.2-Mev proton.

ascribed to the particular properties of certain levels of the residual nucleus. If this is the case, positions of the peaks should remain when incident proton energy is modified. Direct experimental test for the arguments is not available, but there

is some evidence which makes it possible to infer that this will not be true as far as the broad peak is concerned (Fig. 2). In Fig. 2, the peaks appear less prominent, since these data are taken at scattering angles 150° and give us only an information on the relative level density of Ag. From curves for 16.2–, 18.3–Mev incident proton energy, however, it is easy to see that one curve is displaced by about 2-Mev to the other for the excitation energies higher than about 6-Mev. One can see a peak (shown by an arrow *a*) at about 6-Mev excitation energy on the 16.2-Mev curve, and the corresponding one (shown by an arrow *b*) at about 8-Mev excitation energy on the 18.3-Mev. This peak can be identified with the broad peak in Fig. 1. Therefore, we can conclude that the identified peak is displaced by about 2-Mev

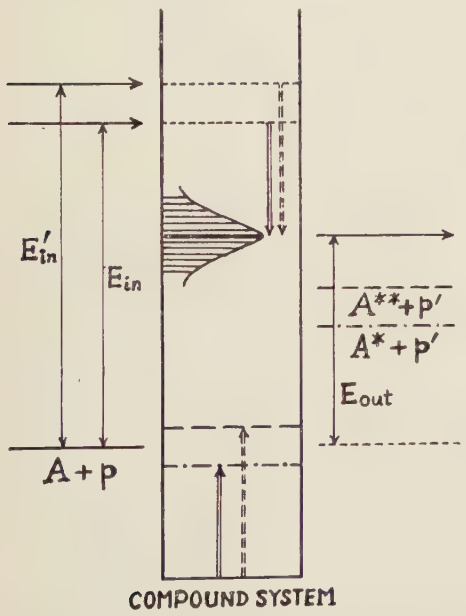

Fig. 3. A schematic picture of inelastic scattering of protons by nucleus A. The excitation energies of residual nucleus A* and A** are $E_{in}-E_{out}$ and $E_{in}'-E_{out}$ respectively.

corresponding the change of the incident energies from 16.2-Mev to 18.3-Mev. On the other hand, the other peak is not displaced at all. Obviously, it is difficult to ascribe the broad peak to particular properties of certain levels of the residual nucleus. This is the reason why a new model must be introduced.

For the purpose of illustrating the model proposed here, a schematic picture of inelastic scattering is shown in Fig. 3. Target nucleus *A* is bombarded by proton to form the compound system, but not the compound nucleus which is usually considered to imply the strong-coupling model. In the compound system formed, some amount of the total excitation energy is shared from the proton to excite the core nucleus *A*. Therefore, there are the cases where incident proton transfers its energy to the core to fall into a certain single-particle resonance level.[3] In these cases, the proton is emitted with large probability due to the resonance, and consequently leaves residual nucleus in the excitation energy by which the proton has been lost. Thus, we should consider that the broad peak of Fig. 1 can be explained by such effects. The appearance of this type of peak may be limited to the cases where the residual nucleus is left at sufficiently high excitation with considerable dense levels from the energetical consideration.

When incident energy somewhat increases (decreases), this model leads to the conclusions that the energy of emitted proton corresponding to the peak still remains, and rather the excitation energy of the residual nucleus increases (decreases) by the very increment (decrement). In other words, the position of the peak should be

displaced by the increment (decrement), if the distribution is plotted against the residual excitation energy. It is rather favorable to this model that two curves in Fig. 2 are coincident with each other if the distribution vs. the energy of emitted protons is plotted. Thus, this model can explain why the observed energy variation of the level densities is a function of emitted protons. From the above arguments, it seems that single-particle resonance in the compound system plays the essential role also in the other aspects of inelastic scattering of nucleons by nuclei. A full note will be published later.

1) P. C. Gugelot, Phys. Rev. **93** (1954), 425.
2) P. C. Gugelot, *Statistical Aspects of the Nucleus* (Brookhaven, 1955).
3) The model proposed here may be equivalent to the intermediate coupling model by Lane, Thomas, and Wigner for the giant-resonance interpretation of nucleon-nucleus interaction. Thus, the single-particle resonance level may be the *p*-group level (nuclear size resonance).

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN

Volume 12, Number 1, January 1957

CONTENTS

Masatoshi YAMAZAKI: Group-Theoretical Treatment of the Energy Bands in Metal Borides MeB_6 1
Kunio FUJIWARA: On the Period of Out-of-step of Ordered Alloys with Anti-phase Domain Structure 7
Kyozaburo KAMBE: Study of Simultaneous Reflexion in Electron Diffraction by Crystals I. Theoretical Treatment 13
Kyozabro KAMBE: The Phase Effect in Simultaneous Reflexion in Electron Diffracions II. Experimental Confirmation 25
Akira ISHIHARA and Ryuzo KOYAMA: Intermolecular Potential between Large Molecules 32
Takeshi OKA, Hiromasa HIRAKAWA and Akira MIYAHARA: Dependence of Quadrupole Coupling Constant on the Vibrational State in ICN Molecule 39
Shōzō SEKINO and Tetsuji NISHIKAWA: Microwave Spectrum of Vinylidene Chloride. .. 43
Yoshitaka KÔI: Line Width of the Pure Quadrupole Resonance of Bromine in $NaBrO_3$. .. 49
Jien NISHIWAKI: Gloss of Machine Glazed Papers 53
Isao IMAI: Application of the M^2-Expansion Method to Compressible Flow past Isolated and Lattice Aerofoils 58
Hidenori HASIMOTO: Boundary Layer Growth on a Flat with Suction or Injection 68
Yaso'o MATUNOBU: Application of the Thin-Wing-Expansion Method to the Compressible Flow past an Elliptic Cylinder, with Special Reference to its Fourth Approximation. 72
Otohiko NOMOTO: Phenomenological Theory of the Molecular Absorption and Dispresion of Sound in Fluids and the Relation between the Relaxation Time of the Internal Energy and the Relaxation Time of the Internal Specific Heat 85

SHORT NOTES

Yoshiro MORIGUCHI and Yutaka KOGA: Electrical Properties of GeTe 100
Toshinosuke MUTO and Sei-ichi OYAMA: Electronic Structure of a Moving Exciton in Alkali Halides 101
Michiru YASUHARA: Asymptotic Behaviour of the Laminar Compressible Boundary Layer over a Circular Cylinder with Uniform Suction 102
Yasuo MATUKURA: Carrier Concentration Changes in p-Si Induced by Heat Treatment 103
Hirosi HASEGAWA, Sigeru NAKANO and Natsuki HASHITSUME: On the Hall Effect in Bismuth at Low Temperature 104
Makoto KIKUCHI and Seiichi DENDA: Spiral Etch Pits on Germanium Surface 105
Tetsuji IMAI and Isao MITSUFUJI: On the Release of Alkaline Earth Metals by the Electron Dissociation of Oxide Films Evaporated from the Oxide Coated Cathodes 106
Yasaburo YOKOZAWA and Isao MIYASHITA: Temperature Dependence of Electron Spin Resonance Absorption of Thiophenol in Sulfuric Acid 107
Toshinosuke MUTO and Hiroshi OKUNO: On the Electronic Structure of Localized Exciton in Alkali Chlorides 108

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN

Volume 12, Number 2, February 1957

CONTENTS

J. MUTO, E. TAKAKOSHI, T. NAKAMURA, A. IMAMURA and Y. TSUNEOKA: The Multipolarity of γ-ray Absorption in ^{12}C $(\gamma, 3\alpha)$ Reaction produced by (Li, p) γ-rays 109

Saburo MIYAKE, Kensaku HINOTANI and Kan-ichi NUNOGAKI: Intensity and Energy Spectrum of Fast Neutrons in Cosmic Radiation 113

Masahiro KODAMA, Kazuaki MURAKAMI and Yataro SEKIDO: Observation of Cosmic-Ray Neutron Intensity at Geomagnetic Latitude 25°N, Part I. Atomospheric Effects 122

Yasuo KANAI and Riro NII: Electrical Properties of Nickel Doped Germanium at Low Temperatures 125

Makoto KIKUCHI: Peripheral Inhomogeneties of the Alloyed Germanium p-n Junction .. 133

Einosuke FUKUSHIMA and Akiya OOKAWA: Observation of the Grain Boundary in Soap Bubble Raft. Part. III. The Grain Boundary in Vibrating State..... 139

Shozo SAWADA, Ine TAKOO and Shin'ichi FUJII: An X-Ray Tube with a Rotating Anode and an Earthed Cathode 147

Maso KURIYAMA, Kazutaka KOHRA and Satio TAKAGI: Electron Diffraction Study on Polish Layers of Gold, I 151

Tokutaro HIRONE and Kengo ADACHI: On the Magnetic Properties of Nickel Arsenide Crystals..... 156

Kazuko MOTIZUKI; Theory of the Para-Ortho Conversion in Solid Deuterium 163

Kazuo HORIUCHI: Electromagnetic Fields due to Current Flowing Parallel to Interface of Two Different Media 170

Michiru YASUHARA: On the Hypersonic Viscous Flow past a Flat Plate with Suction or Injection 177

Yutaka SHIGEMITSU: Experimental Studies on Laminar Sub-layer in Turbulent Boundary Layer involving Separation..... 183

Yûkichi NOMURA and Shigetoshi KATSURA: Diffraction of Electromagnetic Waves by Ribbon and Slit, I 190

Teruo IKEBE and Tosio KATO: Application of Variational Method to the Thomas-Fermi Equation 201

Tasaburô YAMAGUTI: A New Method of Electro-Machining of Hard Metals 204

Tikao HUZITA: On the Ricochet Motion of a Projectile Striking a Water Surface..... 208

SHORT NOTES

Noriaki ITO and Tokuo SUITA: Effect of Plastic Deformation on the Dielectric Breakdown of Silver Chloride Crystals 214

Y. TANAKA, K. SUGA and S. MITANI: The Energy Spectrum of Nucleons in the Air Shower 215

Shigeo NAGAO: Effects of Shock Wave upon Gaseous Discharge 216

Eijirô HAGA: On the Electrical Properties of Degenerate Semiconductors 217

Kazuo KAWABE and Shozo SAWADA: Preparation of $BaTiO_3$ Single Crystals in Coal Gas Atmosphere 218

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An Attempt of Generalizing the Hypothesis of Charge Independence, I

——Selection Rule for the Decay Process——

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An attempt is made of generalizing the hypothesis of charge independence to the weak interaction. The fundamental assumption is that the strangeness quantum number is not a scalar but the third component of an angular momentum vector in the charge space, and the sum of this new vector and the usual isotopic spin is conserved in all processes. The case when the usual charge independence holds is assumed to correspond to the special case when these two vectors are conserved separately. It is shown that this assumption leads, when combined with the composite hypothesis proposed by Markov and Sakata, to a consistent level scheme for hyperons and heavy mesons, and moreover gives a new selection rule for the decay process which is essentially equivalent to $|\Delta I|=\frac{1}{2}$ recently suggested by many authors. It gives also other results which can be compared with experiments directly. Among them a preliminary discussion is given on a new selection rule expected in the annihilation of nucleon-antinucleon pairs. A comment on the b-field proposed by Yang and Mills is also given.

§ 1. Introduction

The hypothesis of charge independence has played an important role in the phenomena of pion and of heavy unstable particles. Its validity is, however, restricted and we can not apply it to decay processes. If this hypothesis forms an integral part of the theory, it would be expected that it should have a more universal character. We propose in this note an attempt which, by generalizing the concept of the isotopic spin, aims to evade this restriction.

The starting point of the discussion is the conservation law of charge. Consider, for example, the simplest case of pion. Its charge is given by

$$Q=eI_3\,, \tag{1·1}$$

where I_3 is the third component of the isotopic spin. In this case, therefore, the conservation law of charge is equivalent to the conservation law of I_3. This fact and the charge independence suggest a simple picture for the charge space. That is, in the charge space only the direction of the third component is characterized by the electromagnetic field, therefore it is symmetric under rotations around its third axis, and can be regarded as effectively isotropic in phenomena where the electromagnetic field does not take part or in the approximation in which it is neglected. It is clear that under such a picture the conservation law of charge and the charge independence are nothing but the expressions of these symmetries.

As has been pointed out by many authors, the gauge transformation (of the first kind)

$$\psi \to \psi e^{i\alpha}, \quad \psi^* \to \psi^* e^{-i\alpha}$$

(where α is an arbitrary real constant)

can be regarded as a rotation in the charge space around its third axis. This fact also seems to suggest this picture strongly.

Such a picture is, however, not so clear in the general case, where the charge is given by

$$Q = \begin{cases} e(I_3 + S/2 + n/2) & \text{for fermion*,} \\ e(I_3 + S/2) & \text{for boson,} \end{cases} \tag{1·2}$$

where S means the strangeness introduced by Nishijima and Gell-Mann,[1] and n the particle number. Therefore, if the above picture, especially the character of the conservation law of charge that it is the conservation law of the third component of the isotopic spin is taken as serious, it seems necessary to assume that $S/2$ and $n/2$ are also the third components of some angular momentum vectors in the charge space. Thus, introducing angular momenta $\boldsymbol{J}$ and $\boldsymbol{K}$ in the charge space other than $\boldsymbol{I}$, we assume that the total isotopic spin is given by

$$\boldsymbol{L} = \begin{cases} \boldsymbol{I} + \boldsymbol{J} + \boldsymbol{K} & \text{for fermion,} \\ \boldsymbol{I} + \boldsymbol{J} & \text{for boson.} \end{cases} \tag{1·3}$$

The charge is given by

$$Q = eL_3 = \begin{cases} e(I_3 + J_3 + K_3) & \text{for fermion,} \\ e(I_3 + J_3) & \text{for boson.} \end{cases} \tag{1·4}$$

Therefore J_3 and K_3 are related to S and n by

$$J_3 = S/2, \tag{1·5}$$

and

$$K_3 = n/2. \tag{1·6}$$

For one particle state K_3 should be restricted to $\pm 1/2$.** Here we assume

(i) L_3 is conserved in all processes,

(ii) in processes where the electromagnetic field does not take part, $(\boldsymbol{I} + \boldsymbol{J})$ is conserved,

(iii) in processes caused by the strong interaction, $\boldsymbol{I}$ and $\boldsymbol{J}$ are conserved separately.

(iii) corresponds to the assumption of usual charge independence. But it is in general more stringent in that it requires the conservation of $\boldsymbol{J}$.

* As for fermion we consider baryons exclusively.

** In the following discussions antibaryons are not included. Therefore we shall disregard $\boldsymbol{K}$ sometimes.

(i) is nothing but the conservation law of charge, but (ii) and (iii) give results which have not been got heretofore. For example, they enable us to predict the branching ratios of various decay processes such as

$$\sum^{+} \rightarrow \begin{cases} p+\pi^{0} \\ n+\pi^{+} . \end{cases}$$

This is because as long as we assume that the dacay interaction is a primary interaction and the electromagnetic field does not take part there, $\boldsymbol{I}+\boldsymbol{J}$ is conserved according to (ii). Also it is possible in some cases to make more precise the conclusion from the usual charge independence by (iii). These predictions can be compared with experiments directly, and the validity of our assumptions can be checked by comparing them with experiments. This will be done in the following.

§ 2. The assignment of the level scheme

As assumed in §1, $\boldsymbol{I}$ and $\boldsymbol{J}$ are mutually commutable in our formulation. Therefore, I, I_3, J and J_3 can be diagonalized simultaneously, and the assignment of these quantum numbers gives level scheme for hyperons and heavy mesons. An example of such an assignment is*

fermions

	$-e$	0	e	$2e$	I	J
Ξ	$\underline{\Xi^{-}(-\frac{1}{2}, -1)}$	$\underline{\Xi'^{0}(-\frac{1}{2}, 0)}$ $\underline{\Xi^{0}(\frac{1}{2}, -1)}$	$\underline{\Xi'^{+}(\frac{1}{2}, 0)}$ $\underline{\Xi^{+}(-\frac{1}{2}, 1)}$	$\underline{\Xi^{++}(\frac{1}{2}, 1)}$	1/2	1
$\sum$	$\underline{\sum^{-}(-1, -\frac{1}{2})}$	$\underline{\sum'^{0}(-1, \frac{1}{2})}$ $\underline{\sum^{0}(0, -\frac{1}{2})}$	$\underline{\sum'^{+}(0, \frac{1}{2})}$ $\underline{\sum^{+}(1, -\frac{1}{2})}$	$\underline{\sum^{++}(1, \frac{1}{2})}$	1	1/2
Λ		$\underline{\Lambda^{0}(0, -\frac{1}{2})}$	$\underline{\Lambda^{+}(0, \frac{1}{2})}$		0	1/2
N		$\underline{n(-\frac{1}{2}, 0)}$	$\underline{p(\frac{1}{2}, 0)}$		1/2	0

bosons

	$-e$	0	e	I	J
K	$\underline{K^{-}(-\frac{1}{2}, -\frac{1}{2})}$	$\underline{K^{0}(-\frac{1}{2}, \frac{1}{2})}$ $\underline{\bar{K}^{0}(\frac{1}{2}, -\frac{1}{2})}$	$\underline{K^{+}(\frac{1}{2}, \frac{1}{2})}$	1/2	1/2
π	$\underline{\pi^{-}(-1, 0)}$	$\underline{\pi^{0}(0, 0)}$	$\underline{\pi^{+}(1, 0)}$	1	0 .

(I_3 and J_3 are written in each bracket)

* This level scheme is the same as the one proposed by Pais in connection with his four dimensional ω-space.[2)]

As already mentioned by Pais, this scheme contains many unwanted levels.* Among them, those written by dotted lines decay rapidly into lower levels by emitting γ-rays, since the selection rule for γ-transition is given by

$$\Delta I = \pm 1,\ 0\ ; \quad \Delta I_3 = 0,$$

and

$$\Delta J = \pm 1,\ 0\ ; \quad \Delta J_3 = 0.$$

The γ-instability does not, however, eliminate all unwanted levels, and in particular the survival of doubly-charged particles such as Ξ^{--} and Σ^{--} seems to be a serious difficulty. It should be noted, however, that it is not right to consider that this scheme is the one corresponding to the actual data directly. For example, some of them may destroy the condition of dynamical stability as the result of their interactions with other fields and disappear, and some of them may form composite new levels if strong attractive forces exist between them. The possibility that the structure of the interactions is such that some of them are not produced in ordinary collisions must also be noted.

In the following will be discussed two trials of reconciling it with the actual data.

(a) The simplest way of eliminating doubly-charged particle is to assume that the electromagnetic mass correction Δm is not so small, and can amount to about 90 m_e. If this is the case, the mass correction for doubly-charged particles will be able to exceed pion mass, since in the lowest order of the weak coupling approximation Δm is proportional to the square of the coupling constant, and for doubly-charged particles it will be given by $\simeq 4 \times 90 m_e = 360 m_e$. Thus the doubly-charged particles can decay into lower levels by emitting pions:

$$\Xi^{--} \to \Xi^{-} + \pi^{-},$$

and

$$\Sigma^{++} \to \Sigma^{+} + \pi^{+}.$$

Ξ^{+} can decay by either of the following processes

$$\Xi^{-} \to \begin{cases} \Lambda^0 + \pi \\ \Lambda^+ + \pi^0. \end{cases}$$

In the latter case the masses of charged particles (Ξ^{-}, Λ^{+}) are much larger than that of the neutral one (π^0), and moreover the Q value is small (about 20 Mev). Therefore the deflection of the ionizing track will be very small, and the whole track will be observed merely as the decay of Λ^{0}. This will make the detection of Ξ^{-} difficult, if the branching ratio to the latter is large.**

* Pais assumed that unwanted levels go very high, and decay into lower levels by emitting K-mesons. This is impossible in our case, since for this to be realized it is necessary to assume that the hamiltonian contains J_3, which contradicts with our previous assumptions.

** This branching ratio is calculated in § 3. The result contains a parameter which can not be determined by kinematical considerations alone. The former mode is, however, the process with $\Delta S = 3$ and will be forbidden.

Σ'^0 will be observed as anomalous Λ^0 since its decay scheme will be

$$\Sigma'^0 \to \begin{cases} p+\pi^- \\ n+\pi^0 . \end{cases}$$

If we assume that its mass is smller than $\Sigma'^{\pm}$ by electromagnetic correction, its Q value will be about 70 Mev. Thus it might be possible to identify anomalous Λ^0 of C.I.T. group as Σ'^0.[3]

Λ^{+} will be observed as anomalous Σ'^+ since its decay will be

$$\Lambda^+ \to \begin{cases} p+\pi^0 \\ n+\pi^+ . \end{cases}$$

Assuming that it is shifted upwards by about 90 m_e by electromagnetic correction, its Q value will be about 80 Mev. In emulsions there have been some evidences for charged hyperons of low Q value (about 60 Mev) which have been considered as Ξ^-.[4] In these cases, however, no decay events due to secondary Λ^0 have been observed, and it might be possible to identify some of them as Λ^+.

Summarizing, in this scheme appear in addition to N.G.'s

(i) anomalous Σ'^+ with Q value about 80 Mev,

(ii) anomalous Λ^0 with Q value about 70 Mev,

(iii) possibly Ξ^+.

(b) Although doubly-charged particles were eliminated in the preceeding scheme, there still remain hyperons with $J_z = +1/2$ or $+1$, which make the following processes possible :

$$\pi + N \to \Lambda^+ + K^- ,$$

and

$$K^+ + N \to \Lambda^+ + \pi \qquad \text{etc.}$$

Table I. Sakata's model of hyperons and heavy mesons.

particle	composition	particle number	isotopic spin	strangeness
N		1	$\frac{1}{2}$	0
Λ^0		1	0	-1
Σ	$N\bar{N}\Lambda^0$	1	1	-1
Ξ	$\bar{N}\Lambda^0\Lambda^0$	1	$\frac{1}{2}$	-2
$K^{+,0}$	$N\bar{\Lambda}^0$	0	$\frac{1}{2}$	$+1$
$K^{-,0}$	$\bar{N}\Lambda^0$	0	$\frac{1}{2}$	-1
π	$N\bar{N}$	0	1	0

($\bar{N}$, $\bar{\Lambda}^0$ mean antinucleon and anti-Λ^0 respectively)

Thus, in this scheme, in contrast to N.G.'s, asymmetries between K^+ and K^- do not appear in their production rates[5] or in nuclear interactions[6]. Although the evidence is not yet conclusive, recent data do not seem to favour this conclusion.

One way of avoiding this difficulty is to adopt composite hypothesis. The composite theory of heavy unstable particles was proposed by Markov[7] and by Sakata[8] et al. as an extension of the nucleon-antinucleon pair theory of pion proposed by Fermi and Yang[9], and in particular Sakata showed that, using nucleons and Λ^0 as "Urteilchen" it is possible to reproduce N.G.'s scheme (see Table I).

Thus, if we adopt Sakata's hypothesis, exact N.G.'s is reproduced if it is possible to eliminate all levels except for nucleons and Λ^0. The simplest way for this is to assume that for strange particles (particles with $S \neq 0$) the electromagnetic mass correction is very large, and can exceed K-meson mass.* ** *** If this is the case, all higher levels except for Λ^0**** decay into lower levels by emitting K-mesons, and we can proceed in the way exactly same as that of Sakata. The point that distinguishes our theory from that of Sakata in this case is that in our theory the relation between the charge, the third component of the isotopic spin, the strangeness and the particle number (1·2) and the selection rule for the decay process (see § 3) are derived from a unified principle of generalized charge independence, while they had to be introduced *ado hoc* in Sakata's theory.*****

§ 3. Decay processes

In usual theories the hypothesis of charge independence can not be applied to decay processes. In our theory, however, the sum $(\boldsymbol{I}+\boldsymbol{J})$ is conserved in this case also, and this enables us to calculate the branching ratio in some cases.

We introduce the following notations:

$\mathcal{K}_I^{I_3}$: eigenstate of $\boldsymbol{I}$ belonging to the eigenvalue I and I_3.

$\mathcal{Y}_J^{J_3}$: eigenstrte of $\boldsymbol{J}$ belonging to the eigenvalue J and J_3.

$\mathcal{Z}_L^{L_3}(I, J)$: eigenstate of $\boldsymbol{L}$ belonging to the eigenvalue L and L_3. I and J written in the bracket are to express that this state is composed of eigenstates of $\boldsymbol{I}$ and $\boldsymbol{J}$ belonging to the eigenvalues I and J respectively.

---

* Recent calculations by Landau et al.[10] show that the masses of charged particles can be entirely of electromagnetic origin. Therefore, this assumption will not contradict the smallness of e.

** Such a situation can occur, for example, if the radii of strange particles are much smaller than that of normal ones.

*** We must expect that this large mass correction destroys the charge independence in some cases. In actual problems, however, it will hold comparatively good, since the mass corrections for Urteilchen are all small. Deviations from it come from virtual processes including charged strange particles. Therefore their effect will be at most of the order of ten percents.

**** Of course other neutral hyperons such as Ξ^0 can survive. They can be used as Urteilchen as in Markov's model, or can be eliminated by assuming that their mass is larger than that of Λ^0 by more than K-meson mass.

***** New possibilities pointed out in § 4 are of course other points that distinguish our theory from that of Sakata.

Initial and final states specified by four quantum numbers I_i, I_{3i}, J_i and J_{3i} and I_f, I_{3f}, J_f and J_{3f} respectively can be expressed in terms of Clebsch-Gordan coefficients as

$$\begin{aligned}\Psi_i &= \mathcal{K}_{I_i}^{I_{3i}} \mathcal{Y}_{J_i}^{J_{3i}} = \sum_L C_{I_i J_i}(L, L_3; I_{3i}, J_{3i})\, \mathcal{Z}_L^{L_3}(I_i, J_i) \\ \Psi_f &= \mathcal{K}_{I_f}^{I_{3f}} \mathcal{Y}_{J_f}^{J_{3f}} = \sum_L C_{I_f J_f}(L, L_3; I_{3f}, J_{3f})\, \mathcal{Z}_L^{L_3}(I_f, J_f),\end{aligned} \tag{3·1}$$

and the transition matrix can be written in the form

$$\begin{aligned}\langle \Psi_f | S | \Psi_i \rangle = \sum_L & C_{I_f J_f}(L, L_3; I_{3f}, J_{3f}) \\ & \times C_{I_i J_i}(L, L_3; I_{3i}, J_{3i}) S_L(I_f, J_f | I_i, J_i).\end{aligned} \tag{3·2}$$

The explicit form of $S_L(I_f, J_f | I_i, J_i)$ can not be given by kinematical considerations alone. They are left as undetermined parameters.

(i) *The decay of Λ^+ and Λ^0.*

Λ^+ and Λ^0 can decay by either of the two modes

$$\Lambda^+ \to \begin{cases} p+\pi^0 \\ n+\pi^+, \end{cases} \quad \text{and} \quad \Lambda^0 \to \begin{cases} p+\pi^- \\ n+\pi^0. \end{cases}$$

Using (3·1), the explicit form of the initial and final states are given as follows.

initial states :

$$\begin{aligned}\Lambda^+ &= \mathcal{K}_0 \mathcal{Y}_{1/2}^{1/2} = \mathcal{Z}_{1/2}^{1/2}(0, 1/2), \\ \Lambda^0 &= \mathcal{K}_0 \mathcal{Y}_{1/2}^{-1/2} = \mathcal{Z}_{1/2}^{-1/2}.\end{aligned}$$

final states :

$$\begin{aligned}p+\pi^0 &= \mathcal{K}_1^0 \mathcal{K}_{1/2}^{1/2} \mathcal{Y}_0 \mathcal{Y}_0 = \sqrt{2/3}\, \mathcal{Z}_{3/2}^{1/2}(3/2, 0) - (1/\sqrt{3})\, \mathcal{Z}_{1/2}^{1/2}(1/2, 0), \\ n+\pi^+ &= \mathcal{K}_1^1 \mathcal{K}_{1/2}^{-1/2} \mathcal{Y}_0 \mathcal{Y}_0 = (1/\sqrt{3})\, \mathcal{Z}_{3/2}^{1/2} + \sqrt{2/3}\, \mathcal{Z}_{1/2}^{1/2}, \\ p+\pi^- &= \mathcal{K}_1^{-1} \mathcal{K}_{1/2}^{1/2} \mathcal{Y}_0 \mathcal{Y}_0 = (1/\sqrt{3})\, \mathcal{Z}_{3/2}^{-1/2} - \sqrt{2/3}\, \mathcal{Z}_{1/2}^{-1/2}, \\ n+\pi^0 &= \mathcal{K}_1^0 \mathcal{K}_{1/2}^{-1/2} \mathcal{Y}_0 \mathcal{Y}_0 = \sqrt{2/3}\, \mathcal{Z}_{3/2}^{-1/2} + (1/\sqrt{3})\, \mathcal{Z}_{1/2}^{-1/2}.\end{aligned}$$

In this case the participating element of the S-matrix is $S_{1/2}(1/2, 0|0, 1/2)$ only, and we have the following results.

Table II

processes	coefficient of $(1/\sqrt{3})S_{1/2}$	branching ratio
$\Lambda^+ \to \begin{cases} p+\pi^0 \\ n+\pi^+ \end{cases}$	-1	1
	$\sqrt{2}$	2
$\Lambda^0 \to \begin{cases} p+\pi^- \\ n+\pi^0 \end{cases}$	$-\sqrt{2}$	2
	1	1

These branching ratios are independent of the energies of Λ^+ and Λ^0.

(ii) *The decay of $\sum^{\pm}$ and $\sum'^0$.*

The possible decay modes are

$$\sum^+ \to \begin{cases} p+\pi^0 \\ n+\pi^+, \end{cases} \quad \sum^- \to n+\pi^-,$$

and

$$\Sigma^{\prime 0} \to \begin{cases} p+\pi^- \\ n+\pi^0 . \end{cases}$$

The initial states are given by

$$\Sigma^+ = \mathcal{K}_1^1 \mathcal{Y}_{1/2}^{-1/2} = (1/\sqrt{3})\, \mathcal{Z}_{3/2}^{1/2}(1, 1/2) + \sqrt{2/3}\, \mathcal{Z}_{1/2}^{1/2}(1, 1/2),$$

$$\Sigma^- = \mathcal{K}_1^{-1} \mathcal{Y}_{1/2}^{-1/2} = \mathcal{Z}_{3/2}^{-3/2},$$

$$\Sigma^{\prime 0} = \mathcal{K}_1^{-1} \mathcal{Y}_{1/2}^{1/2} = (1/\sqrt{3})\, \mathcal{Z}_{3/2}^{-1/2} - \sqrt{2/3}\, \mathcal{Z}_{1/2}^{-1/2} .$$

Definite branching ratio can not be given for these processes, since two elements of the S-matrix

$$S_{3/2}(3/2, 0|1, 1/2) \quad \text{and}$$

$$S_{1/2}(1/2, 0|1, 1/2)$$

participate in this case. The coefficients of these matrix elements are given in Table III. This gives the branching ratios

Table III

processes	coefficients of $(1/3)S_{3/2}$	coefficients of $(1/3)S_{1/2}$
$\Sigma^+ \to p+\pi^0$	$\sqrt{2}$	$-\sqrt{2}$
$\Sigma^+ \to n+\pi^+$	1	2
$\Sigma^- \to n+\pi^-$	3	0
$\Sigma^{\prime 0} \to p+\pi^-$	1	2
$\Sigma^{\prime 0} \to n+\pi^0$	$\sqrt{2}$	$-\sqrt{2}$

$$\begin{aligned} &P(\Sigma^+ \to p+\pi^0) : P(\Sigma^+ \to n+\pi^+) : P(\Sigma^- \to n+\pi^-) \\ &= [\,(2/9)\,|S_{3/2}|^2 + (2/9)\,|S_{1/2}|^2 - (4/9)\,\mathrm{Re}\, S_{3/2}^* S_{1/2}\,] \\ &\quad : [\,(1/9)\,|S_{3/2}|^2 + (4/9)\,|S_{1/2}|^2 + (4/9)\,\mathrm{Re}\, S_{3/2}^* S_{1/2}\,] \\ &\quad : |S_{3/2}|^2, \end{aligned} \tag{3·3}$$

and

$$\begin{aligned} &P(\Sigma^{\prime 0} \to n+\pi^0) : P(\Sigma^{\prime 0} \to p+\pi^-) \\ &= P(\Sigma^{\prime +} \to p+\pi^0) : P(\Sigma^{\prime +} \to n+\pi^+). \end{aligned} \tag{3·4}$$

Some informations concerning the lifetimes of Σ^+ and Σ^- are given from (3·3). For example,

$$P(\Sigma^- \to n+\pi^-) \leqq 3\,[P(\Sigma^+ \to p+\pi^0) + P(\Sigma^+ \to n+\pi^+)\,],$$

or

$$\tau_{\Sigma^-} \geqq (1/3)\,\tau_{\Sigma^+} . \tag{3·5}$$

Also we get

$$P(\Sigma^+ \to p+\pi^0) \leqq P(\Sigma^+ \to n+\pi^+) + P(\Sigma^- \to n+\pi^-). \tag{3·6}$$

(iii) *The decay of* $\Xi^\pm$ *and* Ξ^0.

They decay by either of the following modes

$$\Xi^+ \to \begin{cases} \Lambda^+ + \pi^0 \\ \Lambda^0 + \pi^+, \end{cases}$$

$$\Xi^- \to \Lambda^0 + \pi^-,$$

and

$$\Xi^0 \to \begin{cases} \Lambda^+ + \pi^- \\ \Lambda^0 + \pi^0. \end{cases}$$

Initial and final states are given as follows.

initial states :

$$\Xi^+ = \mathcal{K}_{1/2}^{-1/2}\mathcal{Y}_1^1 = (1/\sqrt{3})\,\mathcal{Z}_{3/2}^{1/2}(1/2, 1) - \sqrt{2/3}\,\mathcal{Z}_{1/2}^{1/2}(1/2, 1),$$

$$\Xi^- = \mathcal{K}_{1/2}^{-1/2}\mathcal{Y}_1^{-1} = \mathcal{Z}_{3/2}^{-3/2},$$

$$\Xi^0 = \mathcal{K}_{1/2}^{1/2}\mathcal{Y}_1^{-1} = (1/\sqrt{3})\,\mathcal{Z}_{3/2}^{-1/2} + \sqrt{2/3}\,\mathcal{Z}_{1/2}^{-1/2}.$$

final states :

$$\Lambda^+ + \pi^0 = \mathcal{K}_0\mathcal{K}_1^0\mathcal{Y}_{1/2}^{1/2}\mathcal{Y}_0 = \sqrt{2/3}\,\mathcal{Z}_{3/2}^{1/2}(1, 1/2) - (1/\sqrt{3})\,\mathcal{Z}_{1/2}^{1/2}(1, 1/2),$$

$$\Lambda^0 + \pi^+ = \mathcal{K}_1^1\mathcal{Y}_{1/2}^{-1/2} = (1/\sqrt{3})\,\mathcal{Z}_{3/2}^{1/2} + \sqrt{2/3}\,\mathcal{Z}_{1/2}^{1/2},$$

$$\Lambda^0 + \pi^- = \mathcal{K}_1^{-1}\mathcal{Y}_{1/2}^{-1/2} = \mathcal{Z}_{3/2}^{-3/2},$$

$$\Lambda^+ + \pi^- = \mathcal{K}_1^{-1}\mathcal{Y}_{1/2}^{1/2} = (1/\sqrt{3})\,\mathcal{Z}_{3/2}^{-1/2} - \sqrt{2/3}\,\mathcal{Z}_{1/2}^{-1/2},$$

$$\Lambda^0 + \pi^0 = \mathcal{K}_1^0\mathcal{Y}_{1/2}^{-1/2} = \sqrt{2/3}\,\mathcal{Z}_{3/2}^{-1/2} + (1/\sqrt{3})\,\mathcal{Z}_{1/2}^{-1/2}.$$

Participating elements of the S-matrix are $S_{3/2}(1, 1/2|1/2, 1)$ and $S_{1/2}(1, 1/2|1/2, 1)$, and their coefficients are listed in Table IV.

If $S_{3/2}$ and $S_{1/2}$ satisfy the relation

$$S_{3/2} = 2S_{1/2}, \tag{3·7}$$

the process $\Xi^+ \to \Lambda^0 + \pi^+$ is forbidden, and it will be difficult to detect Ξ^+ (see § 2). Moreover, if (3·7) is satisfied, the process $\Xi^0 \to \Lambda^+ + \pi^-$ is forbidden also, and Ξ^0 in scheme (a) reduces effectively to that in N.G.'s scheme. It is interesting to note that processes which are forbidden by (3·7) are both the ones with $\Delta S = 3$ from the view-point of N.G.'s scheme*.

Table IV

processes		coefficients of $(1/3)S_{3/2}$	$(1/3)S_{1/2}$
$\Xi^+ \to$	$\Lambda^+ + \pi^0$	$\sqrt{2}$	$\sqrt{2}$
	$\Lambda^0 + \pi^+$	1	-2
$\Xi^- \to$	$\Lambda^0 + \pi^-$	3	0
$\Xi^0 \to$	$\Lambda^+ + \pi^-$	1	-2
	$\Lambda^0 + \pi^0$	$\sqrt{2}$	$\sqrt{2}$

* The interaction hamiltonian consists of the part which transforms like $\sim \mathcal{K}_{1/2}{}^m{}^* \mathcal{Y}_{1/2}{}^m$ and the part $\sim \mathcal{K}_{3/2}{}^m{}^* \mathcal{Y}_{3/2}{}^m$. The transitions with $|\Delta I_3| = |\Delta J_3| = 3/2$ are caused by the latter part. Under the condition (3·7), the former part alone is allowed. In the cases of the decay of Λ, Σ and θ, it is impossible to construct the quantity which transforms like $\sim \mathcal{Y}_{3/2}{}^m$ in the interaction hamiltonian at all (see the footnote on p. 323(*)).

(iv) *The decay of* $\theta^{\pm}$ *and* θ^0.

The possible modes are

$$\theta^{\pm} \to \pi^{\pm} + \pi^0,$$

and

$$\theta^0 \to \begin{cases} \pi^+ + \pi^- \\ 2\pi^0. \end{cases}$$

In this case the final state must be symmetrized including variables in the charge space. The spin and the parity of θ must be (even, even) or (odd, odd) as it decays into two pions. Thus the final states are given by

$$\pi^+ + \pi^0 = \begin{cases} (1/\sqrt{2})[\mathcal{K}_1^1 \mathcal{K}_1^0 + \mathcal{K}_1^0 \mathcal{K}_1^1] = \mathcal{Z}_2^1(2, 0) & \text{for } (e, e), \\ (1/\sqrt{2})[\mathcal{K}_1^1 \mathcal{K}_1^0 - \mathcal{K}_1^0 \mathcal{K}_1^1] = \mathcal{Z}_1^1(1, 0) & \text{for } (o, o), \end{cases}$$

$$\pi^+ + \pi^- = \begin{cases} (1/\sqrt{3})\, \mathcal{Z}_2^0(2, 0) + \sqrt{2/3}\, \mathcal{Z}_0^0(0, 0) & \text{for } (e, e), \\ -\mathcal{Z}_1^0(1, 0) & \text{for } (o, o), \end{cases}$$

$$2\pi^0 = \begin{cases} \sqrt{2/3}\, \mathcal{Z}_2^0(2, 0) - (1/\sqrt{3})\, \mathcal{Z}_0^0(0, 0) & \text{for } (e, e), \\ 0 & \text{for } (o, o). \end{cases}$$

The initial states are given by

$$\theta^+ = \mathcal{K}_{1/2}^{1/2} \mathcal{Y}_{1/2}^{1/2} = \mathcal{Z}_1^1(1/2, 1/2),$$

$$\theta^0 = \mathcal{K}_{1/2}^{-1/2} \mathcal{Y}_{1/2}^{1/2} = (1/\sqrt{2})[\mathcal{Z}_1^0(1/2, 1/2) - \mathcal{Z}_0^0(1/2, 1/2)],$$

$$\bar{\theta}^0 = \mathcal{K}_{1/2}^{1/2} \mathcal{Y}_{1/2}^{-1/2} = (1/\sqrt{2})[\mathcal{Z}_1^0 + \mathcal{Z}_0^0],$$

$$\theta^- = \mathcal{K}_{1/2}^{-1/2} \mathcal{Y}_{1/2}^{-1/2} = \mathcal{Z}_1^{-1}.$$

Or, introducing θ_1^0 and θ_2^0 according to

$$\theta^0 = (1/\sqrt{2})\,(\theta_1^0 + i\theta_2^0),$$

$$\bar{\theta}^0 = (1/\sqrt{2})\,(\theta_1^0 - i\theta_2^0),$$

we find

$$\theta_1^0 = \mathcal{Z}_1^0,$$

$$\theta_2^0 = i\,\mathcal{Z}_0.$$

Also in this case two matrix elements

$$S_1(1, 0|1/2, 1/2) \quad \text{and} \quad S_0(0, 0|1/2, 1/2)$$

Table V

processes	(even, even) coefficient of S_0	(odd, odd) coefficient of S_1
$\theta^{\pm} \to \pi^{\pm}+\pi^0$	0	1
$\theta_1^0 \to \pi^+ + \pi^-$	0	-1
$\theta_1^0 \to 2\pi^0$	0	0
$\theta_2^0 \to \pi^+ + \pi^-$	$i\sqrt{2/3}$	0
$\theta_2^0 \to 2\pi^0$	$-i/\sqrt{3}$	0

participate. But they are associated with θ(odd, odd) and θ(even, even) separately. Their coefficients are listed in Table V.

If θ is assumed as (o, o), we get

$$\tau_{\theta\pm} = \tau_{\theta 1}. \qquad (3\cdot 8)$$

Recent observations show, however, that $\tau_{\theta\pm}$ is about one hundred times longer than $\tau_{\theta 0}$.[11] This is in contradiction with (3·8), and to assume that θ is (odd, odd) seems to be inadmissible.

If θ is assumed as (even, even), $\theta^{\pm} \to \pi^{\pm} + \pi^0$ is forbidden, and we get as the branching ratio for the decay of θ

$$P(\theta^0 \to \pi^+ + \pi^-) : P(\theta^0 \to 2\pi^0) = 2 : 1 \qquad (3\cdot 9)$$

The decay of $\theta^{\pm}$ may be ascribed to "non" charge independent interactions which can appear, for example, from the mass difference of $\pi^{\pm}$ and π^0 which has been neglected thus far.

Summarizing, the main result of our discussion is*

(i) the branching ratio of Λ^0 decay is given by

$$(\Lambda^0 \to p + \pi^-) / (\Lambda^0 \to n + \pi^0) = 2,$$

(ii) the lifetimes of $\Sigma^{\pm}$ satisfy

$$\tau_{\Sigma^+} \leqq 3\tau_{\Sigma^-},$$

(iii) and it is likely that θ is (even, even), and in this case the branching ratio of θ^0 decay is given by

$$(\theta^0 \to \pi^+ + \pi^-) / (\theta^0 \to 2\pi^0) = 2.$$

At first sight it seems difficult to detect the modes $\Lambda^0 \to n + \pi^0$ or $\theta^0 \to 2\pi^0$. However, if we assume that these particles are always produced in pairs, it can be done by simply counting the numbers of actual pair productions and apparent single productions.** Therefore these predictions can be compared with experiments.

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* These results agree with those recently obtained by many authors under the assumption that for decay processes a new selection rule $|\Delta I| = 1/2$ holds.[12] The reason of the agreement is obvious since in cases here considered $|\Delta J| = 1/2$ and the conservation of $\boldsymbol{I}+\boldsymbol{J}$ leads just to $|\Delta I| = 1/2$.

** This suggestion was made by Wentzel[13] at first. Also it seems of interest to seek for the hyperfragments of very small A which decay with the emission of π^0.

§ 4. Concluding remarks

Thus far we have shown a possibility of generalizing the hypothesis of charge independence to the weak interaction. It is expected that such a generalized scheme can be applied to leptons also. If, in this case, the lepton number is not conserved, $\boldsymbol{I}$, $\boldsymbol{J}$ or even $\boldsymbol{I}+\boldsymbol{J}$ is not conserved, and this will distinguish the situation from that encountered in the case of baryons and heavy mesons.

In conclusions, a few comments will be added.

(i) We have assumed that the particle number was expressed by K_3. This means that for states with given particle number there can be many states corresponding to different values of the magnitnde of $\boldsymbol{K}$. Consider, for example, a system composed of a proton and an anti-proton. Its particle number is zero. But according to our scheme, there can be two states corresponding to $K=1$ and 0 respectively. The latter can annihilate by emitting pions, but for the former this is impossible, and it can annihilate only by emitting γ-rays. Such a prediction may also be checked experimentally.

(ii) Recently Yang and Mills[14] pointed out that the locality of the field concept cannot be made compatible with the hypothesis of charge independence so long as one does not introduce a new field which they named b-field. In our theory, however, the rotation in the charge space is identified to the gauge transform. Therefore these two requirements can be made compatible with each other without introducing the b-field.

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Effect of Hard Cores on the Binding Energies of H^3 and He^3, II

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Theoretically calculated binding energies of H^3 and He^3 were reported in our preceding paper, I, where two-body forces of exponential type with hard core cut-offs were assumed. In the present paper we show the results of similar calculations with two-body forces of the Yukawa type which are cut off by hard cores. Only central forces are used and the force parameters are determined from the two-nucleon data at low energies as was done in the preceding paper. This change in the radial form of the nuclear force does not alter the qualitative features of the effects of hard cores: i.e. 1) Hard cores reduce the binding energies of H^3 and He^3. 2) Hard cores push out the wave functions of H^3 and He^3, so that the Coulomb energy of He^3, which is too large without the hard cores, is brought into agreement with the experimental value of the difference between the binding energies of H^3 and He^3.

In the preceding paper[1] (referred to as I hereafter) we calculated the binding energies of H^3 and He^3 assuming two-body central forces with hard cores. The radial shape outside the hard core was taken as exponential. In order to see how much the results depend on the outer shape of the well, it seemed desirable to repeat the calculations with another potential.

In the present paper we assume a *Yukawa type* potential with hard core cut-off. Otherwise the same assumptions, the same notation and the same method of calculations are used as in I. Therefore, we present only numerical results with brief comments.

As for the nuclear force we assume two-body central forces only, which can be derived from the following potential,

$$V_{ij}=(3+\sigma_i\sigma_j)V_t(r_{ij})/4+(1-\sigma_i\sigma_j)V_s(r_{ij})/4\,.$$

Charge independence of the nuclear forces is assumed, and the spin-triplet and spin-singlet potentials, $V_t(r)$ and $V_s(r)$, may include Majorana exchange operators. For the even state we take $V_i(r)$ $(i=t, s)$ as

$$V_i^{(\mathrm{even})}(r)=\begin{cases}-B_i e^{-\beta_i(r-D)}/r & : r>D\\ \infty & : r<D\,.\end{cases} \tag{1}$$

The four parameters B_i and β_i are determined so as to give experimental two-nucleon data at low energies:

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Binding energy of deuteron $=2.226$ Mev,

Triplet scattering length of n-p system $=0.5378\times10^{-12}$ cm,

Singlet scattering length of n-p system $=-2.369\times10^{-12}$ cm,

Singlet effective range of n-p system $=(0.27\times10^{-12}$ cm or 0.24×10^{-12} cm).

Since the singlet effective range of n-p system is not yet determined accurately, we assume two values for it. As for the radius of the hard core, we take $D=0$, 0.2, and 0.6×10^{-13} cm. B_t and β_t thus calculated are listed in Table I. The accuracy of these values is somewhat inferior to those in I. Especially the adjustment to the singlet effective range contains errors within 5%.

Table I. Force parameters in (1)
D: in 10^{-13} cm, B: in Mev-10^{-13} cm, β: in 10^{12} cm^{-1}

	D	0	0.2	0.6
	B_t	77.6	134.5	664.0
	β_t	8.14	10.84	28.43
	B_s	55.9	91.2	309.8
$r_{os}\fallingdotseq0.27\times10^{-12}$ cm	β_s	8.54	10.50	19.36
	B_s	66.4	108.0	459.3
$r_{os}\fallingdotseq0.24\times10^{-12}$ cm	β_s	9.53	12.03	25.29

We use the usual variational method to calculate the binding energy of H^3, and take the same trial function $\psi^{(t)}$ as in I,

$$\psi^{(t)}=\psi_s^{(t)}[\alpha(1)\beta(2)-\alpha(2)\beta(1)]\alpha(3)/\sqrt{2},$$

where

$$\psi_s^{(t)}=\begin{cases}\prod_i[e^{-\mu(r_i-D)}-e^{-\nu(r_i-D)}] & : \text{all } r_i>D\\ 0 & : \text{otherwise,}\end{cases}$$

$$(i=1,\ 2,\ 3).$$

The binding energy and the values of parameters μ and ν determined with variational method are given in Table II and III where their values for the exponential type potential are also listed for comparison. All the values for the Yukawa type potential are considered

Table II. Variationally computed binding energy of H^3. Values in parentheses are those for the exponential type potential in I

Hard core radius D in 10^{-13} cm	B. E. (H^3) in Mev	
	$r_{os}\fallingdotseq2.7\times10^{-13}$ cm	$r_{os}\fallingdotseq2.4\times10^{-13}$ cm
0.0	12.5 (10.26)	16.0 (11.38)
0.2	8.2 (7.87)	8.9 (8.88)
0.6	5.4 (4.78)	6.2 (6.19)
Experimental value	8.49	

Table III. Adjusted values of μ and ν in $10^{13}\,\mathrm{cm}^{-1}$. Values in parentheses are those for the exponential type potential in I

D in 10^{-13} cm	$r_{os} \fallingdotseq 2.7\times10^{-13}$ cm		$r_{os} \fallingdotseq 2.4\times10^{-13}$ cm	
	μ	ν	μ	ν
0.0	0.61 (0.4787)	∞ (∞)	0.66 (0.5033)	∞ (∞)
0.2	0.43 (0.4444)	5.9 (5.004)	0.47 (0.4672)	6.2 (5.128)
0.6	0.44 (0.4385)	4.3 (4.279)	0.48 (0.4846)	4.4 (4.407)

to be about one order of magnitude less precise than those for the exponential type potential.

Since He^3 is the mirror nucleus of H^3 and we are assuming charge independence of nuclear forces, we may calculate the Coulomb energy of He^3 by perturbation method. The results are given in Table IV.

Table IV. Coulomb energy of He^3. Values in parentheses are those for the exponential type potential in I

Hard core radius D in 10^{-13} cm	Coulomb energy of He^3 in Mev	
	$r_{os} \fallingdotseq 2.7\times10^{-13}$ cm	$r_{os} \fallingdotseq 2.4\times10^{-13}$ cm
0.0	1.25 (0.986)	1.36 (1.037)
0.2	0.80 (0.810)	0.86 (0.845)
0.6	0.68 (0.676)	0.72 (0.723)
Experimental value	0.764	

We can see in Tables II, III and IV that the general features of the effects of hard cores are not much different in the two cases of exponential and Yukawa potentials. Namely, 1) hard cores reduce the binding energy of H^3, and 2) hard cores push out the wave functions of H^3 and He^3, so that the Coulomb energy of He^3, which is too large without the hard cores, is brought into agreement with the experimental value of the difference between the binding energies of H^3 and He^3. The hard core effects are slightly enhanced for the Yukawa potential, especially near $D=0$, possibly because of the rapid change of the potential form in this region.

Discussions in I about the approximations are applied also to the present case, and so the values of B. E. (H^3) are expected to be about 1 Mev larger than the values in Table II while the Coulomb energies may be several per cent smaller than the values in Table IV. If we take the hard core radius D as 0.2×10^{-13} cm for $r_{os}=2.7\times10^{-13}$ cm or $D=0.4\times10^{-13}$ cm for $r_{os}=2.4\times10^{-13}$ cm, the best fit to the experimental values is obtained.

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Appendix 1. Explicit formula for potential energy $U_i (i=t, s)$

Expressions for the normalization integral N and the kinetic energy are the same as those in I. The only difference is the potential energy, which can be written as

$$\begin{aligned} NU_i &= -(3/2)B_i \iiint \psi_s^{(i)*} \psi_s^{(i)} e^{-\beta_i (r_1 - D)} r_2 r_3 dr_1 dr_2 dr_3 \\ &= -(3/2) e^{-\beta_i D} B_i [\{DA_1(0) + A_2(0)\}^2 A_1(\beta_i) \\ &\quad - U_i''(2\mu) + 2U_i''(\mu+\nu) - U_i''(2\nu)], \end{aligned}$$

where

$$A_n(x) = -\frac{1}{(x+2\mu)^n} - \frac{2}{(x+\mu+\nu)^n} + \frac{1}{(x+2\nu)^n},$$

and

$$\begin{aligned} U_i''(x) &= 2[\{A_2(x+\beta_i) + (2D+1/x)A_1(x+\beta_i)\}\{A_2(x) + DA_1(x)\} \\ &\quad + A_1(x+\beta_i)\{2A_3(x) + DA_2(x)\}]e^{-xD}/x \\ &\quad + \{A_2(x+\beta_i) + DA_1(x+\beta_i)\}^2 e^{-(x+\beta_i)D}/(x+\beta_i). \end{aligned}$$

Appendix 2. Details of the results of calculation

In order to get the minimum of the expectation value of energy, we have used the quadratic function approximation as in I. In the following tables we give the detailed values which have appeared in the course of our calculation. The values with asterisks are used to obtain Tables II, III and IV. Table V is expected to be useful for calculations using somewhat different force parameters from ours.

Table Va. No hard core. ($\nu=\infty$)

μ in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev, $r_{0s}=2.7\times10^{-13}$ cm	$-U_s$ in Mev, $r_{0s}=2.4\times10^{-13}$ cm	K in Mev	B. E. (H^3) in Mev, $r_{0s}=2.7\times10^{-13}$ cm	B. E. (H^3) in Mev, $r_{0s}=2.4\times10^{-13}$ cm
0.55	38.92	27.04	29.56	53.75	*12.22	14.73
0.6	45.08	31.38	34.48	63.97	*12.49	*15.58
0.65	51.44	35.89	39.58	75.07	*12.26	*15.96
0.7	57.99	40.52	44.86	87.07	11.45	*15.79

Table Vb. Hard core radius $D=0.2\times10^{-13}$ cm

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N
			$r_{os}\fallingdotseq 2.7\times10^{-13}$ cm	$r_{os}\fallingdotseq 2.4\times10^{-13}$ cm		$r_{os}\fallingdotseq 2.7\times10^{-13}$ cm	$r_{os}\fallingdotseq 2.4\times10^{-13}$ cm	
0.55	4.5	41.3	28.8	29.9	64.0	6.1	7.1	0.005945
0.5	6.5	40.5	28.3	29.4	61.4	*7.3	8.5	0.011327
0.5	5.5	38.5	26.9	27.9	57.7	*7.7	*8.7	0.010975
0.5	5	37.2	26.0	26.9	55.6	7.6	*8.5	0.010723
0.5	4.5	35.7	25.0	25.8	53.3	*7.3	8.1	0.010393
0.45	6	33.2	23.2	24.0	48.3	8.1	*8.9	0.020312
0.45	5.5	32.3	22.7	23.3	46.9	*8.1	*8.7	0.020033
0.45	5	31.3	21.9	22.5	45.4	7.9	*8.5	0.019672
0.45	4.5	30.1	21.1	21.6	43.7	*7.5	8.0	0.019194
0.45	4	28.7	20.1	20.5	41.9	6.9	7.4	0.018546
0.4	4.5	24.7	17.3	17.6	34.9	*7.1	*7.4	0.037981

Table Vc. Hard core radius $D=0.6\times10^{-13}$ cm

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N
			$r_{os}\fallingdotseq 2.7\times10^{-13}$ cm	$r_{os}\fallingdotseq 2.4\times10^{-13}$ cm		$r_{os}\fallingdotseq 2.7\times10^{-13}$ cm	$r_{os}\fallingdotseq 2.4\times10^{-13}$ cm	
0.5	4.5	51.1	40.5	42.0	86.9	*4.7	*6.2	0.025025
0.5	4	46.7	37.7	38.6	79.3	*5.0	*5.9	0.023497
0.5	3.5	42.0	34.6	34.9	72.0	*4.6	*4.9	0.021423
0.5	3	36.4	30.8	30.3	63.8	3.4	2.9	0.018909
0.45	4.5	42.6	34.2	35.1	71.6	*5.3	*6.1	0.04371
0.4	5.5	39.2	31.2	32.2	66.0	4.4	5.5	0.08576
0.4	5	36.9	29.7	30.4	61.6	5.0	5.7	0.08382
0.4	4.5	34.3	28.1	28.4	57.2	*5.2	*5.6	0.08132
0.4	4	31.4	26.2	26.3	52.7	*4.9	*5.0	0.07815
0.35	4.5	26.5	22.1	22.1	44.0	4.6	4.6	0.16429

Reference

1) T. Kikuta, M. Morita and M. Yamada, Prog. Theor. Phys. **15** (1956), 222.

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On the Origin of the Magnetic Anisotropy Energy of Ferrites

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Possible sources of the anisotropy energy are investigated for ferromagnetic ferrites, i.e., Ni-, Co-, Fe- and Mn-ferrites. The cubic anisotropy energy which arises from the anisotropy of the crystalline field acting on magnetic ions and is described by the function of the spin operator of one ion identically vanishes for ions with spin less than 2, so that this kind of anisotropy energy vanishes for Ni^{2+} and Co^{2+} ions. The anisotropy energy arising from the magnetic dipole-dipole interaction which appears for cubic crystals in its second order perturbation was calculated for Mn- and Ni-ferrites and that arising from the anisotropic exchange interaction calculated for Ni-ferrite but they were found to be too small to account for the experimental values. Therefore, it is concluded that the major part of the anisotropy energy for Ni-ferrite arises from the anisotropy energy of Fe^{2+} ions, for magnetite from the anisotropy energy of Fe^{2+} and Fe^{3+} ions, and for Mn-ferrite from the anisotropy energy of Fe^{3+} and Mn^{2+} ions. Especially it is shown that the experimental anisotropy energy of magnetite extrapolated to the absolute zero of temperature is in good agreement with the sum of the experimental anisotropy energy of Ni-ferrite and the calculated value of the anisotropy energy of Fe^{2+} ions. For Co-ferrite, its large anisotropy energy seems to come from the pseudo-quadrupole and anisotropic exchange interactions among Co and Fe ions, but the situation is too complex to carry out the calculation of the anisotropy energy in this case.

Finally, the temperature dependence of the cubic anisotropy energy is calculated for Mn-ferrite, and is shown to decrease as $(T_c-T)^2$ near the Curie temperature. Further, the sum of the coefficient, a of the cubic anisotropy Hamiltonian for Mn^{2+} and Fe^{3+} ions of the octahedral sites and that for Fe^{3+} ion of the tetrahedral sites have been determined for Mn-ferrite by adjusting the calculated anisotropy energy *vs.* temperature curve so as to fit the experimental one, and it is found, neglecting the small a-value of Mn^{2+} ions, that the a-value of Fe^{3+} ions on octahedral sites has five times as large an absolute value as that of Fe^{3+} ions on tetrahedral sites and that the former has the opposite sign to the latter.

§ 1. Introduction

Numerous experimental investigations have been done on the magnetic anisotropy energy of ferromagnetic ferrites with the static torque method, magnetization curve method and microwave resonance method.[1-19] Experimental data are summarized in the table of the paper by Bozorth, Tilden and Williams.[17] As seen from this table, Co-ferrite has much larger anisotropy energy compared with other ferromagnetic ferrites, namely Ni, Fe- and Mn-ferrites, and further the anisotropy constant K_1 for the latter ferrites is negative and their easy axis of magnetization lies in [111] direction, except for magnetite at low temperatures, whereas K_1 of Co-ferrite is positive and its easy axis is in the [100] direction. The anisotropy energy of ferrites is quite significantly influenced by a small added quantity of Co ions. Especially, Bickford, Pappis and Stull[10] showed that a small quantity

of cobalt ions mixed in magnetite can shift the temperature at which the easy axis of magnetization changes from [111] to [100] to a markedly high temperature. Bozorth, Beatrice, Cetlin, Galt, Merritt and Yager[19] have reported that, for Ni-ferrite with Fe^{2+} ions as impurities, the microwave method and the static method of measuring anisotropy energy give somewhat different values. Therefore, in order to know the intrinsic value of the anisotropy energy, experimental data for pure substances are needed.

The anisotropy energy of cubic crystal is in general small compared with that of crystal with low symmetry and, at the same time, its temperature dependence is very intense. Thus, the room temperature value can be very different from that at the absolute zero of temperature, especially in Mn-ferrite and Mn-Zn ferrite which have a comparatively low Curie temperature.

The following table shows the anisotropy constant K_1 of ferromagnetic ferrites at low temperature. The anisotropy constant of magnetite in this table is the value at the

Table I

Substance	K_1 erg/cc	K_1 cm^{-1}/molecule.	Reference
$NiFe_2O_4$	-1.18×10^5	-4.3×10^{-2}	(18)
$CoFe_2O_4$	$4.4\sim17.5\times10^6$	$1.6\sim6.4$	(13)
Fe_3O_4	-2.0×10^5	-7.4×10^{-2}	(11)
$MnFe_2O_4$	-2.2×10^5	-8.5×10^{-2}	(2)

absolute zero of temperature obtained by extrapolating the values at temperatures higher than the temperature at which the anisotropy constant changes its sign. The purpose of this paper is to investigate the sources of the anisotropy energy of ferrites and to clarify its nature.

The magnetic ions in ferrites are Fe^{3+}, Mn^{2+}, Fe^{2+}, Co^{2+} and Ni^{2+}. Fe^{3+} and Mn^{2+} ions have no orbital moment, while Fe^{2+}, Co^{2+} and Ni^{2+} have non zero orbital moment in their free states. The latter divalent ions occupy the octahedral sites. The crystalline field acting on the octahedral site arises from the surrounding oxygen ions and metal ions. The part from the surrounding oxygen ions has a cubic symmetry neglecting a slight shift of the position of the oxygen ions, whereas the part from the metal ions has a trigonal symmetry whose axis lies in either of the four {111} directions, so far as we do not distinguish the difference between Fe^{2+} and divalent ions. If we discriminate this difference, this part has a lower symmetry and differs from ion to ion. Therefore, we can consider that the orbital moment of these divalent ions is quenched, namely the ground orbital state is not degenerate, so that the treatment of the anisotropy energy becomes considerably simple though the crystals under consideration have cubic symmetry. This fact has been proved by that g-values near 2 were found by the microwave resonance absorption.

The sources of the anisotropy energy of ferrites are the following interactions: the magnetic dipole-dipole interaction, the effective anisotropy Hamiltonian of each spin in

the crystalline field, and the anisotropic exchange interaction which is produced by the combined action of the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling and the exchange interaction. These were pointed out by Keffer[20] and Moriya and Yosida[21] in the case of antiferromagnetism. In the case of cubic crystals like ferrites, the magnetic dipole-dipole interaction does not give any cubic anisotropy energy in the classical treatment, but its higher order effect contributes to the cubic anisotropy as shown by Van Vleck.[22] For the anisotropy energy arising from the other sources, the situations are the same and higher order calculations are needed, so that the treatment becomes somewhat troublesome compared with the case of lower symmetry. We shall begin with the consideration of the simplest part of the anisotropy energy, namely the anisotropy energy coming from the anisotropy of the crystalline field.

§ 2. Anisotropy spin Hamiltonian for ions under the action of the crystalline field

The cubic spin Hamiltonian of ions whose orbital moment is quenched by the crystalline field is represented by

$$\frac{a}{6}(S_x^4+S_y^4+S_z^4), \tag{2·1}$$

where S_x, S_y and S_z are the x-, y- and z-components of the spin angular momentum $\mathbf{S}$ (x, y and z are the cubic principal axes). The constant a may be calculated by the perturbation method starting from the states which are split by the crystalline field and taking the spin-orbit coupling and intra spin-spin interaction as perturbations[23),24)]. We shall here deal with the calculation of the average value of (2·1) in the case in which the Weiss molecular field is acting in a certain direction, ζ, whose direction cosines referred to the cubic axes will be denoted by (n_1, n_2 and n_3).

If we take ξ- and η-axes perpendicular to the ζ-axis and also to each other and denote their direction cosines referred to the cubic axes by (l_1, l_2, l_3) and (m_1, m_2, m_3), respectively, S_x, S_y and S_z can be expressed in terms of S_ξ, S_η and S_ζ by the following relations:

$$\begin{aligned} S_x &= l_1 S_\xi + m_1 S_\eta + n_1 S_\zeta \\ S_y &= l_2 S_\xi + m_2 S_\eta + n_2 S_\zeta \\ S_z &= l_3 S_\xi + m_3 S_\eta + n_3 S_\zeta . \end{aligned} \tag{2·2}$$

With the use of (2·2) the average value of S_x^4 can be expressed as

$$\begin{aligned} \langle S_x^4\rangle = {} & (l_1^4+m_1^4)\langle S_\xi^4\rangle + n_1^4\langle S_\zeta^4\rangle \\ & + l_1^2 m_1^2\{\langle\{S_\xi S_\eta\}^2\rangle + \langle\{S_\xi^2 S_\eta^2\}\rangle\} \\ & + (l_1^2+m_1^2)n_1^2\{\langle\{S_\xi S_\zeta\}^2\rangle + \langle\{S_\xi^2 S_\eta^2\}\rangle\}, \end{aligned} \tag{2·3}$$

where $\{AB\}$ is the abbreviation of $AB+BA$ and we have taken into account that the averages of the terms including odd powers of S_ξ and S_η vanish and that $\langle S_\xi^4\rangle$ and

$\langle S_\xi^2 S_\zeta^2\rangle$, etc., are respectively equal to $\langle S_\eta^4\rangle$ and $\langle S_\eta^2 S_\zeta^2\rangle$, etc. Adding to (2·3) the corresponding expressions for $\langle S_y^4\rangle$ and $\langle S_z^4\rangle$, we obtain

$$\begin{aligned}\langle S_x^4+S_y^4+S_z^4\rangle=&\langle S_\xi^4\rangle+\langle\{S_\xi S_\zeta\}^2\rangle+\langle\{S_\xi^2 S_\zeta^2\}\rangle\\&+(l_1^2m_1^2+l_2^2m_2^2+l_3^2m_3^2)\{\langle\{S_\xi S_\eta\}^2\rangle+\langle\{S_\xi^2 S_\eta^2\}\rangle-2\langle S_\xi^4\rangle\}\\&+(n_1^4+n_2^4+n_3^4)\{\langle S_\zeta^4\rangle+\langle S_\xi^4\rangle-\langle\{S_\xi S_\zeta\}^2\rangle-\langle\{S_\xi^2 S_\zeta^2\}\rangle\}.\end{aligned}\tag{2·4}$$

If we use the relations

$$\langle\{S_\xi S_\eta\}^2\rangle=2\langle\{S_\xi^2 S_\eta^2\}\rangle+5\langle S_\zeta^2\rangle-2S(S+1),$$
$$\langle\{S_\xi S_\zeta\}^2\rangle=2\langle\{S_\xi^2 S_\zeta^2\}\rangle-(5/2)\langle S_\zeta^2\rangle+(1/2)S(S+1),$$

in (2·4), we obtain

$$\begin{aligned}\langle S_x^4+S_y^4+S_z^4\rangle=&\langle S_\xi^4\rangle+3\langle\{S_\xi^2 S_\zeta^2\}\rangle+(1/2)S(S+1)-(5/2)\langle S_\zeta^2\rangle\\&+(l_1^2m_1^2+l_2^2m_2^2+l_3^2m_3^2)\{3\langle\{S_\xi^2 S_\eta^2\}\rangle-2S(S+1)+5\langle S_\zeta^2\rangle-2\langle S_\xi^4\rangle\}\\&+(n_1^4+n_2^4+n_3^4)\{\langle S_\zeta^4\rangle+\langle S_\xi^4\rangle-3\langle\{S_\xi^2 S_\zeta^2\}\rangle-(1/2)S(S+1)+(5/2)\langle S_\zeta^2\rangle\},\end{aligned}\tag{2·5}$$

Here we will express $\langle\{S_\xi^2 S_\eta^2\}\rangle$, $\langle\{S_\xi^2 S_\zeta^2\}\rangle$ and $\langle S_\xi^4\rangle$ in terms of $\langle S_\zeta^4\rangle$ and $\langle S_\zeta^2\rangle$. There are following two relations between these three quantities, namely

$$2\langle S_\xi^4\rangle+\langle S_\zeta^4\rangle+2\langle\{S_\xi^2 S_\zeta^2\}\rangle+\langle\{S_\xi^2 S_\eta^2\}\rangle=S^2(S+1)^2,\tag{2·6}$$

$$2\langle S_\xi^4\rangle+\langle\{S_\xi^2 S_\eta^2\}\rangle+\langle\{S_\xi^2 S_\zeta^2\}\rangle=S^2(S+1)^2-S(S+1)\langle S_\zeta^2\rangle.\tag{2·7}$$

The first relation is obtained by squaring the relation $S_\xi^2+S_\eta^2+S_\zeta^2=S(S+1)$ and the second by adding two expressions which are obtained by multiplying this relation with S_ξ^2 from the right and from the left. $\langle S_\xi^4\rangle$ can directly be calculated as

$$\begin{aligned}\langle S_\xi^4\rangle=&(3/8)\langle S_\zeta^4\rangle-(3/4)S(S+1)\langle S_\zeta^2\rangle+(3/8)S^2(S+1)^2\\&-(1/4)S(S+1)+(5/8)\langle S_\zeta^2\rangle.\end{aligned}\tag{2·8}$$

From (2·6), (2·7) and (2·8) we see that the coefficient of $(l_1^2m_1^2+l_2^2m_2^2+l_3^2m_3^2)$ in (2·5) vanishes and (2·5) can be expressed completely in terms of $\langle S_\zeta^4\rangle$ and $\langle S_\zeta^2\rangle$ as

$$\begin{aligned}\langle S_x^4+S_y^4+S_z^4\rangle=&-(21/8)\langle S_\zeta^4\rangle+(9/4)S(S+1)\langle S_\zeta^2\rangle+(3/8)S^2(S+1)^2\\&-(15/8)\langle S_\zeta^2\rangle+(1/4)S(S+1)\\&+(n_1^4+n_2^4+n_3^4)[(35/8)\{\langle S_\zeta^4\rangle-(1/5)[S^2(S+1)^2-(1/3)S(S+1)]\}\\&-(15/4)\{\langle S_\zeta^2\rangle-(1/3)S(S+1)\}\{S(S+1)-5/6\}].\end{aligned}\tag{2·9}$$

This result can also be obtained by a more elegant method, namely by adding three Keffer's expressions[25]

$$\langle r_{ij}^{-4}(\mathbf{S}_i\cdot\mathbf{r}_{ij})^4\rangle=cP_0(\theta_{ij})+dP_2(\theta_{ij})+eP_4(\theta_{ij})$$

with $\cos\theta_{ij}=n_1$, n_2 and n_3.

Above the Curie temperature $\langle S_\zeta{}^4\rangle=(1/5)\{S^2(S+1)^2-(1/3)S(S+1)\}$ and $\langle S_\zeta{}^2\rangle=(1/3)S(S+1)$ so that the part including $(n_1{}^4+n_2{}^4+n_3{}^4)$ in (2·9) vanishes. On the other hand, at the absolute zero of temperature $\langle S_\zeta{}^4\rangle=S^4$ and $\langle S_\zeta{}^2\rangle=S^2$ so that the anisotropic part becomes

$$S(S-\tfrac{1}{2})(S-1)(S-\tfrac{3}{2})(n_1{}^4+n_2{}^4+n_3{}^4). \qquad (2\cdot10)$$

This result shows that for this kind of the anisotropy energy the quantum effect is significant and it makes the anisotropy energy identically vanish for S less than 2 in the whole temperature range. This was remarked by Van Vleck[22] as early as 1937. Thus, the anisotropy energy of cubic symmetry arising from the anisotropy of the crystalline field vanishes for Ni^{2+} and Co^{2+} ions. For Fe^{2+}, with $S=2$, the quantum effect reduces the anisotropy constant to one tenth of the classical value.

§ 3. Anisotropy energy in Mn- and Ni-ferrites arising from the dipole-dipole interaction

As mentioned in the preceding section, the anisotropy energy arising from the crystalline field in ferrites remains only for Fe^{3+}, Mn^{2+} and Fe^{2+} ions. Besides this anisotropy energy we can consider those arising from the magnetic dipole-dipole interaction and the anisotropic exchange interaction.

The cubic anisotropy energy arising from the magnetic dipole-dipole interaction at the absolute zero of temperature can be calculated by the perturbation method, taking the magnetic dipole-dipole interaction as the perturbing Hamiltonian. The dipole-dipole interaction between two spins can be divided into the following four parts, as done by Bloembergen, Purcell and Pound[26]:

$$V_{ij}=g^2\mu_B{}^2r_{ij}^{-3}(A+B+C+E), \qquad (3\cdot1)$$

$$A=S_{\zeta i}S_{\zeta j}(1-3\cos^2\theta_{ij}),$$

$$B=-(1/4)\left[(S_{\xi i}-iS_{\eta i})(S_{\xi j}+iS_{\eta j})+(S_{\xi i}+iS_{\eta i})(S_{\xi j}-iS_{\eta j})\right](1-3\cos^2\theta_{ij}), \qquad (3\cdot2)$$

$$C=-(3/2)\left[(S_{\xi i}+iS_{\eta i})S_{\zeta j}+(S_{\xi j}+iS_{\eta j})S_{\zeta i}\right]\sin\theta_{ij}\cos\theta_{ij}e^{-i\varphi_{ij}}+\text{conj. comp.},$$

$$E=-(3/4)(S_{\xi i}+iS_{\eta i})(S_{\xi j}+iS_{\eta j})\sin^2\theta_{ij}e^{-2i\varphi_{ij}}+\text{conj. comp.},$$

where the ζ-axis is taken in the direction of the spontaneous magnetization. The part A has only a diagonal element in the representation which diagonalizes the spin component S_ζ of each ion, and it gives rise to the ordinary classical dipole-dipole anisotropy energy, which evidently vanishes in cubic crystals as shown by Kaplan.[27] The part B has off-diagonal elements between the ground state with absolute saturation magnetization and those excited states in which one spin of octahedral sites and one spin of tetrahedral sites are reversed. The matrix element between the ground state and the excited state in which the ζ-component of the i-th spin of the octahedral site is S_B-1 and that of the j-th spin of the tetrahedral site is $-S_A+1$ is given by

$$-(g^2\mu_B^2/2)\cdot\sqrt{S_B}\,\sqrt{S_A}\,r_{ij}^{-3}(1-3\cos^2\theta_{ij}), \qquad (3\cdot3)$$

where S_B and S_A are, respectively, the magnitudes of the spins of octahedral and tetrahedral sites. Therefore, the second order perturbed energy per one octahedral ion becomes

$$-\frac{g^4\mu_B^4 S_B S_A}{4(\Delta E_B+\Delta E_A)}\sum_{BAj}\left(\frac{r_{ij}^2-3\zeta_{ij}^2}{r_{ij}^5}\right)^2, \qquad (3\cdot4)$$

where ΔE_B is an increase of energy due to the change of the ζ-component of an octahedral spin by one and ΔE_A is the corresponding quantity for a tetrahedral spin, and $\sum_{BAj}'$ means the summation over the tetrahedral j-sites surrounding the octahedral i-site.

If we here transform the coordinate system from ξ, η and ζ to x, y and z of the cubic axes by

$$\begin{aligned}\xi&=l_1x+l_2y+l_3z,\\ \eta&=m_1x+m_2y+m_3z,\\ \zeta&=n_1x+n_2y+n_3z,\end{aligned} \qquad (3\cdot5)$$

the summand in $(3\cdot4)$ becomes

$$\begin{aligned}\left(\frac{r_{ij}^2-3\zeta_{ij}^2}{r_{ij}^5}\right)^2=&\frac{1}{r_{ij}^6}-6\left\{n_1^2\frac{x_{ij}^2}{r_{ij}^8}+n_2^2\frac{y_{ij}^2}{r_{ij}^8}+n_3^2\frac{z_{ij}^2}{r_{ij}^8}+2n_1n_3\frac{x_{ij}y_{ij}}{r_{ij}^8}\right.\\
&\left.+2n_2n_3\frac{y_{ij}z_{ij}}{r_{ij}^8}+2n_3n_1\frac{z_{ij}x_{ij}}{r_{ij}^8}\right\}+9\left\{n_1^4\frac{x_{ij}^4}{r_{ij}^{10}}+n_2^4\frac{y_{ij}^4}{r_{ij}^{10}}\right.\\
&+n_3^4\frac{z_{ij}^4}{r_{ij}^{10}}+6n_1^2n_2^2\frac{x_{ij}^2y_{ij}^2}{r_{ij}^{10}}+6n_2^2n_3^2\frac{y_{ij}^2z_{ij}^2}{r_{ij}^{10}}+6n_3^2n_1^2\frac{z_{ij}^2x_{ij}^2}{r_{ij}^{10}}\\
&+4n_1^3n_2\frac{x_{ij}^3y_{ij}}{r_{ij}^{10}}+4n_1n_2^3\frac{x_{ij}y_{ij}^3}{r_{ij}^{10}}+4n_2^3n_3\frac{y_{ij}^3z_{ij}}{r_{ij}^{10}}+4n_2n_3^3\frac{y_{ij}z_{ij}^3}{r_{ij}^{10}}\\
&+4n_3^3n_1\frac{z_{ij}^3x_{ij}}{r_{ij}^{10}}+4n_3n_1^3\frac{z_{ij}x_{ij}^3}{r_{ij}^{10}}+12n_2^2n_3n_1\frac{y_{ij}^2z_{ij}x_{ij}}{r_{ij}^{10}}\\
&\left.+12n_3^2n_1n_2\frac{z_{ij}^2x_{ij}y_{ij}}{r_{ij}^{10}}+12n_1^2n_2n_3\frac{x_{ij}^2y_{ij}z_{ij}}{r_{ij}^{10}}\right\}. \qquad (3\cdot6)\end{aligned}$$

We concentrate our attention to an octahedral site whose trigonal axis of symmetry lies in the $[111]$ direction. Then, the summation, $\sum_{ij}'x_{ij}^2/r_{ij}^5$, $\sum_{ij}'x_{ij}y_{ij}/r_{ij}^5$, $\sum_{ij}'x_{ij}^4/r_{ij}^{10}$, $\sum_{ij}'x_{ij}^2y_{ij}^2/r_{ij}^{10}$, $\sum_{ij}'x_{ij}^3y_{ij}/r_{ij}^{10}$ and $\sum_{ij}'x_{ij}^2y_{ij}z_{ij}/r_{ij}^{10}$ are unchanged by cyclic changes among x, y and z. The corresponding expressions for the other three kinds of octahedral sites can be obtained by changing the signs of two of n_1, n_2 and n_3. Thus, apart from the parts independent of the direction cosines, the average of the sum of $(3\cdot6)$ over the four kinds of octahedral sites becomes

$$9(n_1^4+n_2^4+n_3^4)\sum_{BAj}'x_{ij}^4/r_{ij}^{10}+54(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2)\sum_{BAj}'x_{ij}^2y_{ij}^2/r_{ij}^{10}. \qquad (3\cdot7)$$

With the use of (3·4) and (3·7), we obtain the anisotropy constant per molecule arising from the B interaction as

$$9\ \frac{g^4\mu_B{}^4 S_B S_A}{(\Delta E_B+\Delta E_A)}\sum_{BA_j}\left\{\frac{x_{ij}^4-3x_{ij}^2y_{ij}^2}{r_{ij}^{10}}\right\}. \tag{3·8}$$

The part E in (3·2) has off-diagonal elements between the ground state and those excited states in which two octahedral spins or two tetrahedral spins are reversed. Using similar notation to that used in the case of the B interaction, the matrix element between the ground state and the excited state in which the ζ-components of the i-th and j-th spins of the octahedral sites are equal to S_B-1, is given by

$$-(3/2)S_B g^2\mu_B{}^2 r_{ij}^{-3}\sin^2\theta_{ij}e^{2i\varphi_{ij}}. \tag{3·9}$$

Therefore, the second order energy per two octahedral spins becomes

$$-\frac{9}{4}\frac{g^4\mu_B{}^4 S_B{}^2}{2\Delta E_B}\sum_{BB_j}\left(\frac{\xi_{ij}^2+\eta_{ij}^2}{r_{ij}^5}\right)^2. \tag{3·10}$$

Expressing the summand here in terms of x, y and z, taking average over four kinds of octahedral spins, and picking out the terms including $(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2)$, we obtain the anisotropy constant per molecule as

$$\frac{9}{4}\frac{g^4\mu_B{}^4 S_B{}^2}{\Delta E_B}\sum_{BB_j}\left\{\frac{x_{ij}^4-3x_{ij}^2y_{ij}^2}{r_{ij}^{10}}\right\}. \tag{3·11}$$

Similarly, we obtain the anisotropy constant per molecule arising from the excited states in which two i and j spins of the tetrahedral sites have the ζ-components equal to $-S_A+1$ as follows:

$$\frac{9}{8}\frac{g^4\mu_B{}^4 S_A{}^2}{\Delta E_A}\sum_{AA_j}\left\{\frac{x_{ij}^4-3x_{ij}^2y_{ij}^2}{r_{ij}^{10}}\right\}. \tag{3·12}$$

The C term in (3·2) has off-diagonal elements between the ground state and those excited states in which one spin of the octahedral sites has ζ-component equal to S_B-1 or one spin of the tetrahedral sites has ζ-component equal to $-S_A+1$. The corresponding matrix elements are, respectively,

$$-\frac{3}{2}\sqrt{2S_B}g^2\mu_B{}^2\{S_B\sum_{BB_j}r_{ij}^{-5}\zeta_{ij}(\xi_{ij}+i\eta_{ij})-S_A\sum_{BA_j}r_{ij}^{-5}\zeta_{ij}(\xi_{ij}+i\eta_{ij})\} \tag{3·13a}$$

and

$$-\frac{3}{2}\sqrt{2S_A}g^2\mu_B{}^2\{-S_A\sum_{AA_j}r_{ij}^{-5}\zeta_{ij}(\xi_{ij}-i\eta_{ij})+S_B\sum_{AB_j}r_{ij}^{-5}\zeta_{ij}(\xi_{ij}-i\eta_{ij})\}. \tag{3·13b}$$

The summation in (3·13a) for an octahedral spin whose trigonal axis is in the direction [111] can be written as

$$\left(S_B\sum_{BB_j}\frac{\zeta_{ij}(\xi_{ij}+i\eta_{ij})}{r_{ij}^5}-S_A\sum_{BA_j}\frac{\zeta_{ij}(\xi_{ij}+i\eta_{ij})}{r_{ij}^5}\right)$$

$$= (l_1 n_2 + l_2 n_1 + l_2 n_3 + l_3 n_2 + l_3 n_1 + l_1 n_3)\left(S_B \sum_{\widetilde{BB}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} - S_A \sum_{\widetilde{BA}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} \right)$$

$$+ i(m_1 n_2 + m_2 n_1 + m_2 n_3 + m_3 n_2 + m_3 n_1 + m_1 n_3)\left(S_B \sum_{\widetilde{BB}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} - S_A \sum_{\widetilde{BA}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} \right). \tag{3·14}$$

Using (3·14) in the calculation of the second order energy to be obtained from (3·13a) and averaging the obtained expression with respect to the four kinds of spins, we have the following value of the anisotropy constant per two octahedral spins:

$$36 \frac{g^4 \mu_B^4 S_B}{\Delta E_B} \left\{ S_B \sum_{\widetilde{BB}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} - S_A \sum_{\widetilde{BA}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} \right\}^2. \tag{3·15}$$

Similarly, we get the anisotropy constant per molecule from (3·13b),

$$18 \frac{g^4 \mu_B^4 S_A}{\Delta E_A} \left\{ S_B \sum_{\widetilde{AB}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} - S_A \sum_{\widetilde{AA}_j} \frac{x_{ij} y_{ij}}{r_{ij}^5} \right\}^2, \tag{3·16}$$

where, however, both summations are found to vanish because of the symmetry of the crystal structure.

The values of the latter sums appearing in (3·8), (3·11), (3·12) and (3·15) are calculated numerically by the direct sum method up to eighth neighbors. In units of $a/8$, a being the lattice parameter, they are as follows:

$$\sum_{\widetilde{BA}_j} (x_{ij}^4 - 3x_{ij}^2 y_{ij}^2)/r_{ij}^{10} = 0.000350\,, \quad \sum_{\widetilde{BB}_j} (x_{ij}^4 - 3x_{ij}^2 y_{ij}^2)/r_{ij}^{10} = -0.00104$$

$$\sum_{\widetilde{AA}_j} (x_{ij}^4 - 3x_{ij}^2 y_{ij}^2)/r_{ij}^{10} = -0.000533\,, \quad \sum_{\widetilde{BB}_j} x_{ij} y_{ij}/r_{ij}^5 = 0.038\,,$$

$$\sum_{\widetilde{BA}_j} x_{ij} y_{ij}/r_{ij}^5 = -0.017\,. \tag{3·17}$$

The excitation energy ΔE_B and ΔE_A can be expressed, using Néel's[2,3] notatins, as

$$\Delta E_B = g\mu_B H_B = g\mu_B n(\beta\mu M_B + \lambda M_A), \quad \Delta E_A = g\mu_B H_A = g\mu_B n(\alpha\lambda M_A + \mu M_B), \tag{3·18}$$

where H_B and H_A are the molecular fields acting on the octahedral and tetrahedral sites and are expressible as linear combinations of the magnetizations of the A and B sites, M_A and M_B, per N ions, N being the Avogadro number.

In the case of Mn-ferrite, the constants n, α and β in (3·18) are estimated from the experimental magnetization *vs.* temperature curve in Sec. 6. Using the values of (6·13) and (6·14), $S_B = S_A = 5/2$ and $\lambda = 1/3$ and $\mu = 2/3$, the excitation energies become

$$\Delta E_B = 251\ \mathrm{cm}^{-1}, \quad \Delta E_A = 471\ \mathrm{cm}^{-1}. \tag{3·19}$$

Inserting (3·19) and (3·17) in (3·8), (3·11), (3·12) and (3·15), we obtain for Mn-ferrite the following values of the anisotropy constants per molecule coming from the B, E and C parts:

$$K_B = 5.48 \times 10^{-5}\ \mathrm{cm}^{-1},$$
$$K_E = -1.17 \times 10^{-4}\ \mathrm{cm}^{-1} - 1.60 \times 10^{-5}\ \mathrm{cm}^{-1},$$
$$K_C = 1.3 \times 10^{-2}\ \mathrm{cm}^{-1}, \quad \text{(Mn-ferrite)} \tag{3·20}$$

where the value of $a = 8.55$ Å is used. This result shows that the anisotropy constant from the dipole-dipole interaction has the opposite sign to that observed and that it amounts to about 15 per cent of the latter.

In the case of Ni-ferrite, we use the values determined by Néel[39] for n, α and β, namely

$$n = 720, \quad \alpha = -0.21 \text{ and } \beta = -0.15. \tag{3·21}$$

Then, we have the following values of ΔE_B and ΔE_A:

$$\Delta E_B = 503\ \mathrm{cm}^{-1}, \quad \Delta E_A = 745\ \mathrm{cm}^{-1}. \tag{3·22}$$

Using (3·21) and (3·17), $a = 8.36$ Å, and the mean value of 5/2 and 1 (i.e. 7/4) for S_B, we obtain the following value of the anisotropy constant for Ni-ferrite:

$$K_B = 2.56 \times 10^{-5}\ \mathrm{cm}^{-1},$$
$$K_E = -3.33 \times 10^{-5}\ \mathrm{cm}^{-1} - 1.16 \times 10^{-5}\ \mathrm{cm}^{-1},$$
$$K_C = 3.41 \times 10^{-3}\ \mathrm{cm}^{-1}. \quad \text{(Ni-ferrite)} \tag{3·23}$$

This result shows that also in Ni-ferrite the anisotropy constant from this source is about ten per cent of the observed value.

We would better use the spin wave theory for a more accurate calculation, as Keffer[25] and Tessman[26] did in discussing the anisotropy energy of metallic ferromagnets. However, in order to know the order of magnitude, the present method would suffice, though there are some ambiguities in determining the values of ΔE_A and ΔE_B, especially for Ni-ferrite.

§ 4. Anisotropy energy arising from the anisotropic exchange interaction in Ni-ferrite

Another source of the anisotropy energy in Ni-ferrite is the anisotropic exchange interaction. We shall calculate this anisotropy energy. The orbital degeneracy of Ni^{2+} ion in its free state is lifted partially by the cubic crystalline field, which is the main part of the crystalline field acting on the octahedral sites, the ground state being singlet in this case. We shall here neglect the effect of a small trigonal part of the crystalline field. Like the case of the magnetic dipole-dipole interaction we shall deal with the exchange interaction, together with the crystalline field energy, as the unperturbed energies. The perturbing Hamiltonian in this case is the spin-orbit interaction of one Ni^{2+} ion. We shall calculate the perturbed energy of the ground state for completely aligned spins.

The orbital states of Ni^{2+} ions subjected to an octahedral cubic crystalline field are split into the following three groups:

$$
\begin{array}{lll}
\Gamma_2 & \Gamma_5 & \Gamma_4 \\
 & \varphi_1=(1/\sqrt{2})(\psi_2+\psi_{-2}), & \varphi_4=\psi_0, \\
\varphi_0=(1/\sqrt{2})(\psi_2-\psi_{-2}), & \varphi_2=\sqrt{5/8}\,\psi_1-\sqrt{3/8}\,\psi_{-3}, & \varphi_5=\sqrt{3/8}\,\psi_1+\sqrt{5/8}\,\psi_{-3}, \\
 & \varphi_3=\sqrt{5/8}\,\psi_{-1}-\sqrt{3/8}\,\psi_3, & \varphi_6=\sqrt{3/8}\,\psi_{-1}+\sqrt{5/8}\,\psi_3, \\
E=0, & E=JE_1, & E=JE_2,
\end{array}
\tag{4·1}
$$

where ψ_m means the wave function of the state in which the z-component of the orbital angular momentum is m. As mentioned before, we shall neglect the effect of the trigonal field and further we shall put the effect of ^3P state out of our consideration. The matrix elements of $(\boldsymbol{L}\cdot\boldsymbol{S})$ with respect to (4·1) are calculated as

	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6
φ_0	0	$2S_z$	$\sqrt{2}\,S^-$	$-\sqrt{2}\,S^+$	0	0	0
φ_1	$2S_z$	0	$\frac{1}{2\sqrt{2}}S^-$	$\frac{1}{2\sqrt{2}}S^+$	0	$\frac{\sqrt{15}}{2\sqrt{2}}S^-$	$\frac{\sqrt{15}}{2\sqrt{2}}S^+$
φ_2	$\sqrt{2}\,S^+$	$\frac{1}{2\sqrt{2}}S^+$	$-\frac{1}{2}S_z$	0	$\frac{\sqrt{15}}{2\sqrt{2}}S^-$	$\frac{\sqrt{15}}{2}S_z$	0
φ_3	$-\sqrt{2}\,S^-$	$\frac{1}{2\sqrt{2}}S^-$	0	$\frac{1}{2}S_z$	$\frac{\sqrt{15}}{2\sqrt{2}}S^+$	0	$-\frac{\sqrt{15}}{2}S_z$
φ_4	0	0	$\frac{\sqrt{15}}{2\sqrt{2}}S^+$	$\frac{\sqrt{15}}{2\sqrt{2}}S^-$			
φ_5	0	$\frac{\sqrt{15}}{2\sqrt{2}}S^+$	$\frac{\sqrt{15}}{2}S_z$	0			
φ_6	0	$\frac{\sqrt{15}}{2\sqrt{2}}S^-$	0	$-\frac{\sqrt{15}}{2}S_z$			

(4·2)

Now we suppose that in the ground state each ion is in Γ_5' state and its spin points to a direction with direction cosines (n_1, n_2, n_3) referred to the cubic axes. Taking the spin orbit coupling as the perturbation, the anisotropy energy is given by the perturbed energy which includes the biquadratic form of the direction cosines $(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2)$. Taking the ξ- and η-axes perpendicular to the ζ-axis and to each other and denoting their direction cosines by (l_1, l_2, l_3) and (m_1, m_2, m_3), there are following relations between the spin components $S^\pm=S_x\pm iS_y$ and S_z and those referred to the ξ-, η- and ζ-axes, $S'^\pm$ and S_ζ':

$$
\begin{aligned}
S^+ &= (1/2)\{(l_1-im_1)+i(l_2-im_2)\}S'^+ + (1/2)\{(l_1+im_1)+i(l_2+im_2)\}S'^- \\
&\quad + (n_1+in_2)S_\zeta', \\
S^- &= (1/2)\{(l_1-im_1)-i(l_2-im_2)\}S'^+ + (1/2)\{(l_1+im_1)-i(l_2+im_2)\}S'^- \\
&\quad + (n_1-in_2)S_\zeta',
\end{aligned}
\tag{4·3}
$$

$$S_z=(1/2)(l_3-im_3)S'^{+}+(1/2)(l_3+im_3)S'^{-}+n_3S_\zeta'.$$

As seen from this expression, the fourth order terms with respect to n_1, n_2 and n_3 come from the fourth order perturbation of $\lambda(\boldsymbol{L}\cdot\boldsymbol{S})$. In the case in which the diagonal element of the perturbing Hamiltonian vanishes in the ground state, the fourth order perturbed energy can be expressed by

$$\begin{aligned}\Delta E_4=&\sum_{n\neq g}\frac{(g|V|n)(n|V|g)}{\Delta E_n^{\,2}}\sum_{n'}\frac{(g|V|n')(n'|V|g)}{\Delta E_{n'}}\\&-\sum_{n,n',n''\neq g}\frac{(g|V|n)(n|V|n')(n'|V|n'')(n''|V|g)}{\Delta E_n\,\Delta E_{n'}\,\Delta E_{n''}},\end{aligned}\tag{4·4}$$

where g and n, n', n'' mean the ground and excited states. The first term of this expression does not give any anisotropy because each factor of it is of the second order with respect to the direction cosines and so it is reduced to a mere constant in cubic case.

The perturbation processes which contribute to the second term in (4·4) can be divided into the following two groups as seen from (4·2):

1) Those which include only Γ_5' states as the excited states.

1.	(0−1−2−1−0)	7.	(0−1−2−2−0)
2.	(0−1−3−1−0)	8.	(0−1−3−3−0)
3.	(0−2−1−2−0)	9.	(0−2−2−1−0)
4.	(0−2−1−3−0)	10.	(0−3−3−1−0)
5.	(0−3−1−3−0)	11.	(0−2−2−2−0)
6.	(0−3−1−2−0)	12.	(0−3−3−3−0)

(4·5)

2) Those which include both Γ_5' and Γ_4' states.

1.	(0−1−5−1−0)	7.	(0−2−4−3−0)
2.	(0−1−6−1−0)	8.	(0−2−5−2−0)
3.	(0−1−5−2−0)	9.	(0−3−4−2−0)
4.	(0−1−6−3−0)	10.	(0−3−6−1−0)
5.	(0−2−4−2−0)	11.	(0−3−4−3−0)
6.	(0−2−5−1−0)	12.	(0−3−6−3−0)

(4·6)

The perturbed energy denominator for the first group is $\Delta E_1^{\,3}$, while that for the second group is $\Delta E_1^{\,2}\Delta E_2$ provided we do not take account of the change of the exchange energy. The numerator is given by the product of λ^4 and four matrix elements of $(\boldsymbol{L}\cdot\boldsymbol{S})$ corresponding to each process. The matrix elements of the numerators for the cases of (4·5) and (4·6) can be written with the use of (4·2) as

$$
\begin{aligned}
1)\quad & 1.\ (1/2)S_z S^- S^+ S_z && 7.\ -(1/2)S_z S^- S_z S^+\\
& 2.\ (1/2)S_z S^+ S^- S_z && 8.\ -(1/2)S_z S^+ S_z S^-\\
& 3.\ (1/4)S^- S^+ S^- S^+ && 9.\ -(1/2)S^- S_z S^+ S_z\\
& 4.\ -(1/4)S^- S^+ S^+ S^- && 10.\ -(1/2)S^+ S_z S^- S_z\\
& 5.\ (1/4)S^+ S^- S^+ S^- && 11.\ (1/2)S^- S_z S_z S^+\\
& 6.\ -(1/4)S^+ S^- S^- S^+ && 12.\ (1/2)S^+ S_z S_z S^-,\\
2)\quad & 1.\ (15/2)S_z S^- S^- S_z && 7.\ -(15/4)S^- S^- S^- S^-\\
& 2.\ (15/2)S_z S^+ S^- S_z && 8.\ (15/2)S^- S_z S_z S^+\\
& 3.\ (15/2)S_z S^- S_z S^+ && 9.\ -(15/4)S^+ S^+ S^+ S^+\\
& 4.\ (15/2)S_z S^+ S_z S^- && 10.\ (15/2)S^+ S_z S^- S_z\\
& 5.\ (15/4)S^- S^- S^+ S^+ && 11.\ (15/4)S^+ S^+ S^- S^-\\
& 6.\ (15/2)S^- S_z S^+ S_z && 12.\ (15/2)S^+ S_z S_z S^-.
\end{aligned}
\qquad (4\cdot 7),\ (4\cdot 8)
$$

In the ground state the spin component of Ni^{++} ion along the z-axis is S, and possible processes of the change of the spin component in the fourth order perturbation and spin operators producing those processes are limited to the following six cases:.

$$
\begin{aligned}
&\mathrm{A}.\quad (S)-(S)-(S)-(S)-(S):\ S_z' S_z' S_z' S_z'\\
&\mathrm{B}_1.\quad (S)-(S)-(S)-(S-1)-(S):\ S_z' S_z' S'^- S'^+\\
&\mathrm{B}_2.\quad (S)-(S)-(S-1)-(S)-(S):\ S_z' S'^- S'^+ S_z'\\
&\mathrm{B}_3.\quad (S)-(S-1)-(S)-(S)-(S):\ S'^- S'^+ S_z' S_z'\\
&\mathrm{C}.\quad (S)-(S-1)-(S)-(S-1)-(S):\ S'^- S'^+ S'^- S'^+\\
&\mathrm{D}.\quad (S)-(S-1)-(S-2)-(S-1)-(S):\ S'^- S'^- S'^+ S'^+
\end{aligned}
\qquad (4\cdot 9)
$$

where the quantities in the parentheses represent the z-component of the spin. Other processes have zero matrix elements in the case of spin one.

Now we shall turn to the calculation of the fourth order perturbed energy. First we express the products of four S's in (4·7) and (4·8) in terms of those of four primed S''s with the use of (4·3) and add them over twelve terms of (4·7) and (4·8). Then we find that all the coefficients of the products of four primed S''s corresponding to processes described by (4·9) for the group (1) vanish. Thus, the anisotropy energy appears from the group (2) alone. The calculated anisotropy energy arising from the second group for each of the six processes of (4·9) is as follows:

$$
\mathrm{A}.\quad -\frac{60S^4\lambda^4}{JE_1^2 JE_2}(n_1^2 n_2^2+n_2^2 n_3^2+n_3^2 n_1^2),
$$

$$
\mathrm{B}_1.\quad \frac{30S^3\lambda^4}{JE_1 JE_2(JE_1+JE_2)}(n_1^2 n_2^2+n_2^2 n_3^2+n_3^2 n_1^2),
$$

$$B_2. \quad -\frac{15\lambda^4 S^3}{\Delta E_1^2(\Delta E_2+\Delta E_x)}\{1-2(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2)\}, \tag{4·10}$$

$$B_3. \quad \frac{30\lambda^4 S^3}{(\Delta E_1+\Delta E_x)\Delta E_1\Delta E_2}(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2),$$

$$C. \quad -\frac{15\lambda^4 S^2}{(\Delta E_1+\Delta E_x)^2\Delta E_2}(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2),$$

$$D. \quad -\frac{30\lambda^4 S(S-(1/2))}{(\Delta E_1+\Delta E_x)^2(\Delta E_2+2\Delta E_x)}\{1+(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2)\},$$

where ΔE_x means the change of the exchange energy of the ion due to the change of its spin component by one in the Weiss molecular field produced by the neighboring spins. Summing up the above six terms of (4·10) and putting $S=1$, we obtain the coefficient of $(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2)$, namely the anisotropy constant per molecule, as

$$K_1=-15\lambda^4\left\{\frac{4}{\Delta E_1^2\Delta E_2}-\frac{2}{\Delta E_1\Delta E_2(\Delta E_1+\Delta E_x)}-\frac{2}{\Delta E_1^2(\Delta E_2+\Delta E_x)}\right.$$
$$-\frac{2}{(\Delta E_1+\Delta E_x)\Delta E_1\Delta E_2}+\frac{1}{(\Delta E_1+\Delta E_x)^2\Delta E_2}$$
$$\left.+\frac{1}{(\Delta E_1+\Delta E_x)^2(\Delta E_2+2\Delta E_x)}\right\}. \tag{4·11}$$

If we expand (4·11) with respect to $\Delta E_x/\Delta E_{1,2}$, we obtain

$$K_1=-30\frac{\lambda^4}{\Delta E_1^2\Delta E_2}\left(\frac{\Delta E_x}{\Delta E_1}+\frac{\Delta E_x}{\Delta E_2}\right)^2. \tag{4·12}$$

This anisotropy energy tends the spin to point to the {111} direction. It is inversely proportional to the fifth power of the energy separation ΔE in the cubic field so that it depends sensitively upon the strength of the cubic field. If we treat the exchange interaction as the perturbation together with the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling, we obtain the same result in its sixth order perturbation. The above calculation is concerned only with the excitation of one Ni ion, but it can be easily shown that processes which include the excitation of two Ni ions give only an isotropic energy up to the fourth order with respect to $\lambda(\boldsymbol{L}\cdot\boldsymbol{S})$.

The g-factor of a Ni ion becomes

$$g-2=-(8\lambda/\Delta E_1). \tag{4·13}$$

For the energy separation due to the cubic field the ratio of ΔE_1 to ΔE_2 is equal to 5/9. If we use this relation, (4·12) becomes

$$K_1=-40.3\cdot(\lambda^4/\Delta E_1^3)\cdot(\Delta E_x/\Delta E_1)^2. \tag{4·14}$$

Considering exchange interactions only between octahedral ions and tetrahedral ions, the change of the exchange energy ΔE_x at absolute zero using the molecular field approximation can be put as

$$\Delta E_x = 2Jz S_{Fe}\,, \tag{4·15}$$

where J is the exchange integral and z the number of tetrahedral sites neighboring an octahedral ion and S_{Fe} means the spin of an Fe^{3+} ion, namely $5/2$. $2Jz$ can be estimated from the Curie temperature of Ni-ferrite in the same approximation as

$$T_c = (2Jz/3k)\left[\{S_{Fe}(S_{Fe}+1)+S_{Ni}(S_{Ni}+1)\}S_{Fe}(S_{Fe}+1)\right]^{1/2}. \tag{4·16}$$

Using $T_c = 860°K$ and $S_{Ni}=1$, we obtain

$$\Delta E_x = 465\ \text{cm}^{-1}, \quad 2Jz = 186\ \text{cm}^{-1}. \tag{4·17}$$

This value of ΔE_x is less than $\Delta E_{//}$ of (3·22) by ten per cent. Inserting ΔE_x of (4·17) into (4·14) and taking the value of λ equal to $-335\ \text{cm}^{-1}$ in (4·14) and (4·13), we obtain the following table:

Table II

$\Delta E_1\ \text{cm}^{-1}$	5000	7000	10000	15000
$-K_1\ \text{cm}^{-1}$/mole.	3.5×10^{-2}	6.6×10^{-3}	1.1×10^{-3}	1.5×10^{-4}
g	2.54	2.38	2.27	2.18

Considering the experimental g-value equal to about 2.2, it would be reasonable to assume $\Delta E_1 = 1\sim1.5\times10^4\ \text{cm}^{-1}$. Then, we might conclude that the anisotropy constant arising from the anisotropic exchange interaction is less than $10^{-3}\ \text{cm}^{-1}$ per molecule.

As another source of the anisotropy energy we can consider the electric quadrupole-quadrupole interaction between two Ni ions. As seen from the above calculations, we must in this case also consider the perturbed energy quartic in $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling because the quadrupole-quadrupole interaction does not include the direction cosines of the spins. Such energies would have an order of $(\lambda/\Delta E)^4 H_Q$, where H_Q represents the quadrupole-quadrupole interaction in the free state of the ions. This part of anisotropy is estimated as $10^{-4}\ \text{cm}^{-1}$ if we put $H_Q = 10^2\ \text{cm}^{-1}$.

Thus, it becomes very probable that the anisotropy energy found experimentally in Ni-ferrite arises from Fe^{3+} ions which occupy the tetrahedral and octahedral sites. Using the value of the anisotropy constant of Ni-ferrite in Table I, we obtain the fine structure constant a of Fe^{3+} ion as

$$a^B_{Fe} + a^A_{Fe} = 1.87\times10^{-2}\ \text{cm}^{-1}, \tag{4·18}$$

where A and B mean the tetrahedral and octahedral sites respectively.

§ 5. Anisotropy energy of Fe^{2+} ions in magnetite

In magnetite, anisotropy energy arising from Fe^{2+} ions exists besides that arising from Fe^{3+} ions. In this section, we shall estimate this part of the anisotropy energy. The orbital state of Fe^{2+} ion occupying an octahedral site is considered to be split by the

large cubic part of the crystalline field. The ground state is triply degenerate. If we assume, for simplicity, that the charges of Fe^{2+} and Fe^{3+} ions on the octahedral sites are replaced by their average, the low symmetry part of the crystalline field becomes trigonal and its axis lies in one of the four {111} directions. Therefore, the Fe^{2+} ions are divided into four kinds of ions according as the direction of the axis of the field to which the ions are subjected.

We shall concentrate our attention to one Fe^{2+} ion whose trigonal axis coincides with the [111] direction. If we denote the five degenerate orbital wave functions of Fe^{2+} ion in its free state by φ_0, $\varphi_{\pm1}$ and $\varphi_{\pm2}$, where suffixes represent the z-components of the orbital moment, the cubic field splits these states into doubly degenerate states Γ_3': φ_0, $(1/\sqrt{2})(\varphi_2+\varphi_{-2})=\varphi_{2+}$ and triply degenerate states Γ_5': $(1/\sqrt{2}\,i)(\varphi_2-\varphi_{-2})=\varphi_{xy}$, $-(1/\sqrt{2}\,i)(\varphi_1+\varphi_{-1})=\varphi_{yz}$, $-(1/\sqrt{2})(\varphi_1-\varphi_{-1})=\varphi_{zx}$. The energy separation between these two groups will be denoted by ΔE. These states are further split by the trigonal field, whose expressions are, following Abragam and Pryce,[31] represented by

$$V_T=T_2(yz+zx+xy)+T_4\{(x^3y+y^3z+z^3x+x^3z+y^3x+z^3y)-6(x^2yz+y^2zx+z^2xy)\}\cdots\cdots. \tag{5·1}$$

The matrix element of this trigonal field with respect to the five cubic wave functions can be calculated as follows:

$$(V_T)=\begin{array}{c|ccccc} & \varphi_{yz} & \varphi_{zx} & \varphi_{xy} & \varphi_{2+} & \varphi_0 \\ \varphi_{yz} & 0 & a & a & -\sqrt{3}\,b & b \\ \varphi_{zx} & a & 0 & a & \sqrt{3}\,b & b \\ \varphi_{xy} & a & a & 0 & 0 & -2b \\ \varphi_{2-} & -\sqrt{3}\,b & \sqrt{3}\,b & 0 & 0 & 0 \\ \varphi_0 & b & b & -2b & 0 & 0 \end{array}, \tag{5·2}$$

where

$$a=\frac{1}{7}\langle r^2\rangle_{\mathrm{av.}}T_2-\frac{4}{21}\langle r^4\rangle_{\mathrm{av.}}T_4,$$
$$b=\frac{1}{7\sqrt{3}}\langle r^2\rangle_{\mathrm{av.}}T_2+\frac{1}{7\sqrt{3}}\langle r^4\rangle_{\mathrm{av.}}T_4. \tag{5·3}$$

The matrix (5·2) including the cubic part can be brought into the diagonal form by the following unitary transformation S:

$$S=\begin{vmatrix} \frac{1}{\sqrt{3}}, & \frac{1}{\sqrt{3}}, & -\frac{1}{\sqrt{3}}, & 0, & 0 \\ -\frac{1}{\sqrt{6}}\sin\theta, & -\frac{1}{\sqrt{6}}\sin\theta, & \frac{2}{\sqrt{6}}\sin\theta, & 0, & \cos\theta \\ \frac{1}{\sqrt{6}}\cos\theta, & \frac{1}{\sqrt{6}}\cos\theta, & -\frac{2}{\sqrt{6}}\cos\theta, & 0, & \sin\theta \end{vmatrix} \tag{5·4}$$

$$\begin{matrix} \frac{1}{\sqrt{2}}\sin\theta\,, & -\frac{1}{\sqrt{2}}\sin\theta\,, & 0\,, & \cos\theta\,, & 0 \\ -\frac{1}{\sqrt{2}}\cos\theta\,, & \frac{1}{\sqrt{2}}\cos\theta\,, & 0\,, & \sin\theta\,, & 0 \end{matrix}\Bigg]$$

where

$$\tan 2\theta = -\frac{2\sqrt{6}\,b}{(a+\Delta E)}\,. \tag{5·5}$$

The eigenfunctions and their eigenvalues are

$$S\varphi = \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}(\varphi_{yz}+\varphi_{zx}+\varphi_{xy}) \\ \frac{1}{\sqrt{6}}\sin\theta(2\varphi_{xy}-\varphi_{yz}-\varphi_{zx})+\cos\theta.\,\varphi_0 \\ -\frac{1}{\sqrt{6}}\cos\theta(2\varphi_{xy}-\varphi_{yz}-\varphi_{zx})+\sin\theta.\varphi_0 \\ \frac{1}{\sqrt{2}}\sin\theta(\varphi_{yz}-\varphi_{zx})+\cos\theta.\varphi_{2+} \\ -\frac{1}{\sqrt{2}}\cos\theta(\varphi_{yz}-\varphi_{zx})+\sin\theta.\,\varphi_{2+} \end{bmatrix} \tag{5·6}$$

and

$$SVS^{-1} = \begin{bmatrix} 2a & 0 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 & 0 \\ 0 & 0 & E_2 & 0 & 0 \\ 0 & 0 & 0 & E_1 & 0 \\ 0 & 0 & 0 & 0 & E_2 \end{bmatrix}, \tag{5·7}$$

where

$$\begin{aligned} E_1 &= \Delta E\cos^2\theta - a\sin^2\theta - 2\sqrt{6}\,b\sin\theta\cos\theta\,, \\ E_2 &= -a\cos^2\theta + \Delta E\sin^2\theta + 2\sqrt{6}\,b\sin\theta\cos\theta\,. \end{aligned} \tag{5·8}$$

As shown by these results, the trigonal field splits the energy levels into one singlet and two doublets. From the distribution of Fe ions around the considering Fe^{2+} ion, it may be reasonable to assign the ground state to a singlet. This fact is consistent with the g-value near two obtained by ferromagnetic resonance absorption. Otherwise, the enormously large anisotropy would come out, because of the surviving orbital moment in the doublet states. The actual crystalline field acting on an individual Fe^{2+} ion has a lower symmetry than trigonal and is different from ion to ion so that the trigonal field assumed here must be interpreted as the average representation.

The matrix elements of the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling with respect to the basic functions of (5·6) omitting a factor λ are obtained as

$$(\boldsymbol{L}\cdot\boldsymbol{S})=\begin{array}{|c|c|c|c|}
\hline
0 & i(S_x-S_y)\times\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right) & -i(S_x-S_y)\times\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right) & \frac{i}{\sqrt{3}}(S_x+S_y-2S_z)\times\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right) \\
\hline
 & 0 & \frac{i}{\sqrt{2}}(S_x-S_y) & -\frac{i}{\sqrt{3}}(S_x+S_y+S_z)\times(\sin\theta\cos\theta-2\sqrt{2}\sin^2\theta)-\frac{3i}{\sqrt{6}}(S_x+S_y) \\
\hline
 & & 0 & \frac{i}{\sqrt{3}}(S_x+S_y+S_z)\times(\cos^2\theta-2\sqrt{2}\cos\theta\sin\theta) \\
\hline
 & \text{conjugate complex} & & 0 \\
\hline
\end{array}\qquad(5\cdot9)$$

With the use of (5·7) and (5·9) we can calculate the g-value of Fe^{2+} ion, namely

$$g=2(1-\lambda\Lambda),$$

$$\Lambda_{\mu\nu}=\sum_n(0\,|\,L_\mu\,|\,n)(n\,|\,L_\nu\,|\,0)/\Delta E_n \qquad (5\cdot10)$$

where λ is the coefficient of the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling, and

$$\Lambda_{xx}=\Lambda_{yy}=\Lambda_{zz}=\frac{4}{3}\cdot\frac{1}{\Delta E_1}\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2+\frac{4}{3}\cdot\frac{1}{\Delta E_2}\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2,$$

$$\Lambda_{xy}=\Lambda_{yz}=\Lambda_{zx}=-\frac{2}{3}\cdot\frac{1}{\Delta E_1}\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2-\frac{2}{3}\cdot\frac{1}{\Delta E_2}\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2, \qquad (5\cdot11)$$

where $\Delta E_1=E_1-2a$ and $\Delta E_2=E_2-2a$. Thus, the g-value has a trigonal symmetry. The g-values of the other three kinds of ions for which the trigonal axes are in $[\bar{1}\bar{1}1]$, $[1\bar{1}\bar{1}]$ and $[\bar{1}1\bar{1}]$ directions respectively can be derived by changing the signs of (xy), (yz) and (zx). Averaging (5·11) over these four values, the off-diagonal part vanishes and we obtain the following result:

$$g-2=-\frac{8}{3}\frac{\lambda}{\Delta E_1}\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2-\frac{8}{3}\frac{\lambda}{\Delta E_2}\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2. \qquad (5\cdot12)$$

Compared with the calculation of the g-value, the calculation of the cubic anisotropy constant fo Fe^{2+} ions is considerably complicated. The perturbation in this case consists

of two parts, one is the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling and the other is the intra spin-spin interaction which has been expressed as

$$-\rho[(\boldsymbol{L}\cdot\boldsymbol{S})^2+(1/2)(\boldsymbol{L}\cdot\boldsymbol{S})-(1/3)L(L+1)S(S+1)] \qquad (5\cdot13)$$

by Pryce and Abragam[23,24] who also have estimated ρ to be 0.95 cm^{-1} for Fe^{2+} ion. The latter interaction is very small in its magnitude, but it contains $(\boldsymbol{L}\cdot\boldsymbol{S})^2$ so that the cubic anisotropy possibly arises from the lower order perturbation than the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling alone which needs the fourth order calculation to produce the cubic anisotropy, because it has a quartic form of the spin operator $\boldsymbol{S}$.

We will omit the detailed description of this perturbation calculation, but its gist is as follows. We select the part including the fourth order of the spin $\boldsymbol{S}$ from each order of perturbed energy, because the second order term with respect to $\boldsymbol{S}$ vanishes when it is averaged over the four kinds of ions. Such parts appear in the second order perturbation quadratic only in the intra spin-spin coupling (5·13), the third order perturbation linear in the intra spin-spin coupling and quadratic in the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling and the fourth order perturbation quartic only in the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling. In these calculations, we can ignore the non-commutability between the components of $\boldsymbol{S}$ because surplus terms produced by the commutation have a lower order of $\boldsymbol{S}$. The calculated result for the coefficient of $(S_x^2S_y^2+S_y^2S_z^2+S_z^2S_x^2)$ is as follows:

$$\begin{aligned}
&-\frac{16}{9}\lambda^4\left\{\frac{1}{\Delta E_1^2}\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2+\frac{1}{\Delta E_2^2}\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2\right\}\\
&\qquad\times\left\{\frac{3}{2}\frac{\rho}{\lambda^2}+\frac{1}{\Delta E_1}\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2+\frac{1}{\Delta E_2}\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2\right\}\\
&+\frac{8}{9}\lambda^4\left(\frac{1}{\Delta E_1^3}+\frac{3\rho}{\lambda^2}\frac{1}{\Delta E_1^2}+\frac{\rho^2}{\lambda^4}\frac{1}{\Delta E_1}\right)\\
&\qquad\times\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2(\sin^2\theta+2\sqrt{2}\sin\theta\cos\theta)^2\\
&+\frac{8}{9}\lambda^4\left(\frac{1}{\Delta E_2^3}+\frac{3\rho}{\lambda^2}\frac{1}{\Delta E_2^2}+\frac{\rho^2}{\lambda^4}\frac{1}{\Delta E_2}\right)\\
&\qquad\times\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2(\cos^2\theta-2\sqrt{2}\cos\theta\sin\theta)^2\\
&+\lambda^4\left\{\frac{1}{\Delta E_1^2\Delta E_2}+\frac{\rho}{\lambda^2}\left(\frac{1}{\Delta E_1^2}+\frac{2}{\Delta E_1\Delta E_2}\right)+\frac{\rho^2}{\lambda^4}\frac{1}{\Delta E_2}\right\}\\
&\qquad\times\left\{\frac{8}{9}(\sin\theta\cos\theta-2\sqrt{2}\sin^2\theta)^2+\frac{56\sqrt{2}}{9}(\sin\theta\cos\theta-2\sqrt{2}\sin^2\theta)+\frac{88}{9}\right\}\\
&\qquad\times\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)^2
\end{aligned}$$

$$+\lambda^4\left\{\frac{1}{\Delta E_1^{\,2}\Delta E_2}+\frac{\rho}{\lambda^2}\left(\frac{1}{\Delta E_1^{\,2}}+\frac{2}{\Delta E_1\Delta E_2}\right)+\frac{\rho^2}{\lambda^4}\frac{1}{\Delta E_1}\right\}$$
$$\times\left\{\frac{16}{9}(\sin\theta\cos\theta-2\sqrt{2}\sin^2\theta)-\frac{8\sqrt{2}}{3}\right\}(\sin^2\theta+2\sqrt{2}\sin\theta\cos\theta)$$
$$\times\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right)\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)$$
$$+\lambda^4\left\{\frac{1}{\Delta E_1\Delta E_2^{\,2}}+\frac{\rho}{\lambda^2}\left(\frac{1}{\Delta E_2^{\,2}}+\frac{2}{\Delta E_1\Delta E_2}\right)+\frac{\rho^2}{\lambda^4}\frac{1}{\Delta E_1}\right\}$$
$$\times\left\{\frac{8}{9}(\sin\theta\cos\theta+2\sqrt{2}\cos^2\theta)^2-\frac{56\sqrt{2}}{9}(\sin\theta\cos\theta+2\sqrt{2}\cos^2\theta)+\frac{88}{9}\right\}$$
$$\times\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)^2$$
$$+\lambda^4\left\{\frac{1}{\Delta E_1\Delta E_2^{\,2}}+\frac{\rho}{\lambda^2}\left(\frac{1}{\Delta E_2^{\,2}}+\frac{2}{\Delta E_1\Delta E_2}\right)+\frac{\rho^2}{\lambda^4}\frac{1}{\Delta E_2}\right\}$$
$$\times\left\{\frac{16}{9}(\sin\theta\cos\theta+2\sqrt{2}\cos^2\theta)+\frac{8}{3}\sqrt{2}\right\}(\cos^2\theta-2\sqrt{2}\cos\theta\sin\theta)$$
$$\times\left(\frac{1}{\sqrt{2}}\cos\theta+\sin\theta\right)\left(\frac{1}{\sqrt{2}}\sin\theta-\cos\theta\right). \qquad (5\cdot14)$$

In order to evaluate the above derived anisotropy constant, it is necessary to know the magnitudes of the cubic separation ΔE and the strength of the trigonal field a and b. From the point charge approximation, ΔE and the trigonal separation are expected to be about 10^4 cm^{-1} and 10^3 cm^{-1}, respectively, and in (5·3) the first term is considerably larger than the second term so that a is approximately $\sqrt{3}$ times as large as b. Putting

$$-(2\sqrt{6}/\Delta E)b=\beta\,,\quad -a/\Delta E=\alpha\,, \qquad (5\cdot15)$$

and using $\lambda=-100$ cm^{-1} and $\rho=0.95$ cm^{-1}, we obtain the following table which shows the g-value calculated from (5·12) for several combinative sets of α and β:

Table III

E	α	β	$g-2$
10^4 cm^{-1}	0.05	0.15	0.13
,,	0.08	0.24	0.10
,,	0.10	0.30	0.09
,,	0.10	0.50	0.11

The experimental g-value obtained by Bickford at high temperature range from 2.1 to 2.09. Thus, the values of $\alpha=0.08$ and $\beta=0.24$ would be considered appropriate. We

calculated the anisotropy constant with the use of these two values and obtain the following value :

$$K_1 = -3.56 \times 10^{-2} \text{ cm}^{-1} \text{ per } Fe^{2+} \text{ ion.} \qquad (5 \cdot 16)$$

The anisotropy energy arising from the anisotropic exchange interaction might be considered to come from the higher order perturbation with respect to JE_x/JE, where JE_x is the change of the exchange energy from the view-point of the molecular field treatment mentioned in Sec. 4, in which the exchange energy is treated as an unperturbed energy. Therefore, this contribution would be less than (5·16). Thus, the anisotropy constant is concluded to consist of (5·16) and the anisotropy constant from Fe^{3+} ions. As seen from Table I, the sum of (5·16) and the anisotropy constant of Ni-ferrite is nearly equal to the experimental value of magnetite.

In deriving (5·16) we assumed the uniform charge distribution of Fe^{2+} and Fe^{3+} ions of octahedral sites. At low temperature where the short range order in the arrangement of those two kinds of ions develops, this assumption will not be allowed any longer even as an approximation. The reason of the drastic change of the anisotropy constant in its magnitude and even in its sign might be due partly to the effect of the short range order on the crystalline field.

§ 6. Temperature dependence of the anisotropy energy of Mn-ferrite

From the considerations made in the preceding sections, we arrive at the conclusion that the experimental anisotropy energies of Ni-, Fe- and Mn-ferrites can be accounted for by the fine structure coupling of Fe^{3+}, Mn^{2+} and Fe^{2+} ions with the surrounding crystalline field. For such anisotropy energies, the temperature variation can be described by (2·9). In this section we shall calculate the temperature dependence of the anisotropy energy of Mn-ferrite and determine the fine structure constant a for Fe^{3+} and Mn^{2+} of octahedral and tetrahedral sites. The reason why we choose Mn-ferrite is that all the magnetic ions in this substance have spin of 5/2 and the Curie temperature is relatively low so that we may neglect the effect of thermal expansion.

We shall begin with the determination of the exchange couplings between magnetic ions. Since there has not been any experimental data of the susceptibility above the Curie temperature, we shall utilize the spontaneous magnetization *vs.* temperature curve obtained by Guillaud and Greveaux[12] for the determination of parameters. The total magnetizations of N ions situated in the tetrahedral and octahedral positions, $\mathfrak{M}_A$ and $\mathfrak{M}_B$, can, respectively, be expressed by

$$\mathfrak{M}_A = -Ng\mu_B \langle S_\zeta \rangle_A, \quad \mathfrak{M}_B = -Ng\mu_B \langle S_\zeta \rangle_B \qquad (6 \cdot 1)$$

where g is the g-factor of ions of tetrahedral and octahedral sites, μ_B is the Bohr magneton and $\langle S_\zeta \rangle_A$ and $\langle S_\zeta \rangle_B$ represent the average values of the spin component along the ζ-axis. Here, we neglect the difference between Fe^{3+} and Mn^{2+} ions in the octahedral sites in order to diminish the number of parameters. Introducing λ and μ which express the

fractions of the numbers of tetrahedral and octahedral lattice points, we put the molecular fields acting on the tetrahedral and octahedral sites, after Néel,[28] as

$$\begin{aligned} H_A &= q_{AA}\lambda\mathfrak{M}_A + q_{AB}\mu\mathfrak{M}_B\,, \\ H_B &= q_{BA}\lambda\mathfrak{M}_A + q_{BB}\mu\mathfrak{M}_B\,, \\ \lambda &= 1/3\,, \quad \mu = 2/3\,. \end{aligned} \tag{6·2}$$

Putting $q_{AB}=q_{BA}=-n$, $q_{AA}=n\alpha$ and $q_{BB}=n\beta$ and denoting the absolute values of the magnetizations $\mathfrak{M}_A$ and $\mathfrak{M}_B$ by M_A and M_B, (6·2) can be written as

$$\begin{aligned} H_A &= -n(\alpha\lambda M_A + \mu M_B)\,, \\ H_B &= n(\lambda M_A + \beta\mu M_B)\,, \end{aligned} \tag{6·3}$$

where we take the direction of $\mathfrak{M}_B$ as the positive direction. With the use of (6·3), (6·1) becomes

$$\begin{aligned} M_A &= M_0 B_{5/2}[M_0 n(\alpha\lambda M_A + \mu M_B)/RT]\,, \\ M_B &= M_0 B_{5/2}[M_0 n(\beta\mu M_B + \lambda M_A)/RT]\,, \end{aligned} \tag{6·4}$$

where $M_0 = Ng\mu_B S$ and $R=kN$, and B is the Brillouin function. (6·4) is equivalent to the following set of equations:

$$\begin{aligned} M_A &= M_0 B_{5/2}[u]\,, \quad u = (M_0 n/RT)\cdot(\alpha\lambda M_A + \mu M_B)\,, \\ M_B &= M_0 B_{5/2}[v]\,, \quad v = (M_0 n/RT)\cdot(\beta\mu M_B + \lambda M_A)\,. \end{aligned} \tag{6·5}$$

We define φ and ψ by the relations:

$$M_A/M_B = B_{5/2}[u]/B_{5/2}[v] = \varphi\,, \quad u/v = (\alpha\lambda\varphi+\mu)/(\beta\mu+\lambda\varphi) = \psi\,. \tag{6·6}$$

φ is related to the temperature by the equation of

$$RT/M_0^2 n = (\beta\mu+\lambda\varphi)\,(B_{5/2}[v]/v)\,. \tag{6·7}$$

Denoting the value of φ corresponding to the Curie temperature by φ_c, we obtain from (6·7)

$$\frac{T}{T_c} = \frac{15}{7}\cdot\frac{\beta\mu+\lambda\varphi}{\beta\mu+\lambda\varphi_c}\cdot\frac{B_{5/2}[v]}{v}\,. \tag{6·8}$$

φ_c is determined by the following equation which is derived from (6·6):

$$\lambda\varphi_c^2 + (\beta\mu-\lambda\alpha)\varphi_c - \mu = 0\,. \tag{6·9}$$

Solving this equation, we obtain

$$\varphi_c = (1/2)\{-(2\beta-\alpha) + \sqrt{(2\beta-\alpha)^2+8}\}\,. \tag{6·10}$$

The magnetization per mol is given by

$$M_S = 3(\mu M_B - \lambda M_A) = 3(\mu-\lambda\varphi)M_0 B_{5/2}[v]\,. \tag{6·11}$$

If a value of v is determined for an arbitrary value of φ, the temperature and magnetization for that value of φ can be obtained by (6·7) and (6·11). The value of v is

obtained by solving the simultaneous equations (6·6). As has been shown by Néel,[28] it may be convenient to express them in the logarithmic equations:

$$\log B_{5/2}[u]-\log B_{5/2}[v]=\log\varphi\,,$$

$$\log u-\log v=\log\psi\,,$$

$$\psi=(\alpha\lambda\varphi+\mu)/(\beta\mu+\lambda\varphi)\,. \qquad (6\cdot12)$$

The value of α and β must be taken from the hatched region in Fig. 1 in order to fit the calculated magnetization to the experimental one of Mn-ferrite. We move a point (α, β) in this region with the precision 0.01 and seek a point which fits best to the experimental magnetization curve obtained by Guillaud and Greveaux[32] in the temperature region of 25°K to 290°K. The values of α and β thus determined are

$$\alpha=-0.50\,,\ \beta=-0.10. \qquad (6\cdot13)$$

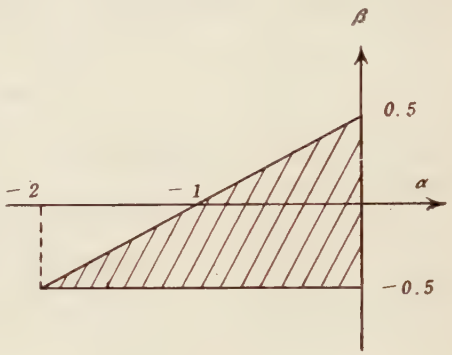

Fig. 1

The magnetization curve calculated with these values of α and β is shown in Fig. 2. The agreement with the experimental values is very good in the whole temperature range.

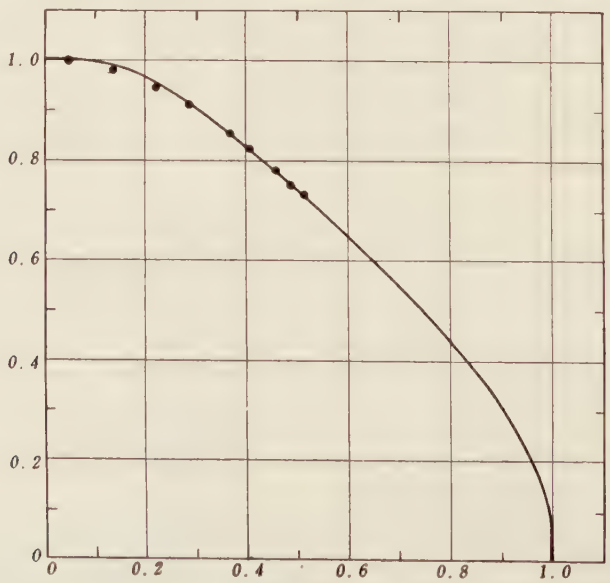

Fig. 2. The calculated magnetization *vs.* temperature curve of Mn-ferrite. The points represent the experimental values by Guillaud and Greveaux.

Another parameter n can be determined from the Curie temperature. We obtain

$$n=361\,. \tag{6·14}$$

The interaction between the ions of tetrahedral sites and that between the ions of octahedral sites are represented by $n\alpha$ and $n_i\beta$. These are shown below together with those of magnetite which have been calculated by Néel.[28]

Table IV

	$n\alpha$	$n\beta$
Mn-ferrite	−181	−36.1
Fe-ferrite	−282	+5.53

Now we shall turn to the calculation of the temperature dependence of the anisotropy constant. As seen from (2·9), the anisotropy constant K_1 in Mn-ferrite is expressed by

$$\begin{aligned} K_1=-\frac{N}{6}\Big[&(a_{\mathrm{Fe}}^{B}+a_{\mathrm{Mn}}^{B})\Big(\frac{35}{4}\langle S_\zeta{}^4\rangle_B-\frac{15}{2}\Big\{S(S+1)-\frac{5}{6}\Big\}\langle S_\zeta{}^2\rangle_B \\ &+\frac{3}{4}S(S+1)\{S(S+1)-2\}\Big) \\ &+a_{\mathrm{Fe}}^{A}\Big(\frac{35}{4}\langle S_\zeta{}^4\rangle_A-\frac{15}{2}\Big\{S(S+1)-\frac{5}{6}\Big\}\langle S_\zeta{}^2\rangle_A \\ &+\frac{3}{4}S(S+1)\{S(S+1)-2\}\Big)\Big], \end{aligned} \tag{6·15}$$

where A and B mean respectively the tetrahedral and octahedral sites. The average value of $\langle S_\zeta\rangle$ is obtained as

$$\langle S_\zeta\rangle=\frac{\sum' S_\zeta\exp(-pS_\zeta)}{\sum'\exp(-pS_\zeta)}=-\Big\{\Big(S+\frac{1}{2}\Big)\coth\Big(S+\frac{1}{2}\Big)p-\frac{1}{2}\coth\frac{p}{2}\Big\},$$

$$p=\frac{g\mu_B H}{kT}\,. \tag{6·16}$$

It can be shown that $\langle S_\zeta^{n+1}\rangle$ satisfies the following recurrence formula:

$$\langle S_\zeta^{n+1}\rangle=(-\partial/\partial p+\langle S_\zeta\rangle)\langle S_\zeta{}^n\rangle\,. \tag{6·17}$$

Therefore, if we use the two relations

$$\frac{\partial}{\partial p}\langle S_\zeta\rangle=\langle S_\zeta\rangle^2-\langle S_\zeta\rangle\coth\frac{p}{2}-S(S+1), \tag{6·18}$$

$$\frac{\partial}{\partial p}\coth\frac{p}{2}=\frac{1}{2}\Big(1-\coth^2\frac{p}{2}\Big), \tag{6·19}$$

together with the above recurrence formula, we can calculate $\langle S_\zeta{}^n\rangle$ for an arbitrary n value. Putting

$$\langle S_\zeta\rangle=x\,,\quad \coth p/2=y\,,$$

(6·17), (6·18) and (6·19) can be expressed in a more simplified form,

$$\begin{aligned}&\langle S_\zeta^{n+1}\rangle=(-\partial/\partial p+x)^n x\\&\partial x/\partial p=x^2-xy-S(S+1)\\&\partial y/\partial p=(1/2)(1-y^2).\end{aligned}\tag{6·20}$$

$\langle S_\zeta^2\rangle$ and $\langle S_\zeta^4\rangle$ calculated with the use of (6·20) are as follows:

$$\langle S_\zeta^2\rangle=S(S+1)+xy,\tag{6·21}$$

$$\langle S_\zeta^4\rangle=3xy^3+2S(S+1)y^2+2\{S(S+1)-1\}xy+S^2(S+1)^2-S(S+1).\tag{6·22}$$

Using (6·21) and (6·22) we obtain

$$\begin{aligned}&\left[\frac{35}{4}\langle S_\zeta^4\rangle-\frac{15}{2}\left\{S(S+1)-\frac{5}{6}\right\}\langle S_\zeta^2\rangle+\frac{3}{4}S(S+1)\{S(S+1)-2\}\right]\\&=\frac{1}{4}[105xy^3+70S(S+1)y^2+\{40S(S+1)-45\}xy\\&+8S^2(S+1)^2-16S(S+1)].\end{aligned}\tag{6·23}$$

If we put

$$z=Sp=g\mu_B SH/kT\tag{6·24}$$

we obtain

$$x=\langle S_\zeta\rangle=-SB_S[z],\tag{6·25}$$

and

$$y=\coth(p/2)=\coth(z/2S).\tag{6·26}$$

Inserting (6·25) and (6·26) into (6·23) and putting $S=5/2$, we get

$$\begin{aligned}&\left[\frac{35}{4}\langle S_\zeta^4\rangle-\frac{15}{2}\left\{S(S+1)-\frac{5}{6}\right\}\langle S_\zeta^2\rangle+\frac{3}{4}S(S+1)\{S(S+1)-2\}\right]\\&=\frac{5}{8}\left[-105B_S[z]\left(\coth\frac{z}{2S}\right)^3+245\left(\coth\frac{z}{2S}\right)^2\right.\\&\left.-305B_S[z]\coth\frac{z}{2S}+189\right]=\frac{5}{8}[z].\end{aligned}\tag{6·27}$$

Then, we can write (6·15) as

$$K_1=-N\frac{5}{48}\{(a_{\mathrm{Fe}}^B+a_{\mathrm{Mn}}^B)[v]+a_{\mathrm{Fe}}^A[u]\},\tag{6·28}$$

or

$$\frac{K_1}{M_S}=-\frac{1}{24}\frac{1}{g\mu_B}\left\{(a_{\mathrm{Fe}}^B+a_{\mathrm{Mn}}^B)\frac{[v]}{m}+a_{\mathrm{Fe}}^A\frac{[u]}{m}\right\},\tag{6·29}$$

where

$$m=M_S/Ng\mu_B S,$$

$[u]$ and $[v]$ defined by (6·27) with $z=u$ and $z=v$ can be calculated with the use of the values of α, β and n determined by (6·13) and (6·14). Dillon, Geschwind and Jaccarino[2] have measured K_1/M_S of Mn-ferrite over the temperature range of 4°K to 300°K. We shall select two points, $T=4.2$°K and $T=300$°K, and determine the values of $(a_{Fe}^B+a_{Mn}^B)$ and a_{Fe}^A so that the calculated K_1/M_S may coincide with the experimental values at these two points. The determined values of $(a_{Fe}^B+a_{Mn}^B)$ and a_{Fe}^A are as follows:

$$\begin{aligned} a_{Fe}^B+a_{Mn}^B&=4.21\times10^{-2}\ \text{cm}^{-1},\\ a_{Fe}^A&=-0.817\times10^{-2}\ \text{cm}^{-1}. \end{aligned} \qquad (6\cdot30)$$

The anisotropy constant *vs.* temperature curve calculated with the use of (6·30) is shown in Fig. 3. The agreement with the experimental value is satisfactory. This curve is very sensitive for a slight change of the a-value.

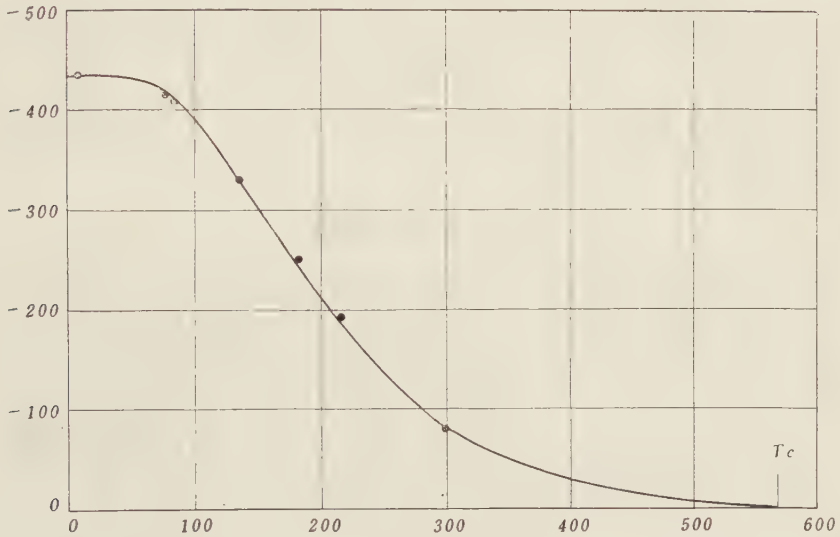

Fig. 3. The calculated temperature dependence of K_1/M_s of Mn-ferrite. The points represent the experimental values obtained by Dillon, Geschwind and Jaccarino.

In the vicinity of the Curie temperature, $\langle S_\zeta^n\rangle$ defined by (6·21) or (6·22) can be expanded in a power series of p or z, because the molecular field there is small, and we obtain

$$(5/8)[z]=5p^4+\cdots=(2^4/5^3)z^4+\cdots. \qquad (6\cdot31)$$

On the other hand, u and v are proportional to $(T_c-T)^{1/2}$ in the neighborhood of the Curie temperature so that the anisotropy constant K_1 is proportional to $(T_c-T)^2$ near the Curie temperature. In the calculation of K_1/M_s in Fig. 3 this expansion formula

has been used in this region.

The fact that the sign of the a-value of the ions of octahedral sites is opposite to that of the ions of tetrahedral sites, as seen from (6·30), seems to be reasonable because the cubic crystalline field acting on the tetrahedral position is opposite in sign to that on the octahedral position. The a-value of Fe^{3+} ion in several alums is about -1.34 to -1.27×10^{-2} cm^{-1} and that of Mn^{2+} ion in Tutton salts and fluosilicates is $+0.10$ to $+0.03\times10^{-2}$ cm^{-1} [33]. The a-value of Mn^{2+} ion is less than one tenth of that of Fe^{3+} ion. Thus, a_{Fe}^{B} has a magnitude nearly three times as large as that of the Fe^{3+} ions in alums and has the opposite sign to the latter if we neglect the contribution of a_{Mn}^{B} in (6·30). Recently, Low[34] has measured the fine structure of the paramagnetic resonance absorption of Fe^{3+} ion in the single crystals of MgO and obtained a considerably large a-value of 1.83×10^{-2} cm^{-1}, though its sign has not been determined. Therefore, the a-value given by (6·30) does not seem to be unreasonable. For Ni-ferrite, the a-value of Fe^{3+} ion of the octahedral site is considered to be about 2.0×10^{-2} cm^{-1} and this is quite reasonable. Very recently Sugiura et al.[35] observed the paramagnetic resonance absorption of Fe^{3+} ion in the mixed ferrite of $ZnOAl_2O_3$ and $ZnOFe_2O_3$ and that of $MgOAl_2O_3$ and $MgOFe_2O_3$ and found that the resonance widths extrapolated to zero content of Fe^{3+} ion are, respectively, about 600 and 180 oersteds. The fact that the ratio is equal to 33 : 1 seems to be favourable for our result (6·30) because Fe^{3+} ions in the former mixed ferrite occupy the octahedral sites and those in the latter occupy the tetrahedral positions.*

§ 7. Conclusion

From the results of calculations made so far, we draw the conclusion that the anisotropy energies of Ni- and Mn-ferrites arise from the fine structure coupling of Fe^{3+} ions which occupy the octahedral and tetrahedral positions. For Mn-ferrite, we determined the a-values of Fe^{3+} ions of the octahedral and tetrahedral sites by analyzing the experimental anisotropy constant *vs.* temperature curve obtained by Dillon, Geschwind and Jaccarino to be $a_{Fe}^{A}=-0.8\times10^{-2}$ cm^{-1} and $a_{Fe}^{B}=4.2\times10^{-2}$ cm^{-1}. This value of a_{Fe}^{B} seems to be somewhat larger than that expected from the a-value of Fe^{3+} ion in MgO. The magnitude of a_{Fe}^{B} in Ni-ferrite is expected from the anisotropy constant of this ferrite to be about 2×10^{-2} cm^{-1}. This value is quite reasonable. The reason for the large a-value of Fe^{3+} ion in Mn-ferrite is not clear at the present stage. However, the temperature dependence of the anisotropy constant of Mn-ferrite calculated with the use of these a-values shows a good agreement with the experiment.

For Co-ferrite, the situation is very complex. It has a very large anisotropy constant,

* G. T. Rado and V. J. Folen have found that at room temperature the anisotropy constant for Fe^{3+} ion at the B site is negative and that for Fe^{3+} ion at the A site is positive and the ratio of these is about 3.1 by the magnetic measurements of single crystals of $(MgO)_{1-x}(FeO)_x Fe_2O_3$. See G. T. Rado and V. J. Folen, Bull. Am. Phys. Soc. Ser. 2, **1** (1956), 132. We would like to thank to Dr. Rado for his private communication.

namely 2 cm^{-1} per molecule and also has a very large magnetostriction of $\lambda_{100}=5.9\times10^{-4}$ and $\lambda_{111}=1.1\times10^{-4}$. Thus, it might be supposed that in this substance the orbital moment is only partially quenched and the combined action of the electric quadrupole-quadrupole interaction, the exchange interaction between Co ions and the spin-orbit interaction would give rise to a large anisotropy energy.

For magnetite, it is considered that the anisotropy energies arising from Fe^{2+} and Fe^{3+} ions are leading terms in the high temperature range. The anisotropy energy below the transition temperature at 120° K is orthorhombic because of the orthorhombic superstructure of the arrangement of Fe^{2+} and Fe^{3+} ions in the octahedral sites. This anisotropy energy may be due to the Fe^{2+} and Fe^{3+} ions and the magnetic dipole-dipole interaction between these magnetic ions. In the temperature region above the transition, where the anisotropy constant decreases drastically and changes its sign from plus to minus, the effect of the short range order may play a certain role in determining the anisotropy energy. In order to account for the behavior of the anisotropy constant over this range, precise knowledge about the nature of the order-disorder transition would be needed.

In conclusion, the authors would like to express their sincere thanks to Professor T. Nagamiya and Mr. J. Kanamori for their valuable discussions.

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Note added in proof

A recent neutron diffraction experiment by J. M. Hastings and L. M. Corliss (Phys. Rev. **104** (1956), 328) on Mn-ferrite shows that the fraction of the tetrahedral sites occupied by manganese ions is 0.8 for three samples used. We, on the other hand, assumed that Mn-ferrite is of the inverse spinel type, i. e. the fraction is zero. To get out of this contradiction, the following three possibilities may be suggested.

1. The fraction of 0.8 is assumed to be valid also in the sample used by Dilllon[2] for measuring the anisotropy constant and in the sample used by Guillaud[32] for measuring the magnetization; Our analysis is based on these two measurements. Then the following values of a^{A}_{Fe} and a^{B}_{Fe} (neglecting the anisotropy constants arising from the manganese ions) can be shown to explain excellently the curve of K_1/M_s versus temperature of Dillon.
However, $a^{A}_{Fe}+a^{B}_{Fe}=-1.8\times10^{-2}\mathrm{cm}^{-1}$ in this case, which seems to be unacceptable since in Ni-ferrite the corresponding value is $+1.87\times10^{-2}\,\mathrm{cm}^{-1}$ (the opposite sign).

2. The fraction of 0.8 and the value of $a^{A}_{Fe}+a^{B}_{Fe}$ found for Ni-ferrite are assumed to be valid for Mn-ferrite, again neglecting the contribution of the manganese ions to the anisotropy constant. Taking the Dillon's[2] experimental value of K_1/M_s at 300°K–Dillon's and Tannenwald's[33] measurements give about the same value at 300°K–we obtain

$$a^{B}_{Fe}=1.37\times10^{-2}\,\mathrm{cm}^{-1},\qquad a^{A}_{Fe}=0.50\times10^{-2}\,\mathrm{cm}^{-1}.$$

The ratio $a^{B}_{Fe}/a^{A}_{Fe}=2.74$ is favourable for the experimental results due to Sugiura[35] and Rado (private communication) but the sign of a^{A}_{Fe} is opposite to that found by Rado. The temperature dependence of K_1/M_s comes nearer to that observed by Tannenwald[33] (Fig. 4).

3. The sample used by Dillon[2] is assumed to have had a different fraction of the manganese ions on the tetrahedral sites. This fraction will be denoted by x. The following numerical values found by our analysis are assumed to be valid:

$$(1-x)\;a^{A}_{Fe}=-0.82\times10^{-2}\,\mathrm{cm}^{-1},\qquad (1+x)a^{B}_{Fe}=4.21\times10^{-2}\,\mathrm{cm}^{-1},$$

$$a^{A}_{Fe}+a^{B}_{Fe}=1.87\times10^{-2}\,\mathrm{cm}^{-1}\quad\text{(for Ni-ferrite)}.$$

Then we obtain

$$x=0.347,\quad a^{A}_{Fe}=-1.26\times10^{-2}\,\mathrm{cm}^{-1},\quad a^{B}_{Fe}=3.13\times10^{-2}\,\mathrm{cm}^{-1}.$$

The ratio $|a^{B}_{Fe}/a^{A}_{Fe}|$ is 2.48 and the sign of a^{A}_{Fe} is in agreement with that found by Rado.

We note that the sublattice magnetizations per ion observed by neutron diffraction experiment and those found by our calculation (based on Guillaud's[32] measurements) are in good agreement:

neutron diffraction study:	our calculation:
$\frac{(m_B)_{300°K}}{(m_B)_{4.2°K}}=0.822$,	$\frac{(m_B)_{300°K}}{(m_B)_{4.2°K}}=0.815$,
$\frac{(m_A)_{300°K}}{(m_A)_{4.2°K}}=0.941$,	$\frac{(m_A)_{300°K}}{(m_A)_{4.2°K}}=0.914$.

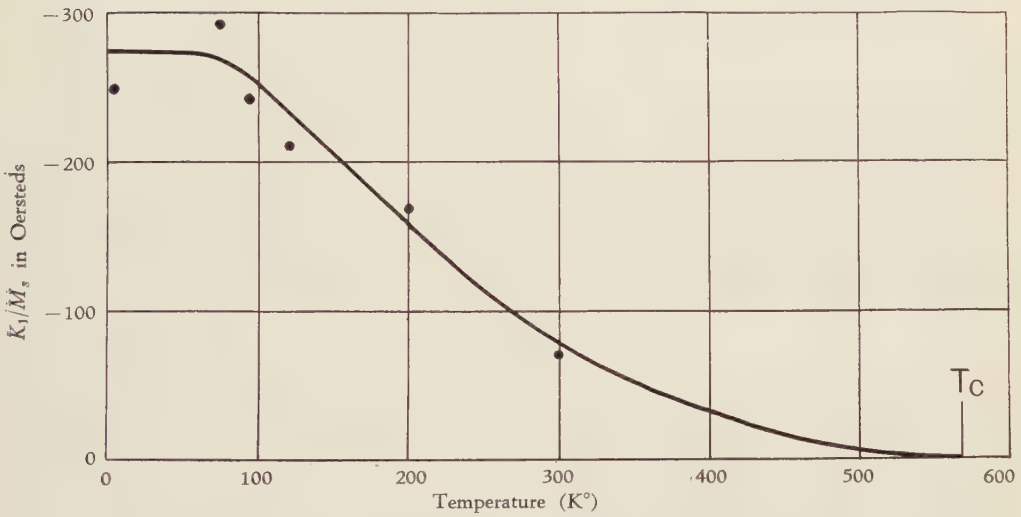

Fig. 4. The calculated temperature dependence of K_1/M_s of M_n-ferrite. The points represents the experimental values obtained by Tannenwald.

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Pion Theory of Three-Body Forces

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Three-body forces among three nucleons are calculated on the basis of the static pion theory. It is assumed that only one pion each is exchanged between nucleon (1) and (2), and between (2) and (3), although any number of pions may be emitted and absorbed by the same nucleon. The validity of this calculation is therefore limited to the case when every nucleon is well separated from the other. This three-body potential has a contribution of about 0.22 Mev to the triton binding. In a heavy nucleus the binding energy due to this potential is attractive and about ten percent of that due to two-body interaction. This effect can be roughly stated as follows. Within a nuclear matter the potential between two nucleons takes a different shape due to many body forces. The depth is increased by 9 percent and the range is decreased by 13 percent.

§ I. Introduction

The existence of many-body forces among nucleons has been stated for a long time.[1] However, no definite estimation has been made on this subject. Recent development indicates that the pion theory of nuclear forces is correct, unless nucleons are very close together.[2] Also it has become possible to calculate higher order processes on a fairly firm basis. It is the purpose of this paper to give the explicit form of the three-body force in terms of the static pion theory.

The process being calculated is this. Particle (1) emits a pion which is scattered by particle (2) and then absorbed by particle (3). Some part of this process is thought of as exchanges of pions between (1) and (2) and independently between (2) and (3), giving the iteration of two-body forces. The other part, which cannot be separated in this way, gives rise to the direct interaction between particles (1), (2) and (3). Exchange of three or more pions among particles are regarded as small unless nucleons are very near to each other, and is not considered in this paper.

For the scattering by particle (2), experimental values can be used. Actually this is a scattering of a virtual pion having zero energy. However, the use of dispersion relation makes it possible to correlate this scattering with real scatterings. In this way all virtual transitions on particle (2) are correctly taken into account.

§ II. Three body forces

The scattering matrix for the zero-energy pion is given by[3]*

* We use units in which $\hbar = c = \mu$ (pion mass) $= 1$.

$$< j, q|S|i, k > = 2\pi i\delta(0)[\{A(0)\tau_i\tau_j + B(0)\tau_j\tau_i\}\sigma\boldsymbol{k}\sigma\boldsymbol{q} + \{B(0)\tau_i\tau_j + A(0)\tau_j\tau_i\}\sigma\boldsymbol{q}\sigma\boldsymbol{k} + 2D(0)\delta_{ij}]\, e^{i\boldsymbol{x}(k-q)}.$$

Notations are as follows. i and k are the isotopic number and the momentum of the incident pion, and j and q are those of the scattered pion. $\boldsymbol{x}$ is the position of the nucleon. $A(0)$, $B(0)$ and $D(0)$ are given by

$$A(0) = \frac{f^2}{-i\mathcal{E}} + \frac{5}{18\pi}\int \frac{dp}{\omega_p^2}\sigma_{33}(p),$$

$$B(0) = \frac{1}{6\pi}\int \frac{dp}{\omega_p^2}\sigma_{33}(p), \tag{1}$$

$$D(0) = \frac{2\pi}{3}(a_1 + 2a_3),$$

where σ_{ij} is the total cross section of the pure $3/2$–$3/2$ state and a_i is the scattering length in the $1/2$ state. In (1), σ_{11}, σ_{13}, σ_1 and σ_3 are neglected. $A(0)$ contains a term proportional to $i/\mathcal{E}$. However, it is easily seen that this term gives rise to the iteration of two-body forces and must be omitted.

The three-body potential is given by*

$$V^{(3)} = V(123) + V(231) + V(312),$$

$$V(123) = -[\{A(\tau^1\tau^2)(\tau^2\tau^3) + B(\tau^3\tau^2)(\tau^2\tau^1)\}(\sigma^1\nabla^1)(\sigma^2\nabla^1)(\sigma^2\nabla^3)(\sigma^3\nabla^3) + \{B(\tau^1\tau^2)(\tau^2\tau^3) + A(\tau^3\tau^2)(\tau^2\tau^1)\}(\sigma^3\nabla^3)(\sigma^2\nabla^3)(\sigma^2\nabla^1)(\sigma^1\nabla^1) + 2D(\tau^1\tau^3)(\sigma^1\nabla^1)(\sigma^3\nabla^3)]\, Y(12)\, Y(23),$$

where

$$Y(ij) = \frac{e^{-\varkappa_{ij}}}{\varkappa_{ij}}, \quad \varkappa_{ij} = |x_i - x_j|,$$

$$A = \frac{f^2}{4\pi}\frac{5}{72\pi^2}\int \frac{\sigma_{33}}{\omega_p^2}\, dp, \quad B = \frac{3}{5}A,$$

$$D = \frac{f^2}{4\pi}\frac{1}{6}(a_1 + 2a_3).$$

Inserting numerical values,

$$\int \sigma_{33}/\omega_p^2 \cdot dp = 3.7, \quad a_1 + 2a_3 = -0.06.$$

Terms containing A and B are due to p-wave scattering and the last term proportional to D is due to s-wave scattering. This s-wave term was derived by Drell and Huang[4]

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* In differentiating $Y(x)$, there appears a term proportional to $\delta(x)$. This contact potential must be omitted.

and by Gelbard.[5] However, since they took Born approximation values for scattering lengths, their potential was about two hundred times larger than this potential. Actually this term is negligibly small.

§ III. Contributions to triton binding energy

For the triton wave function we choose the simplest one, namely,

$$\Phi=\frac{1}{\sqrt{6}}\begin{vmatrix}\alpha(1)n(1) & \alpha(2)n(2) & \alpha(3)n(3)\\ \alpha(1)p(1) & \alpha(2)p(2) & \alpha(3)p(3)\\ \beta(1)n(1) & \beta(2)n(2) & \beta(3)n(3)\end{vmatrix}\varphi(123),$$

where α and β are spin functions, p and n are isotopic spin functions and $\varphi(123)$ is a totally symmetric function of x_{12}, x_{23} and x_{31}. With this wave function,

$$(\Phi V(123)\Phi)=\frac{3}{5}(A+B)\int(3\cos^2\theta_2-1)f(x_{12})f(x_{23})\varphi^2(123)\,d\boldsymbol{x}_1 d\boldsymbol{x}_2 d\boldsymbol{x}_3+\mathrm{O}(D). \quad (3)$$

where

$$\cos\theta_2=(\boldsymbol{x}_{12}\boldsymbol{x}_{23})/x_{12}x_{23},$$

and

$$f(x)=\left(\frac{3}{x^2}+\frac{3}{x}+1\right)e^{-x}.$$

The contribution of the three-body force to the triton binding is

$$\Delta U=(\Phi V^{(3)}\Phi)=3(\Phi V(123)\Phi).$$

For the evaluation of (3), a detailed wave function is necessary since $3\cos^2\theta-1$ is not positive definite. In order to get a rough idea of the magnitude of ΔU, we assume that the triton forms an equilateral triangle. The length of its side may be determined from the Coulomb energy of He^3, and is equal to 1.3. In this case,

$$\Delta U\approx\tfrac{9}{5}(A+B)\,(-\tfrac{1}{4})f^2(1.3)=-0.22\ \text{Mev}.$$

The three-body effect is very small in pure s-state. If there is a considerable amount of p- and higher states in triton, the three-body force becomes important.

§ IV. Three body forces in heavy nuclei

As a simple model for a nucleus, we take a very large Fermi gas with both spin and τ-spin saturated. Its radius is taken as $A^{1/3}$. With this wave function Ψ, we calculate

$$U=(\Psi\textstyle\sum V^{(3)}\Psi),$$

where the summation extends over all nucleon trios. The direct integral vanishes since

spins are saturated. Two exchange integral remains.*

$$U=U_1+U_2\,,$$

$$U_1=-\tfrac{1}{2}\int V(123)\rho(13)\rho(22)\rho(31)\,d\boldsymbol{x}_1 d\boldsymbol{x}_2 d\boldsymbol{x}_3\,,$$

$$U_2=\int V(123)\rho(13)\rho(23)\rho(31)\,d\boldsymbol{x}_1 d\boldsymbol{x}_2 d\boldsymbol{x}_3\,,$$

where ρ is the density matrix of the nucleus,

$$\rho(12)=\sum_{\text{nucleus}}\varphi_n^*(1)\varphi_n(2)\,. \tag{4}$$

U_1 is the exchange energy for the interchange of two of the interacting trio, and U_2 represents the two symmetric permutations of three nucleons among themselves.

We first take U_1 and calculate

$$\Delta V^{(2)}(13)=\int V^{(3)}(123)\rho(22)\,d\boldsymbol{x}_2.$$

This quantity, which is a function of 1 and 3, may be called the change of two body interaction within nucleus. This means that U_1 can be calculated as if there is an additional interaction $\Delta V^{(2)}$ between any pair of nucleons. Substituting (2) and (4) for $V^{(3)}$ and ρ,

$$\Delta^{(2)}(13)=(\tau^1\tau^3)\,(\sigma^1\nabla)\,(\sigma^3\nabla)\left[C\frac{e^{-x_{13}}}{x_{13}}+De^{-x_{13}}\right], \tag{5}$$

where

$$C=2(A+B)=0.007\,,$$

$$D=-3(A+B)=-0.010\,.$$

With this $\Delta^{(2)}$,

$$U_1=-0.3\ \text{Mev}\times A \tag{6}$$

where A is the mass number of the nucleus.

The triple exchange term is found to be

$$U_2=0.1\ A\ \text{Mev}. \tag{7}$$

These values are rather small compared with the actual binding energy per nucleon which is about 15 Mev. Of course (6) and (7) are underestimations of the actual situations due to the crudeness of the model. While tensor forces are important for the nuclear binding, they are completely neglected in the Fermi gas model. For instance, the potential energy calculated from the Yukawa potential

$$V^{(2)}=g^2/4\pi\cdot(\tau\tau)\,(\sigma\nabla)\,(\sigma\nabla)\cdot e^{-x}/x$$

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* The integral over $\boldsymbol{x}$ is assumed to include the summation over spin and τ-spin.

gives only 4 Mev per nucleon. The strong correlation between nucleons which is certainly present within a nucleus will make exchange integrals larger.

The change of two body force $\Delta V^{(2)}$ can be transformed in a simpler form. The effective two body force in a nucleus can be written, since C and D are small,

$$V_{\text{eff}}^{(2)} = V^{(2)} + \Delta V^{(2)} = \frac{g^2}{4\pi} \frac{1}{1-\Delta} (\tau\tau)(\sigma\nabla)(\sigma\nabla) \frac{\exp\{-(1+\Delta\mu)x\}}{x}, \tag{8}$$

where

$$\Delta = (g^2/4\pi)^{-1} C = 0.08,$$

$$\Delta\mu = -(g^2/4\pi)^{-1} D = 0.12.$$

(8) shows that within nuclear matter, depth of the two body potential is increased by 9 percent, and its range is shortened by 13 percent.

Manipulations in (8) may be justified if we consider the effect of many body forces. A virtual pion emitted by nucleon (1) may be directly absorbed by nucleon (3) and give rise to $V^{(2)}$. Or the pion may be scattered by nucleon (2) on its way, and this is the content of $\Delta V^{(2)}$. The pion can be scattered twice or more before it is absorbed by (3). Contributions of these terms may eventually be summed up into the form (8).*

§ V. Conclusions and discussions

Two conclusions can be drawn. One is that the contribution of the three-body interaction is about ten percent of that of the two-body interaction. The other point is that the many body forces due to s-wave scattering is extremely small contrary to the expectation from perturbation theory. The scattering length calculated from the lowest order perturbation theory is

$$\frac{1}{3}(a_1 + 2a_2) = -\frac{g^2}{4\pi}\frac{1}{M} = -2.1,$$

while its experimental value used in this paper is -0.02. Such a small interaction cannot be responsible for the saturation of the nucleus, for which some other reason must be looked for.

The exchange of three or more pions among nucleons is neglected in this paper. For nuclear interactions at short distances these terms will be very important. It seems that the contributions from such short distances is rather large for the nuclear binding. The binding energies calculated in this paper, therefore, cannot be taken too seriously.

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* If there is a scattering term $\lambda\rho p^2 \phi_i^2(p)$ in the interaction Hamiltonian, the total Hamiltonian can be written as

$$H = \tfrac{1}{2}(p^2+1)\phi_i^2(p) + \lambda\rho p^2\phi_i^2(p) + \cdots = \tfrac{1}{2}(1-\Delta)\{p^2 + (1+\Delta\mu)^2\}\phi_i^2(p) + \cdots$$

From this Hamiltonian, nuclear potential of the form (8) is obtained.

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Spin-Orbit Coupling in Heavy Nuclei

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In the preceding paper we have calculated the three-body forces in the static approximation. Using the result a strong spin-orbit coupling, compared with the Thomas term, is derived in this paper. Though it is not sufficient to explain the observed spin-orbit coupling for itself, we expect that a considerable part of the nuclear spin-orbit interaction should be due to the many-body forces.

§ 1. Introduction

Recently experimental evidences have revealed the presence of rather strong spin-orbit interaction in nuclei.

The scattering experiment of neutrons or protons on He^4 can give information about the unbound states of He^5 or Li^5 respectively. Since each of them contains only one nucleon outside the tightly bound α-particle core, Li^5 and He^5 are the nuclei to which the single-particle description can certainly be applied. The resonance peak of type $p_{1/2}$ occurs in the neutron scattering at an energy several Mev higher than that of type $p_{3/2}$. It implies that the spin-orbit splitting in $1p$ state is of the order of several Mev. Similarly the scattering of neutrons on O^{16} suggests[1] that the doublet splitting in $1d$ state is about 5 Mev.

The direct demonstration of the polarization of the scattered protons has also verified the presence of the spin-orbit coupling very clearly. The double scattering experiments[2] have been performed with protons of a few hundred Mev as well as those of several Mev. It was shown[3] that the order of magnitude of the observed polarization is correctly given by the same spin-orbit interaction as at low energies.

From the existence of the strong spin-orbit interaction we can understand immediately the occurrence of magic numbers, on which the nuclear (jj)-coupling shell model is based. The change in the binding energy of the last nucleon at the magic numbers 50 or 82 is about 2 Mev. This means that the doublet splitting of the $1g$ or $1h$ level has at least this magnitude.

Various explanations of the nuclear spin-orbit interaction have been proposed so far, which can be classified grossly into the following three categories : i) the relativistic effect (Thomas term), ii) the higher-order consequence of tensor forces and iii) the two-body spin-orbit coupling due to the nucleon recoil effect.

First the Thomas term has the right sign but is too small in magnitude to explain the observed effects.[4] It is written as :

$$H'=\sum_i(\hbar^2/2M^2c^2)\,(r_i^{-1}dV(r_i)/dr_i)\,(\boldsymbol{l}_i\boldsymbol{s}_i). \tag{1}$$

Here V is an effective potential energy of one nucleon in the field of all the others, M the nucleon mass, and $\boldsymbol{l}_i$ or $\boldsymbol{s}_i$ is the orbital or spin angular momentum of the i-th nucleon. The Thomas precession gives the doublet splitting only of the order of 100 kev. A replacement of one factor M in Eq. (1) by μ, the meson mass, has been suggested by many authors.[5] However, most of their arguments are based on the wrong hypotheses with the type of π-meson.

Second the higher-order effects of phenomenological tensor interaction were discussed by Feingold and Wigner[6] who calculated the level splittings of He^5 and Li^7. Their results are somewhat small but of the right order of magnitude to account for the observed data. However, the possible equivalence of the spin-orbit coupling and the second-order consequence of tensor forces has not completely been clarified yet and it remains to be investigated.

Recently, the third explanation was proposed[7] from the view-point of the meson theory. The calculated two-body spin-orbit interaction is not proportional to M^{-2} as in Eq. (1) but is proportional to M^{-1} and has the correct sign. The result is probably reliable in the region where the distance of two nucleons is not so small. In order to compare it with the empirical one, we shall reduce it to the single-body interaction, using the Fermi gas model and cutting off its strong singularity at the origin, which should not exist in reality. In Sec. 2 we shall find that its magnitude is not sufficient enough to explain the empirical data, though it may be several times larger than the Thomas term.

Now we know many sources which lead to the spin-orbit interaction. However, anyone of them cannot explain the empirical interaction for itself. In Sec. 3 another derivation of the spin-orbit coupling is shown. It originates from many-body forces in the nuclear matter, for which meson-theoretical calculations were performed in the preceding paper.[8] It seems to be another important origin which leads to the observed strong spin-orbit coupling.

§ 2. Reduction of the two-body spin-orbit coupling to the single-body one

The calculated two-body spin-orbit coupling has the expression,[7]

$$V^{(\mathrm{s.o.})}=-(2/M)\,(f^2/4\pi)^2\{(1+x)\,(1+x+x^2)/x^6\}\exp(-2x)$$
$$\times[3+2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)]\cdot[\boldsymbol{L}\cdot\boldsymbol{S}], \tag{2}$$

where $\boldsymbol{S}=(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$, $\boldsymbol{L}=\boldsymbol{x}\times(\boldsymbol{p}_1-\boldsymbol{p}_2)/2$ and $\boldsymbol{x}=\boldsymbol{x}_1-\boldsymbol{x}_2$,

(we put $\hbar=c=\mu=1$ hereafter).

For simplicity we assume that every nucleon distributes uniformly in a sphere with radius R. Then the desired single-body interaction is

$$(\rho/4)\int_{x_2<R} V^{(s,o,)}d\boldsymbol{x}_2 \tag{3}$$

where ρ is the density of nucleons. Provided that the distributions of momenta, spins and isotopic spins in the nucleus are at random, expression (3) is reduced to

$$\rho\int_{x_2\leq R} f(x)[\boldsymbol{x}\times\boldsymbol{p}_1,\ \sigma_1]\,d\boldsymbol{x}_2 \tag{4}$$

in which

$$f(x)=-(3/2M)\,(f^2/4M)^2\{(1+x)\,(1+x+x^2)/x^6\}\exp(-2x) \tag{5}$$

(4) can be written as

$$[\boldsymbol{x}_1\times\boldsymbol{p}_1,\ \sigma_1]\,S(x_1), \tag{6}$$

where

$$S(x_1)=(\rho/x_1^2)\int_{x_2\leq R} f(x)\,(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\boldsymbol{x}_1 d\boldsymbol{x}_2.$$

(6) has similar form as the Thomas spin-orbit coupling (1) since $S(x_1)$ is large only when x_1 is near to the nuclear surface. Instead of obtaining $S(x_1)$, we calculate

$$\int_{-\infty}^{\infty} S(x_1)\,dx_1$$

in order to compare the magnitude of the present spin-orbit interaction with that of the Thomas term. We assume the nuclear size to be sufficiently large compared with the range of interaction. Then the surface of the nucleus can be regarded as a plane, on which we set x and y axes. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx_{1z}\int_{x_{2z}<0} (x_1-x_2)_z\,f(x)\,d\boldsymbol{x}_2\\
&\quad=2\int_0^{\infty} dx_{1z}\int_{x_{2z}<0} (x_1-x_2)_z\,f(x)\,d\boldsymbol{x}_2\\
&\quad=-(4\pi/3)\int_{x_c}^{\infty} f(t)\,t^4 dt
\end{aligned} \tag{7}$$

where x_c is the cut-off radius. Therefore

$$\begin{aligned}
F(x_c)&=-\rho\int_{x_c}^{\infty} f(t)\,t^4 dt\\
&=(3/2M)\,(f^2/4\pi)^2\int_{x_c}^{\infty}\{(1+t^2)\,(1+t+t^2)/t^2\}\exp(-2t)\,dt\\
&=(3/2M)\,(f^2/4\pi)^2\left[\tfrac{1}{2}(x_c^2+2x_c+3)\exp(-2x_c)+\int_{2x_c}^{\infty}(1/t+2/t^2)\exp(-t)\,dt\right].
\end{aligned} \tag{8}$$

The numerical values of the quantity in the parenthesis of expression (8) and $F(x_c)$ is tabulated in Table I.

Table I.

x_c	$\left[\frac{1}{2}(x_c^2+2x_c+3)\exp(-2x_c)+\int_{2x_c}^{\infty}\left(\frac{1}{t}+\frac{2}{t^2}\right)\exp(-t)\,dt\right]$	$F(x_c)$ Mev cm^2
0.3	1.88	$0.72\ 10^{-26}$
0.4	1.26	0.52
0.5	0.91	0.37
0.6	0.68	0.28
0.7	0.53	0.22

In order to compare it with the Thomas term, we integrate Eq. (1) with respect to r.

$$\begin{aligned}&\tfrac{1}{2}\int_0^{\infty}(1/2M^2)\,(r^{-1}dV/dr)\,dr\,(\boldsymbol{l}\cdot\sigma)\\&\quad=-(1/4M^2)\langle V\rangle_{\mathrm{av}}\cdot(\boldsymbol{l}\cdot\sigma)/R.\end{aligned}\tag{9}$$

If we rather arbitrarily take $\langle V\rangle_{\mathrm{av}}=+20$ Mev, we have $-(1/4M^2)\langle V\rangle_{\mathrm{av}}=-0.22\times10^{-26}$ Mev cm^2. On the other hand, the corresponding quantity can be derived from Eq. (6) and (7) :

$$-F(x_c)\cdot(\boldsymbol{l}\cdot\sigma)/R.\tag{10}$$

Therefore, the meson-theoretical spin-orbit coupling is of magnitude about $2\sim3$ times as large as that of the Thomas term for reasonable cut-off radii, $x_c=0.4\sim0.6$.

§ 3. Derivation of spin-orbit coupling from three-body force

The additional interaction between two nucleons in nuclear matter, which arises from three-body force[8], is proportional to

$$^1\int(\sigma^3\nabla^1)\,(\sigma^2\nabla^3)\,(\sigma^2\nabla^1)\,(\sigma^1\nabla^1)\,Y_{12}Y_{21}d\boldsymbol{x}_2.\tag{11}$$

(Rigorously speaking, we must subtract the contact interactions in Exp. (11). However it does not affect the result below.) We define a component of the density matrix,

$$\rho(1,3)=\delta_{13}^{\sigma}\delta_{13}^{\tau}[1/(2\pi)^3]\int\exp\{-i\boldsymbol{q}(\boldsymbol{x}_1-\boldsymbol{x}_3)\}\rho(q)\,d\boldsymbol{q}\tag{12}$$

where

$$\rho(q)=\begin{cases}1 & q\leqq P_F\\ 0 & q>P_F\end{cases}\quad(P_F:\text{ Fermi momentum}).$$

The energy expectation value involves the following term,

$$[C/(2\pi^2)^2]\int_{z_i\geq 0}\bar{\psi}(1)\{(\sigma^2\boldsymbol{k})(\sigma^1\boldsymbol{p})(\boldsymbol{kp})/(k^2+1)(p^2+1)\}$$

$$\times\exp\{-i\boldsymbol{k}\boldsymbol{x}_{12}-i\boldsymbol{p}\boldsymbol{x}_{23}\}\rho(13)\psi(3)\,d\boldsymbol{k}d\boldsymbol{p}d\boldsymbol{x}_1d\boldsymbol{x}_2d\boldsymbol{x}_3$$

$$=\{C/(2\pi^2)^2(2\pi)^3\}\int\bar{\psi}(1)\{i\sigma(\boldsymbol{k}\times\boldsymbol{p})(\boldsymbol{kp})/(k^2+1)(p^2+1)\}$$

$$\times\exp\{-i\boldsymbol{k}\boldsymbol{x}_{12}-i\boldsymbol{p}\boldsymbol{x}_{23}-i\boldsymbol{q}\boldsymbol{x}_{31}\}\rho(q)\psi(3)\,d\boldsymbol{k}d\boldsymbol{q}d\boldsymbol{x}_1d\boldsymbol{x}_2d\boldsymbol{x}_3, \tag{13}$$

when

$$C=12\rho(A+B).$$

In the right-hand side of (13), the term independent of spin is omitted since this term does not contribute to the spin-orbit coupling. Using the Taylor expansion of $\psi(3)$,

$$\psi(3)=\psi(1)+i(\boldsymbol{x}_3-\boldsymbol{x}_1)\boldsymbol{P}\psi(1)+\ldots,$$

(13) can be written as

$$\int\bar{\psi}(1)E_{\text{eff}}(1)\psi(1)\,d\boldsymbol{x}_1\,,$$

where the effective potential E_{eff} for the state ψ is written as

$$E_{\text{eff}}(1)=\{C/(2\pi^2)^2(2\pi)^3\}\int\{i\sigma(\boldsymbol{k}\times\boldsymbol{p})(\boldsymbol{kp})/(k^2+1)(p^2+1)\}\rho(q)$$

$$\times\boldsymbol{P}\cdot\nabla_q\exp\{-i\boldsymbol{k}\boldsymbol{x}_{32}-i\boldsymbol{p}\boldsymbol{x}_{21}-i\boldsymbol{q}\boldsymbol{x}_{13}\}\,d\boldsymbol{k}d\boldsymbol{p}d\boldsymbol{q}d\boldsymbol{x}_2d\boldsymbol{x}_3, \tag{14}$$

where $\boldsymbol{P}$ is the momentum of ψ. It can be shown that the first term in the Taylor expansion which is independent of $\boldsymbol{P}$ does not contribute to the integral, and the higher-power terms with $\boldsymbol{P}$ can be reasonably neglected.

We assume the nucleus to be very large and integrate E_{eff} (1) along z-axis.

$$\int_0^\infty E_{\text{eff}}(1)\,dz_1=-\{C(2\pi)/(2\pi^2)^2\}\int i\sigma(\boldsymbol{k}\times\boldsymbol{p})(\boldsymbol{kp})\,\delta(k_x-p_x)\,\delta(k_y-p_y)$$

$$\times\delta(q_x-k_x)\,\delta(q_y-k_y)/(k^2+1)(p^2+1)(k_z-p_z+i\varepsilon)(q_z-k_z+i\varepsilon)(p_z-q_z+i\varepsilon)$$

$$\times\boldsymbol{P}\cdot\nabla_q\rho(q)\,d\boldsymbol{k}d\boldsymbol{p}d\boldsymbol{q}$$

Putting $k+1=k_z^2+\omega_k^2$,

$$=-\{C(2\pi)/(2\pi^2)^2\}\int(\sigma_yq_x-\sigma_xq_y)(q_x^2+q_y^2+k_zp_z)/\{(k^2+\omega_q^2)(p^2+\omega_q^2)$$

$$\times(p_z-q_z+i\varepsilon)(q_z-k_z+i\varepsilon)\}\cdot\{\boldsymbol{P}\cdot\boldsymbol{q}/|q|\}\,\delta(q-P_F)\,dk_zdp_zdq$$

$$=\{C(2\pi)^3/(2\pi^2)^24\}\int\{2(q_x^2+q_y^2)+1/(q_x^2+q_y^2+1)(q^2+1)|q|\}$$

$$\times(\sigma_yP_xq_x^2-\sigma_xq_y^2P_y)\,\delta(|q|-P_F)\,dq$$

$$= \{C(2\pi)^3/8(2\pi^2)^2\} \int \{(2q^2 \sin^2\theta + 1)/(q^2 \sin^2\theta + 1)(q^2+1)\}$$
$$\times q^2 \sin^2\theta \delta(q - P_F) d^3q (\sigma_y P_x - \sigma_x P_y)$$
$$= \{\pi C(2\pi)^3/4(2\pi^2)^2\} \int P_F^3 \sin^2\theta (2P_F^2 \sin^2\theta + 1)/(P_F^2 \sin^2\theta + 1)(P_F^2+1)\}$$
$$\times \sin\theta d\theta (\boldsymbol{P} \times \sigma)_z . \qquad (15)$$

The integration of last equation can be carried out easily and

$$\int_0^\infty E_{\text{eff}}(1) dz_1 = \{C\pi(2\pi)^3/4(2\pi^2)^2\} \{P_F^3/(P_F^2+1)\}$$
$$\times [8/3 - 1/P_F^2 (2 + 1/P_F \sqrt{P_F^2+1} \log|(\sqrt{P_F^2+1} - P_F)/(\sqrt{P_F^2+1} + P_F)|]$$
$$\cdot (\boldsymbol{P} \times \sigma)_z . \qquad (16)$$

Inserting the numerical values, $P_F = 3/2$ and $f^2/4\pi = 0.08$,

$$\int_0^\infty E_{\text{eff}}(1) dz_1 = -0.94 \times 10^{-26} (\boldsymbol{l}\sigma)/R \ \text{(Mev.cm)}. \qquad (17)$$

This result is about 4.3 times as large as that of the Thomas term, as seen from the discussion in Sec. 2.

§ 4. Conclusions and discussions

The derivation of the spin-orbit coupling as stated in Sec. 3 can be understood physically as follows: First let us assume the existence of the spin-dependent three-body force. Or more visually, a meson emitted from nucleon 1 propagates via nucleon 2 to nucleon 3. If the nucleon 2 distributes uniformly in space, such as in the inner region of a heavy nucleus, the path of the meson between nucleon 1 and 2 would be a straight line on an average. However, in the region near the nuclear surface the direction of the meson emitted from nucleon 1 must be different even on an average from that in which the meson arrives at nucleon 3. This fact leads to the spin-orbit coupling, or the different self-energy values of the nucleon according to the relative directions of its spin and orbital angular momenta.*

Conclusions of this paper are:

i) The meson-theoretical two-body spin-orbit coupling has the effect about two or three times as large as the Thomas term. Of course, it may be somewhat increased because of the ambiguity with radial dependence of the interaction in the small distance between two nucleons and possible correlations between their positions.

ii) Our spin-orbit coupling derived from the meson-theoretical three-body force is about 4.3 times larger than the Thomas term. The existence of the correlations may also

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* This kind of spin-orbit coupling was suggested by Prof. Y. Nambu in private communication.

increase this magnitude further. It is probable that the higher-order corrections for three-body forces have a large effect.

iii) The experimental spin-orbit coupling[7] is 40 times larger than the Thomas term with the magnitude as evaluated in Sec. 2. The net effect of Thomas term, two-body spin-orbit coupling and many-body interaction, which have the common correct sign, seems to be only about one fifth of the experimental one. Although we have underestimated the effects of many-body forces or two-body spin-orbit couplings because of the roughness of the Fermi gas model and the ambiguity in the small distance region, a considerable fraction of the strong spin-orbit coupling should probably be attributed to some other causes.

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Effects of the Central Spin on Planetary Motion

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The motion of spinning test-particles in the field of a spinning sphere is investigated. The bending of light rays is calculated, and finally the perturbing forces on the Newtonian elliptical orbit are determined.

§ 1. Introduction

Lense & Thirring[1] calculated the gravitational field due to a homogeneous, uniformly rotating sphere on the basis of Einstein's linearized law of gravitation. They calculated by standard perturbation theory the secular variations in the six Newtonian orbital elements.

In the present paper we shall extend their results upto higher approximation (upto terms containing ω^2) by making use of the exterior gravitational potentials due to a rotating spheroid, given by Clark[5].

We shall also consider the spin of test-particles simultaneously, for which we apply the equations of motion for the spinning test-particles by Papapetrou[6], Corinaldesi and Papapetrou[7].

In a recent paper, Rayner[10] evaluated the external field due to a rotating sphere after Whitehead's law of gravitation. We shall show that for terms beyond the first approximation, Einstein's theory differs from Whitehead's.

§ 2. Solution for a rotating spheroid

The gravitational field is assumed to be weak. Notations x_1, x_2, x_3 are adopted for three spatial co-ordinates and $t=x_4$ for the temporal one.

The metric tensor is then set

$$g_{\mu\nu}=\overset{\circ}{g}_{\mu\nu}+h_{\mu\nu}, \tag{2·1}$$

where $\overset{\circ}{g}_{\mu\nu}$ are the Galilean values and $h_{\mu\nu}$ are small. $\gamma_{\mu\nu}$ are introduced by means of the relations

$$\gamma_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\,\overset{\circ}{g}_{\mu\nu}\,\overset{\circ}{g}^{\sigma\rho}\,h_{\sigma\rho}. \tag{2·2}$$

Einstein *et al.*[3] pointed out that the inverse relations are

$$h_{\mu\nu}=\gamma_{\mu\nu}-\tfrac{1}{2}\,\overset{\circ}{g}_{\mu\nu}\,\overset{\circ}{g}^{\sigma\rho}\,\gamma_{\sigma\rho}. \tag{2·3}$$

Now, from Clark's[5] solution for a spheroid rotating with an angular velocity ω about x_3 axis, the external potentials (non-periodic) are

$$\left.\begin{aligned}
\gamma_{44} &= -16\pi\rho r_0^3/3r_1 + 4\pi\rho\epsilon r_0^2/15[10r_0/r_1 + r_0^3\{-2/r_1^3 + 3(x_1^2+x_2^2)/r_1^5\}],\\
\gamma_{41} &= -16\pi\rho r_0^5\omega x_2/15r_1^3 + 4\pi\rho\epsilon r_0^7\omega x_2/105[28/r_0^2r_1^3 - 12/r_1^5 + 15(x_1^2+x_2^2)/r_1^7],\\
\gamma_{42} &= +16\pi\rho r_0^5\omega x_1/15r_1^3 + 4\pi\rho\epsilon\, r_0^7\omega x_1/105[-28/r_0^2r_1^3 + 12/r_1^5 - 15(x_1^2+x_2^2)/r_1^7],\\
\gamma_{11} &= 16\pi\rho r_0^7\omega^2[1/105r_1^3 - x_2^2/35r_1^5] + 4\pi\rho\epsilon\omega^2[\lambda r_0^7(1/3r_1^3 - x_2^2/r_1^5)\\
&\quad + \nu r_0^9\{-12/r_1^5 + 90x_2^2/r_1^7 + 15(x_1^2+x_2^2)/r_1^7 - 105(x_1^2+x_2^2)x_2^2/r_1^9\}],\\
\gamma_{22} &= 16\pi\rho r_0^7\omega^2[1/105r_1^3 - x_1^2/35r_1^5] + 4\pi\rho\epsilon\omega^2[\lambda r_0^7(1/3r_1^3 - x_1^2/r_1^5)\\
&\quad + \nu r_0^9\{-12/r_1^5 + 90x_1^2/r_1^7 + 15(x_1^2+x_2^2)/r_1^7 - 105(x_1^2+x_2^2)x_1^2/r_1^9\}],\\
\gamma_{12} &= 16\pi\rho r_0^7\omega^2 x_1x_2/35r_1^5 + 4\pi\rho\epsilon\omega^2[\lambda r_0^7\, x_1x_2/r_1^5 + \nu r_0^9\times\\
&\quad \times\{-90x_1x_2/r_1^7 + 105x_1x_2(x_1^2+x_2^2)/r_1^9\}].
\end{aligned}\right\} \tag{2·4}$$

where

$$\left.\begin{aligned}
&\lambda = 162\,\nu,\ 945\nu = 1,\\
&m = 4/3\cdot\pi\rho r_0^3(1-\epsilon/2),\\
&A = 8/15\cdot\pi\rho r_0^5(1-\epsilon),\\
&I = -4/15\cdot\pi\rho r_0^5\epsilon.
\end{aligned}\right\} \tag{2·4a}$$

For a perfect sphere i.e. $\epsilon = I = 0$, periodic terms giving rise to gravitational waves do not come at all. From (2·3), (2·4), (2·4a) the corresponding simplified $h_{\mu\nu}$ becomes

$$\left.\begin{aligned}
h_{44} &= -2m/r_1 + Ar_0^2\omega^2/7r_1^3[2/3 - (x_1^2+x_2^2)/r_1^2],\\
h_{11} &= -2A\omega x_2/r_1^3,\ h_{12} = +2A\omega x_1/r_1^3,\\
h_{11} &= -2m/r_1 + Ar_0^2\omega^2/7r_1^5\cdot(x_1^2 - x_2^2),\\
h_{22} &= -2m/r_1 - Ar_0^2\omega^2/7r_1^5\cdot(x_1^2 - x_2^2),\\
h_{33} &= -2m/r_1 - Ar_0^2\omega^2/7r_1^3[2/3 - (x_1^2+x_2^2)/r_1^2],\\
h_{12} &= 2Ar_0^2\omega^2x_1x_2/7r_1^5.
\end{aligned}\right\} \tag{2·5}$$

§ 3. Equations of motion of spinning test-particles

The co-ordinates employed here are approximately isotropic rectangular co-ordinates, the corresponding isotropic polar co-ordinates are obtained by putting $x_1 = r_1\sin\theta\cos\phi$, $x_2 = r_1\sin\theta\sin\phi$, $x_3 = r_1\cos\theta$.

Accordingly, the fundamental tensor of the gravitational manifold transforms into

$$\left.\begin{aligned}
g_{11} &= -1 - 2m/r_1 - Ar_0^2\omega^2/7r_1^3(1 - 5/3\cdot\cos^2\theta),\\
g_{22} &= -r_1^2(1 + 2m/r_1 - Ar_0^2\omega^2/21r_1^3\cdot\sin^2\theta),\\
g_{33} &= -r_1^2\sin^2\theta(1 + 2m/r_1 + Ar_0^2\omega^2/7r_1^3\cdot\sin^2\theta),\\
g_{44} &= 1 - 2m/r_1 + Ar_0^2\omega^2/7r_1^3(2/3 - \sin^2\theta),\\
g_{12} &= Ar_0^2\omega^2/21r_1^3\cdot\sin 2\theta,\ g_{34} = 2A\omega/r_1\cdot\sin^2\theta.
\end{aligned}\right\} \tag{3·1}$$

Comparing the above set with the corresponding one given by Rayner[10] (equ. (5·3) of his paper), we conclude that Einstein's theory differs from that of Whitehead in the terms containing ω^2. Therefore, perihelion shift, bending of light rays, and other astronomical effects in the field of a rotating sphere will be slightly different from the two theories. It will be worth mentioning here that recently Clark[11] has shown that these two theories do not agree again on the acceleration of the c.g. in a two body system—an effect primarily depending upon the terms beyond the first order.

Now, neglecting terms from $O[m^2/r_1^4]$, the non-vanishing Christoffel's symbols of the second kind are

$$
\begin{aligned}
&\left\{\begin{matrix}1\\11\end{matrix}\right\}=-\tfrac{1}{2}[2m/r_1^2-4m^2/r_1^3+3Ar_0^2\omega^2/7r_1^4\cdot(1-5/3\cdot\cos^2\theta)], && \left\{\begin{matrix}2\\34\end{matrix}\right\}=2A\omega/r_1^3\cdot\sin\theta\cos\theta,\\
&\left\{\begin{matrix}1\\22\end{matrix}\right\}=-r_1[1-m/r_1+2m^2/r_1^2+Ar_0^2\omega^2/7r_1^3\cdot(2/3-3/2\cdot\sin^2\theta)], && \left\{\begin{matrix}3\\24\end{matrix}\right\}=-2A\omega/r_1^3\cdot\cot\theta,\\
&\left\{\begin{matrix}1\\33\end{matrix}\right\}=-r_1\sin^2\theta[1-m/r_1+2m^2/r_1^2-3Ar_0^2\omega^2/14r_1^3\cdot\sin^2\theta], && \left\{\begin{matrix}1\\34\end{matrix}\right\}=-A\omega/r_1^2\cdot\sin^2\theta,\\
&\left\{\begin{matrix}1\\44\end{matrix}\right\}=1/2[2m/r_1^2-4m^2/r_1^3-3Ar_0^2\omega^2/7r_1^4\cdot(2/3-\sin^2\theta)], && \left\{\begin{matrix}3\\14\end{matrix}\right\}=A\omega/r_1^4,\\
&\left\{\begin{matrix}2\\12\end{matrix}\right\}=1/r_1[1-m/r_1+2m^2/r_1^2+Ar_0^2\omega^2/14r_1^3\cdot\sin^2\theta], && \left\{\begin{matrix}4\\13\end{matrix}\right\}=-3A\omega/r_1^2\cdot\sin^2\theta,\\
&\left\{\begin{matrix}3\\13\end{matrix}\right\}=-1/r_1[1-m/r_1+2m^2/r_1^2-3Ar_0^2\omega^2/14r_1^3\cdot\sin^2\theta], && \left\{\begin{matrix}2\\22\end{matrix}\right\}=Ar_0^2\omega^2/21r_1^3\cdot\sin\theta\cos\theta,\\
&\left\{\begin{matrix}4\\14\end{matrix}\right\}=1/2[2m/r_1^2+4m/r_1^3-3Ar_0^2\omega^2/7r_1^4\cdot(2/3-\sin^2\theta)], && \left\{\begin{matrix}2\\44\end{matrix}\right\}=-Ar_0^2\omega^2/7r_1^5\cdot\sin\theta\cos\theta,\\
&\left\{\begin{matrix}2\\33\end{matrix}\right\}=-\sin\theta\cos\theta\,[1+2Ar_0^2\omega^2/7r_1^3\cdot\sin^2\theta], && \left\{\begin{matrix}4\\24\end{matrix}\right\}=Ar_0^2\omega^2/7r_1^3\cdot\sin\theta\cos\theta,\\
&\left\{\begin{matrix}3\\23\end{matrix}\right\}=\cot\theta[1+Ar_0^2\omega^2/7r_1^3\cdot\sin^2\theta], && \left\{\begin{matrix}2\\11\end{matrix}\right\}=-Ar_0^2\omega^2/21r_1^5\cdot\sin\theta\cos\theta,\\
& && \left\{\begin{matrix}1\\12\end{matrix}\right\}=-3Ar_0^2\omega^2/7r_1^3\cdot\sin\theta\cos\theta.
\end{aligned}
\qquad (3\cdot2)
$$

The sets (3·1), (3·2) can easily be transformed into spherical polar co-ordinates by the substitution

$$r=(1+m/2r_1)^2 r_1\,. \tag{3·3}$$

Now, the equation of spinning test-particles in the basic field $g_{\mu\nu}$, as given by Papapetrou[6], is

$$\frac{D}{Ds}\left(mu^\alpha+u_\beta\frac{DS^{\alpha\beta}}{Ds}\right)+1/2\;S^{\mu\nu}\,u^\sigma\,R^\alpha_{\nu\sigma\mu}=0, \tag{3·4}$$

where the tensor $S^{\alpha\beta}$ represents the spin angular momentum of the test-particles, u^α and $R^\alpha_{\nu\sigma\mu}$ as usual denote the velocity and R-C curvature respectively. The quantity $DA_{\cdots}/Ds$ is defined as follows. Let a tensor $A_{\cdots}$ be given on a line L; then $A_{\cdots}$ is extended (arbitrarily) to a tensorfield in a small tube surrounding L. Now denoting by $(A_{\cdots})_{;\nu}$ the covariant derivative of this field, one writes

$$\frac{DA_{\cdots}}{Ds}=(A_{\cdots})_{;\nu}u^\nu.$$

This formula shows directly that $(DA_{\cdots}/Ds)$ will be a tensor.

Inserting the transformed values of (3·2) into (3·4), one finds after some elementary calculations

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2-r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2+m/r^2-m/r^2\left\{3\left(\frac{dr}{dt}\right)^2-2r^2\left(\frac{d\theta}{dt}\right)^2-2r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right\}$$

$$-2m^2/r^3-2A\omega/r^2\cdot\sin^2\theta\frac{d\phi}{dt}-3Ar_0^2\omega^2/14r^4\cdot(2/3-\sin^2\theta)-6Ar_0^2\omega^2/7r^3\cdot\sin\theta\cos\theta\;\frac{dr}{dt}\;\frac{d\phi}{dt}$$

$$-3Ar_0^2\omega^2/14r^4\cdot(1-5/3\cdot\cos^2\theta)\left(\frac{dr}{dt}\right)^2-Ar_0^2\omega^2/7r^2(2/3-3/2\cdot\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2$$

$$+3/14\cdot Ar_0^2\omega^2/r^2\cdot\sin^4\theta\left(\frac{d\phi}{dt}\right)^2=0, \tag{3·4a}$$

$$\frac{d^2\theta}{dt^2}+2/r\frac{dr}{dt}\;\frac{d\theta}{dt}-\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2-2m/r^2\frac{dr}{dt}\;\frac{d\theta}{dt}+4A\omega/r^3\cdot\sin\theta\cos\theta\frac{d\phi}{dt}$$

$$+Ar_0^2\omega^2/21r^3\cdot\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2-2Ar_0^2\omega^2/7r^3\cdot\sin^3\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2-Ar_0^2\omega^2/21r^5\cdot\sin\theta\cos\theta$$

$$\times\left(\frac{dr}{dt}\right)^2-\frac{Ar_0^2\omega^2}{7r^5}\sin\theta\cos\theta$$

$$+\frac{Ar_0^2\omega^2}{7r^4}\sin^2\theta\;\frac{dr}{dt}\;\frac{d\theta}{dt}+\frac{3m}{r^2E}\left[r(1-2m/r)\sin^2\theta\frac{d\phi}{dt}S^{23}+1/r\frac{dr}{dt}\;S^{12}\right]=0, \tag{3·4b}$$

$$\frac{d^2\phi}{dt^2}+2/r\frac{dr}{dt}\;\frac{d\phi}{dt}+2\cot\theta\;\frac{d\theta}{dt}\;\frac{d\phi}{dt}-2m/r^2\frac{dr}{dt}\;\frac{d\phi}{dt}-4A\omega/r^3\cdot\cot\theta\frac{d\theta}{dt}+2A\omega/r^4\frac{dr}{dt}$$

$$+2Ar_0^2\omega^2/7r^3\cdot\sin\theta\cos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}-3Ar_0^2\omega^2/7r^4\cdot\sin^2\theta\frac{dr}{dt}\frac{d\phi}{dt}-3m/r^2E$$

$$\times\left[r(1-2m/r)\frac{d\theta}{dt}S^{23}+1/r\frac{dr}{dt}S^{31}\right]=0, \qquad (3\cdot4c)$$

where E represents the integral of energy. It is to be noted here that the spin-spin product terms are omitted for the sake of simplicity.

§ 4. The advance of perihelion of an equatorial orbit

We shall now discuss whether the set of equations (3·4) admits of any solution representing a motion in a plane passing through the rotating central body. The general condition comes out to be that such a motion is possible only in the plane $\theta=\pi/2$ and with $S^{23}=S^{12}=0$, and $rS^{13}=S_z\neq0$ i.e. the central spin and planetary spin must be perpendicular to the plane of motion and consequently parallel or antiparallel to each other.

For the motion in the equatorial plane we get from (3·4a), (3·4c)

$$\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2+m/r^2=R=m/r^2\left\{3\left(\frac{dr}{dt}\right)^2-2r^2\left(\frac{d\phi}{dt}\right)^2\right\}+2m^2/r^3+2A\omega/r^2\frac{d\phi}{dt}$$

$$+Ar_0^2\omega^2/14r^4+3Ar_0^2\omega^2/14r^4\cdot\left(\frac{dr}{dt}\right)^2-3Ar_0^2\omega^2/14r^2\cdot\left(\frac{d\phi}{dt}\right)^2, \quad (4\cdot1a)$$

$$r\frac{d^2\phi}{dt^2}+2\frac{dr}{dt}\frac{d\phi}{dt}=\Phi=2m/r\frac{dr}{dt}\frac{d\phi}{dt}-2A\omega/r^3\frac{dr}{dt}+3Ar_0^2\omega^2/7r^3\frac{dr}{dt}\frac{d\phi}{dt}$$

$$-\frac{3mS_z}{Er^3}\frac{dr}{dt}, \quad (4\cdot1b)$$

where R and Φ denote the radial and transverse perturbations.

From the standard perturbation theory, the rate of perihelion shift is

$$d\tilde{\omega}/dt=h/me\,[-R\cos(\phi-\tilde{\omega})+\Phi(1+mr/h^2)\sin(\phi-\tilde{\omega})], \qquad (4\cdot2)$$

where the instantaneous elliptic orbit is given by $r^{-1}=m/h^2[1+e\cos(\phi-\tilde{\omega})]$. Using this Newtonian approximation in the perturbation terms and writing $(\phi-\tilde{\omega})=\xi$, eqn. (4·2) goes over into

$$\frac{d\tilde{\omega}}{dt}=1/e[m^2/h^2\cdot(2e+2e\sin^2\xi+2e^2\cos^3\xi-e^2\sin^2\xi\cos\xi)$$

$$-2A\omega m/h^3\cdot\{2e+(1+e^2)\cos\xi\}$$

$$-Ar_0^2\omega^2m/14h^4\cdot(\cos\xi+2e\cos^2\xi+e^2\cos^3\xi)$$

$$+3Ar_0^2\omega^2m^3/14h^6\cdot\{4e+(1+4e+2e^2)\cos\xi+4e^3\cos^2\xi$$

$$+3e^2\sin^2\xi\cos\xi+e^4\cos^3\xi+2e^3\cos^2\xi\sin^2\xi\}$$

$$-3S_zm^2/Eh^3\cdot(2e\sin^2\xi+e^2\sin^2\xi\cos\xi)]\frac{d\xi}{dt}. \qquad (4\cdot3)$$

Treating $S_z (\simeq (1-2m/r)^{-1} S_z)$ as approximate constant of motion, perihelion shift per revolution becomes

$$\Delta\tilde{\omega} = \int_{\tilde{\varphi}=0}^{\tilde{\varphi}=2\pi} d\tilde{\omega} = 6\pi\, m^2/h^2 - 8\pi A\omega\, m/h^3 - \pi A r_0^2\, \omega^2 m/7h^4$$
$$+ 12\pi(1+9e^2/32) A r_0^2 \omega^2 m^3/7h^6 - 6\pi S_z m^2/Eh^3.$$

Reinstating c, perihelion advance per revolution occurs (in circular measure)

$$\Delta\tilde{\omega} = 24\pi^3 a^2/c^2T^2(1-e^2) - 32\pi^2 r_0^2 \omega_\perp/5c^2T(1-e^2)^{3/2} - 2\pi r_0^4 \omega_\perp^2/35c^2a^2(1-e^2)^2$$
$$+ 96\pi^3(1+9e^2/32) r_0^4 \omega_\perp^2/35c^4T^2(1-e^2)^3 - 24\pi^2 r_p^2 \omega_p/5c^2T(1-e^2)^{3/2}, \quad (4\cdot4)$$

where,

$a =$ the semi-major axis of the planetary orbit,

$e =$ the numerical eccentricity,

$c =$ the velocity of light in vacuo,

$T =$ the period of revolution,

$r_0 =$ the radius of the central sphere,

$\omega_\perp =$ the normal component of the central spin w. r. t. orbital plane,

$r_p =$ the radius of the planet,

$\omega_p =$ the normal component of the planetary spin,

The last four terms in (4·4) denote the spin effects. The central spin-corrections are significant only for certain satellite orbits in some planet-moon system. The first order central spin term was first derived by Lense and Thirring[1], and is of almost similar form as that of the planetary spin-term.

Lense and Thirring[1] in their calculation omitted the terms of $O|V^2|$ (where V denotes the maximum velocity of a planet in its approximately elliptical orbit), which is actually of the order of the Einstein advance of perihelion. They counteract this omission by adding the Einstein term afterwards, a questionable procedure, although their final result may well be correct. It is better to adopt a more exact method of calculation, such as that leading to equation (4·4).

§ 5. Deflection of photon-track

The law of propagation of light is characterized by the equation

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = 0. \quad (5\cdot1)$$

To within the approximation which we are using, and in the system of co-ordinates which we have selected, the velocity of light is characterized, according to (5·1) & (3·1) by the relation

$$L=\frac{(dr^2+r_1^2d\phi^2)^{1/2}}{dt}\simeq 1-2m/r_1+2A\omega/r_1^3\left(x_1\frac{dx_2}{dt}-x_2\frac{dx_1}{dt}\right)$$
$$-2Ar_0^2\omega^2/21r_1^3. \qquad (5\cdot 2)$$

Now, for a ray of light, travelling parallel to the x_2 axis in the x_1-x_2 plane, at a distance of r_0 from the origin, we have approximately

$$\left.\begin{aligned} &x_1\simeq r_0\\ &\frac{dx_2}{dt}\simeq 1-m/\sqrt{r_0^2+t^2}\cdot(2-r_0^2/r_0^2+t^2)\simeq 1\\ &x_3=0. \end{aligned}\right]$$

Applying these results, (5·2) becomes

$$L=1-2m/r_1+2Ar_0\omega/r_1^3-2Ar_0^2\omega^2/r_1^3. \qquad (5\cdot 3)$$

We can therefore draw the conclusion from this that the ray of light will be deflected, in all, to the extent $\alpha=\int_{-\infty}^{\infty}\frac{1}{L}\frac{\partial L}{\partial x_1}dx_2$ towards the sun.

On performing the integration we get

$$\alpha=4m/r_0-8A\omega/r_0^2+8A\omega^2/21r_0. \qquad (5\cdot 4)$$

When the spin of the photon is taken into account (the amount of spin is assumed to be $\hbar$ and along the x_3 axis), the deflection goes over into

$$\alpha=4/r_0[m-2A\omega/r_0+2A\omega^2/21][1-\lambda/4\pi r_0],$$

causing a slight dispersion

$$\frac{d\alpha}{d\lambda}=-\frac{1}{\pi r_0^2}[m-2A\omega/r_0+2A\omega^2/21].$$

From (5·4) it is clear that light will bend differently in two opposite sides of the solar axis. This difference is $32GM\omega/5c^3$ radians. Relative magnitude of this correction w.r.t. the ordinary relativistic term is $8r_0\omega/5c\simeq 1.06\times 10^{-5}$ and therefore negligible.

§ 6. Slow rotation of the planetary plane

Let the instantaneous orbital plane be one ϕ-constant plane, so that $(d\phi/dt)$ is negligible compared to $(d\theta/dt)$. Moreover, let the major axis of the approximate ellipse be along x_3-axis, and the non-vanishing component of planetary-spin be $rS^{13}=S_z$. With the above simplifying assumptions, from (3·4c)

$$r\sin\theta\left[\frac{d^2\phi}{dt^2}+2/r\,\frac{dr}{dt}\,\frac{d\phi}{dt}+2\cot\theta\frac{d\theta}{dt}\,\frac{d\phi}{dt}\right]=\Phi\simeq-2A\omega/r^3\cdot\sin\theta\,\frac{dr}{dt}$$
$$+4A\omega/r^2\cdot\cos\theta\frac{d\theta}{dt}-3m/Er^3S_z\cdot\sin\theta\,\frac{dr}{dt}, \qquad (6\cdot 1)$$

where Φ is now acceleration orthogonal to the orbital plane.

Using the standard perturbation theory, the planetary plane will rotate at the rate (See Fig. 1)

$$\left.\begin{aligned} d\Omega/dt &= (\Phi r/h)\cdot \sin\theta, \\ di/dt &= (\Phi r/h)\cdot \cos\theta. \end{aligned}\right\} \qquad (6\cdot 2)$$

Because the instantaneous ellipse is given by $r^{-1}=m/h^2\cdot(1+e\cos\theta)$, from $(6\cdot 1)$ and $(6\cdot 2)$

$$\left.\begin{aligned} &\Delta\Omega = 0, \\ \text{and}\quad &\Delta i = 16\pi r_0^2\omega_{11}/5c^2T(1-e^2)^{3/2} \text{ radians/revolution} \end{aligned}\right\}, \qquad (6\cdot 3)$$

where ω_{11} is parallel component of the central spin w.r.t. the elliptic plane. The spin of test-particle, in this specific orientation, does not contribute to the rotation of orbital plane.

The planetary plane is gradually tilted until the same is oriented normal to the central spin. This suggests an explanation of the fact that almost all the planetary orbits cut the solar axis normally.

§ 7. Determination of the perturbation on the six Newtonian orbital elements

Let a and e be the semi-major axis and eccentricity of the instantaneous elliptical orbit of a planet moving (not necessarily in the equatorial plane) about the central sphere. Let the plane of the instantaneous ellipse intersect the celestial sphere (of centre O) in the great circle $NIIN'$ making angle $i(0\leqq i\leqq\pi)$ with the equatorial plane. Let the angle between the arbitrarily chosen initial line ON joining O to the node N be Ω; let β be the angle $NOII$ where II is the projection of the perihelion on the celestial sphere, and let ψ and χ be respectively the (current) angles NOP and $IIOP$. Finally let R be the component of the disturbing force along the current radius-vector OP, T be the component tangential to the projected orbit (in the direction of ψ increasing), and W be the component orthogonal to the orbital plane, so that R, T, W forms a right-handed triad.

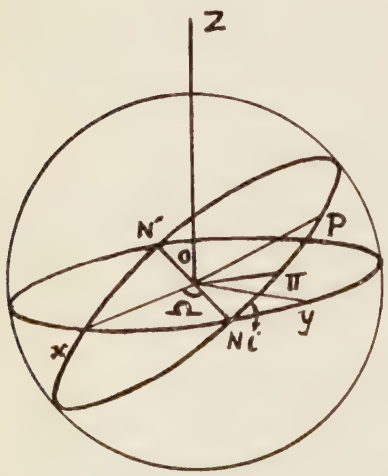

Fig. 1

We shall now derive the perturbation forces R, T, W from the equations of motion which

we have already calculated. Our method is essentially similar to that followed by Rayner[10].

Writing $\frac{dr}{dt}=V_r$, $r\frac{d\theta}{dt}=V_\theta$, $r\sin\theta\frac{d\phi}{dt}=V_\phi$ & $2A\omega=K$, from the set of equations (3·4), the perturbing accelerations come out to be

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2-r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2+m/r^2=R=m/r^2\cdot(3V_r^2-2V_\theta^2-2V_\phi^2)+2m^2/r^3$$
$$+K\sin\theta/r^3\cdot V_\phi+3Kr_0^2\omega/28r^4\cdot\left(\frac{2}{3}-\sin^2\theta\right)$$
$$+3Kr_0^2\omega/7r^4\cdot\sin\theta\cos\theta V_r V_\theta+3Kr_0^2\omega/28r^4\cdot(1-5/3\cdot\cos^2\theta)V_r^2$$
$$+Kr_0^2\omega/14r^4\cdot(2/3-3/2\sin^2\theta)V_\theta^2-3Kr_0^2\omega/28r^4\cdot\sin^2\theta V_\phi^2 \qquad (7\cdot1a)$$

$$r\left[\frac{d^2\theta}{dt^2}+2/r\frac{dr}{dt}\frac{d\theta}{dt}-\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right]=\Theta=2m/r^2\cdot V_r V_\theta-2K/r^3\cdot\cos\theta V_\phi$$
$$-Kr_0^2\omega/42r^4\cdot\sin\theta\cos\theta V_\theta^2+Kr_0^2\omega/7r^4\cdot\sin\theta\cos\theta V_\phi^2+Kr_0^2\omega/42r^4\cdot\sin\theta\cos\theta V_r^2$$
$$-Kr_0^2\omega/14r^4\cdot\sin^2\theta V_r V_\theta$$
$$+Kr_0^2\omega/14r^4\cdot\sin\theta\cos\theta-\frac{3m}{Er}[(1-2m/r)\sin\theta V_\phi S^{23}+1/r\cdot V_r S^{12}], \qquad (7\cdot1b)$$

$$r\sin\theta\left[\frac{d^2\phi}{dt^2}+2/r\frac{dr}{dt}\frac{d\phi}{dt}+2\cot\theta\frac{d\theta}{dt}\frac{d\phi}{dt}\right]=\Phi=2m/r^2\cdot V_r V_\phi+2K\cos\theta/r^3\cdot V_0$$
$$-K\sin\theta/r^3\cdot V_r-Kr_0^2\omega/7r^4\cdot\sin\theta\cos\theta V_\theta V_\phi+3Kr_0^2\omega/14r^4\cdot\sin^2\theta V_r V_\phi$$
$$+3m\sin\theta/Er\cdot[(1-2m/r)V_\theta S^{23}+1/r\cdot V_r S^{31}], \qquad (7\cdot1c)$$

where R is the radial perturbation, Θ and Φ are components of acceleration in the directions tangential to the parametric line of θ and ϕ increasing.

An examination of fig. 1 will show that, if ν be the angle between the component directions Θ and T of the perturbing forces, then

$$\begin{bmatrix}R\\T\\W\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos\nu&\sin\nu\\0&-\sin\nu&\cos\nu\end{bmatrix}\begin{bmatrix}R\\\Theta\\\Phi\end{bmatrix}. \qquad (7\cdot2)$$

Also, from the elementary theory of spherical triangles

$$\left.\begin{aligned}&\sin\theta\cos\nu=-\cos\psi\sin i,\\&\sin\theta\sin\nu=\cos i,\\&\cos\theta=\sin\psi\sin i,\\&\tan(\phi-\Omega)=\tan\psi\cos i.\end{aligned}\right\} \qquad (7\cdot3)$$

Now, denoting the orbit of the planet by $r^{-1}=m/h^2\cdot(1+e\cos\chi)$ and total angular momentum by $\frac{1}{2}r^2\frac{d\chi}{dt}=\frac{1}{2}r^2\frac{d\psi}{dt}=\frac{1}{2}h$,

we obtain from (7·1), (7·2) and (7·3),

$$R = 3m^3e^2/h^2r^2 \cdot \sin^2\chi - 2mh^2/r^4 + 2m^2/r^3 + Kh\cos i/r^4$$
$$+ 3Kr_0^2\omega/7r^4 \cdot [1/6 - 1/4 \cdot (1 - \sin^2 i \sin^2\psi) - me/2r \cdot \sin\chi \sin 2\psi \sin^2 i + m^2e^2/4h^2 \cdot \sin\chi$$
$$\times (1 - 5/3 \cdot \sin^2\psi \sin^2 i)$$
$$+ h^2/6r^2(1 - \sin^2 i \sin^2\psi) \cdot \{2/3 \cdot \sin^2 i \cos^2\psi - 3/2 \cdot (1 - \sin^2 i \sin^2\psi(1 + \sin^2 i)\}], \quad (7\cdot 4a)$$

$$T = 2m^2e/r^3 \cdot \sin\chi - Kme/hr^3 \cdot \cos i \sin\chi + Kr_0^2\omega/7r^4 \cdot \sin^2 i\,[h^2/6r^2 \cdot \cos^3\psi \sin\psi \sin^2 i$$
$$\times 1/(1 - \sin^2 i \sin^2\psi) + me \sin\chi/2r \cdot (3\cot^2 i - \cos^2\psi) - \tfrac{1}{2}\sin\psi\cos\psi(1 + 1/3 \cdot m^2e^2\sin^2\chi/h^2)]$$
$$+ 3m^2e/Ehr^2 \cdot \sin\chi[\cos i \cdot S^{31} + \sin i \cos\psi S^{12}/\sqrt{1 - \sin^2 i \sin^2\psi}], \quad (7\cdot 4b)$$

$$W = Kh \sin i/r^3 \cdot [em/h^2 \cdot \cos\psi \sin\chi + 2/r \cdot \sin\psi]$$
$$- Kr_0^2\omega \sin 2i/14r^4 \cdot [2me/r \cdot \sin\chi \cos\psi + \sin\psi/2 \cdot (1 + m^2e^2/3h^2 \cdot \sin^2\chi)$$
$$+ h^2/r^2 \cdot \sin\psi/(1 - \sin^2 i \sin^2\psi) \cdot \{\cos^2 i - 5/6 \cdot \sin^2 i \cos^2\psi\}]$$
$$+ 3mh/Er^2 \cdot [(1 - 2m/r)\sqrt{1 - \sin^2 i \sin^2\psi}\; S^{23}$$
$$+ me\sin\chi/h^2 \cdot \{S^{12}\cos i/\sqrt{1 - \sin^2 i \sin^2\psi} - S^{31}\sin i \cos\psi\}]. \quad (7\cdot 4c)$$

The first order central spin terms in the last set of eqns. (7·4) are exactly the same as those obtained by Lense and Thirring[1], Rayner[10].

I record my sincere thanks to Dr. T. C. Roy for some illuminating discussions.

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Low's Scattering Equation and S-Matrix

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Without reference to the details of the interaction, Low's scattering theory is formulated including scatterings of bound particles. The Low equation is derived in two forms: the differential form in the Heisenberg representation, and the form like the second order perturbation formula in the Schrödinger representation. Corresponding to what state we use as the comparison state, there exist various types of the Low equation and expressions of the S-matrix. From these Low equations the symmetry properties of the S-matrix are deduced. Using the differential form of the Low equation, it is shown that the Low equation in which the comparison state is the vacuum can be linearized, and has infinitely many solutions as has been pointed out by Castillejo et al.

§ 1. Introduction

Recently a new formulation of the scattering problem in field theory has been proposed by Low.[1] Chew and Low[2] have applied this theory to the fixed source meson model of Chew and made a great success. They have shown that Low's theory has satisfactory features and advantages in many respects. Although many analyses of this theory have been accomplished, many points about its general character still are left unclear as it differs so much from other theories so far introduced.

On the other hand there have been several attempts to formulate and study the S-matrix theory without reference to the details of the interactions, for example that of Lehmann et al.[3] Though the first motivation of Low's theory has not been along such line, it seems to belong to such category in essential points.

The aim of this paper is to formulate Low's theory from such a point of view, and to investigate immediate consequences for the structure of the S-matrix and the properties of the Low equation. In the following derivations we generalize the theory so as to include scatterings of composite particles and derive the Low equation in the two forms, that is, the differential form in the Heisenberg representation and the form like the second order perturbation formula in the Schrödinger representation. Further we show that there exist various types of the Low equation according to the various possible choices of the comparison state, whereby Wick's[4] technique is applied, and using these Low equations the symmetry properties of the S-matrix are derived. Lastly, we show that one type of the Low equations in which the comparison state is the vacuum can be linearized and in the course of its linearization the same arbitrariness that has been found by Castillejo et al.[5] enters into the linearized equation, therefore there exist infinitely many solutions of the Low equation, and the conclusion of Castillejo et al. holds in general.

§ 2. Assumptions and definitions

We consider a closed physical system consisting of interacting fields. We assume for the system that it has the displacement operator $\boldsymbol{P}_\nu$ which has the property

$$[\boldsymbol{P}_\nu,\ O(x)]=i\partial_\nu O(x) \tag{2·1}$$

for any arbitrary Heisenberg operator $O(x)$ of the system. Also we assume the existence of the vacuum state i.e., in the orthonormal complete eigenstates Ψ_M of $\boldsymbol{P}_\nu$ there exists a state Ψ_0 which has the lowest eigenvalue of $H=-i\boldsymbol{P}_4$. We normalize $\boldsymbol{P}_\nu$ (subtracting the eigenvalue of the vacuum state) as $\boldsymbol{P}_\nu \Psi_0=0$, then all other eigenvalues of H are always positive.

The eigenstates Ψ_M may be divided into two groups in accordance with the boundary conditions of the scattering experiments which we suppose to be specified in the remote past or future. There are a group of states which represent one spatially localized entity and suffer no real scattering. For example, the one particle states and isolated bound states belong to this group. We call such states the steady states. The states belonging to the second group initially contain at least two spatially separated parts and suffer real scattering. We assume that the initial condition for the states of the latter type is expressible by a superposition of the former steady states, i.e. it approaches asymptotically to an independent assembly of the steady state particles in the remote past or future. We assume also that we already know all steady states and their eigenvalues of $\boldsymbol{P}_\nu$. Our theory treats the problem of scattering between the steady state particles, but it does not provide any procedure to know what the steady states and their spectrum are.

To proceed further we take a system consisting of the nucleon field, $\psi(x)$ and $\bar{\psi}(x)$ and the charged meson field, $A(x)$ and $A^\dagger(x)$. We begin by defining the following operators for the steady states:

$$\begin{cases} q_a^{(-)}(\sigma)=N_a^{-1}\int_\sigma \bar{\psi}(x)\gamma_\nu(\Psi_0|\psi(x)|\Psi_a)\,d\sigma_\nu, \\ q_a^{(+)}(\sigma)=N_a^{-1}\int_\sigma(\Psi_a|\bar{\psi}(x)|\Psi_0)\gamma_\nu\psi(x)\,d\sigma_\nu. \end{cases} \tag{2·2}$$

Here Ψ_a is the one nucleon state and we describe its all quantum numbers by a single symbol "a" which includes the three components of momentum, the spin and charge indices, and

$$N_a=\int_\sigma(\Psi_a|\bar{\psi}(x)|\Psi_0)\gamma_\nu(\Psi_0|\psi(x)|\Psi_a)\,d\sigma_\nu.$$

Note that N_a does not depend on σ. Other notations follow that of Schwinger.[1]

For the one meson state Ψ_b,

$$\begin{cases} q_b^{(-)}(\sigma)=N_b^{-1}\int_\sigma A^\dagger(x)\overleftrightarrow{\partial}_\nu(\Psi_0|A(x)|\Psi_b)\,d\sigma_\nu, \\ q_b^{(+)}(\sigma)=N_b^{-1}\int_\sigma(\Psi_b|A^\dagger(x)|\Psi_0)\overleftrightarrow{\partial}_\nu A(x)\,d\sigma_\nu, \end{cases} \tag{2·3}$$

with $$N_b=\int_\sigma(\Psi_b|A^\dagger(x)|\Psi_0)\overleftrightarrow{\partial}_\nu(\Psi_0|A(x)|\Psi_b)\,d\sigma_\nu,$$

and $$f(x)\overleftrightarrow{\partial}_\nu g(x)=\partial_\nu f(x)\cdot g(x)-f(x)\partial_\nu g(x).$$

For the two nucleon bound state Ψ_c,

$$\begin{cases} q_c^{(-)}(\sigma) = N_c^{-1} \int\int : \bar{\psi}(x_1)\bar{\psi}(x_2) : \gamma_\nu^{\,1}\gamma_\mu^{\,2}(\Psi_0| : \psi(x_1)\psi(x_2) : |\Psi_c) d\sigma_\nu^{\,1} d\sigma_\mu^{\,2}, \\ q_c^{(+)}(\sigma) = N_c^{-1} \int\int (\Psi_c| : \bar{\psi}(x_1)\bar{\psi}(x_2) : |\Psi_0)\gamma_\nu^{\,1}\gamma_\mu^{\,2} : \psi(x_1)\psi(x_2) : d\sigma_\nu^{\,1} d\sigma_\mu^{\,2}, \end{cases} \tag{2·4}$$

and $N_c = \int\int_\sigma (\Psi_c| : \bar{\psi}(x_1)\bar{\psi}(x_2) : |\Psi_0)\gamma_\nu^{\,1}\gamma_\mu^{\,2}(\Psi_0| : \psi(x_1)\psi(x_2) : |\Psi_c) d\sigma_\nu^{\,1} d\sigma_\mu^{\,2}$,

here the product of field operators is to be taken in the sense of Nishijima's.[7]

By the same way we define operators $q_s^{(-)}(\sigma)$ and $q_s^{(+)}(\sigma)$ for all steady states Ψ_s.

From above definitions it follows immediately that

$$(\Psi_s|q_s^{(-)}(\sigma)|\Psi_0) = 1, \quad (\Psi_0|q_s^{(+)}(\sigma)|\Psi_s) = 1. \tag{2·5}$$

Taking σ as a flat one and using the relation

$$O(\boldsymbol{r}, t) = e^{itH} O(\boldsymbol{r}, o) e^{-itH}, \tag{2·6}$$

we can separate the time dependence of $q_s^{(\pm)}(t)$ as follows;

$$\begin{cases} q_s^{(-)}(t) = e^{itH} q_s^{(-)}(o) e^{-itH - itE_s}, \\ q_s^{(+)}(t) = e^{itH + itE_s} q_s^{(+)}(o) e^{-itH}. \end{cases} \tag{2·7}$$

Using (2·7) we get

$$(\Psi_M|q_s^{(+)}(\pm\infty)|\Psi_0) = \lim_{t\to\pm\infty} (\Psi_M|q_s^{(+)}(t)|\Psi_0)$$

$$= \lim_{t\to\pm\infty} e^{it(E_M+E_s)}(\Psi_M|q_s^{(+)}(0)|\Psi_0) = 0,$$

for an arbitrary state Ψ_M because $E_M + E_s$ is always positive. Then holds identically

$$q_s^{(+)}(\pm\infty)\Psi_0 = 0. \tag{2·8}$$

Also we get

$$(\Psi_M|q_s^{(-)}(\pm\infty)|\Psi_0) = \lim_{t\to\pm\infty} (\Psi_M|q_s^{(-)}(t)|\Psi_0) = \lim_{t\to\pm\infty} e^{it(E_M - E_s)} (\Psi_M|q_s^{(-)}(o)|\Psi_0),$$

so that it does not vanish only when $E_M = E_s$. From the definition of $q_s^{(-)}$ it can be easily shown that $(\Psi_M|q_s(t)|\Psi_0)$ has nonvanishing value only when $P_{Mi} = P_{si}(i = 1, 2, 3,)$ therefore it must be $P_{M\nu} = P_{s\nu}$ $(\nu = 1, 2, 3, 4)$ in order that $(\Psi_M|q_s^{(-)}(\pm\infty)|\Psi_0)$ does not vanish. However, the energy-momentum conservation or some other conservation of quantum numbers between the steady state and the other states fails, therefore it is the case only when $\Psi_M = \Psi_s$ that $(\Psi_M|q_s^{(-)}(\pm\infty)|\Psi_0)$ does not vanish.

From this and (2·5) we get

$$(\Psi_M|q_s^{(-)}(\pm\infty)|\Psi_0) = \delta_{M,s},$$

or

$$q_s^{(-)}(\pm\infty)\Psi_0 = \Psi_s. \tag{2·9}$$

Thus (2·8) and (2·9) show that $q_s^{(-)}(\pm\infty)$ and $q_s^{(+)}(\pm\infty)$ are the creation and annihilation operators of the steady state particles respectively. This interpretation does not hold when $t\neq\pm\infty$, but we use $q_s^{(\pm)}(t)$ as the auxiliary quantities. (2·8), (2·9) and (2·7) show that we can put the commutation relation of $q_s^{(-)}(t)$ and $q_s^{(+)}(t)$ as

$$\begin{cases} [q_s^{(+)}(t),\ q_{s'}^{(-)}(t)]_{\pm}=\delta_{ss'}, \\ [q_s^{(+)}(t),\ q_{s'}^{(+)}(t)]_{\pm}=[q_s^{(-)}(t),\ q_{s'}^{(-)}(t)]_{\pm}=0, \end{cases} \tag{2·10}$$

where the plus commutator corresponds to the states s and s' containing the odd numbers of fermion, the minus commutator to the states containing the even numbers of fermion.

Next we consider the time derivatives of $q_s^{(\pm)}(t)$. From (2·7)

$$\begin{cases} \dfrac{\partial q_s^{(-)}(t)}{\partial t}=i\{[H,\ q_s^{(-)}(t)]_- -E_s\, q_s^{(-)}(t)\}=iV_s^{(-)}(t), \\ \dfrac{\partial q_s^{(+)}(t)}{\partial t}=i\{[H,\ q_s^{(+)}(t)]_- +E_s\, q_s^{(+)}(t)\}=-iV_s^{(+)}(t). \end{cases} \tag{2·11}$$

The forms of time dependence of $V_s^{(-)}(t)$ and $V_s^{(+)}(t)$ are the same as that of $q_s^{(-)}(t)$ and of $q_s^{(+)}(t)$ respectively. The expressions of $V_s^{(\pm)}(t)$ can be obtained easily by using the lemma:

$$\frac{\delta}{\delta\sigma(x)}\int_\sigma O(x)\,d\sigma_\nu=\partial_\nu O(x).$$

For example for the one particle states

$$\begin{cases} V_a^{(-)}(t)=N_a^{-1}\int_t(\gamma_\nu^T\partial_\nu-m_a)\bar{\psi}(x)\cdot(\Psi_0|\psi(x)|\Psi_a)\,d^3x, \\ V_b^{(-)}(t)=N_b^{-1}\int_t(\Box-m_b^2)A(x)\cdot(\Psi_0|A(x)|\Psi_b)\,d^3x, \end{cases} \tag{2·12}$$

where m_a and m_b are the physical masses of the nucleon and the meson and where we have used the fact that $(\Psi_0|\psi(x)|\Psi_a)$ and $(\Psi_0|A(x)|\Psi_b)$ statisfy the Dirac and the Klein-Goldon equations respectively.

§ 3. Scattering state and definition of the S-matrix

In this and next sections we consider the one nucleon-one meson scattering state Ψ_{ab} as an example. Ψ_{ab} satisfies the Schrödinger equation

$$H\Psi_{ab}=(E_a+E_b)\Psi_{ab}.$$

Following Wick[5] we decompose Ψ_{ab} into two parts. Though Low has taken the one nucleon state, as the comparison state his choice is not the only possible one. In the relativistic theory there exist still other possible choices, i.e.

$$\begin{cases} \text{i)} & \Psi_{ab}=q_a^{(-)}(t)\,q_b^{(-)}(t)\,\Psi_0+\chi_{ab}(t), \\ \text{ii)} & \Psi_{ab}=q_b^{(-)}(t)\,\Psi_a+\chi'_{ab}(t), \\ \text{iii)} & \Psi_{ab}=q_a^{(-)}(t)\,\Psi_b+\chi_{ab}''(t). \end{cases} \tag{3·1}$$

The second one is what Low has used. In the right side of (3·1) the first terms may be interpreted as the incident wave, and the second terms, which we determine later, as the scattering wave. In the static nucleon model the case ii) is the only possible one, in the relativistic theory they have equal validity and corresponding to these possibilities there exist various types of the Low equation and expressions of the S-matrix. From the physical point of view the case ii) in which non-relativistic limit can be easily taken is the most important one, but as will be shown in the later sections the case i) has the advantage of more simplicity compared with the others. In the following derivations we treat the case i) mainly for the later use, and for the other cases write the final results only.

Let us determine $\chi_{ab}(t)$. As Ψ_{ab} is the state vector in the Heisenberg representation, it does not depend on t. Therefore differentiating both sides of (3·1)-(i) by t, we get

$$\frac{\partial \Psi_{ab}}{\partial t}=\frac{\partial}{\partial t}(\boldsymbol{q}_{ab}^{(-)}(t))\,\Psi_0+\frac{\partial}{\partial t}\chi_{ab}(t)=0,$$

here $\boldsymbol{q}_{ab}^{(-)}(t)=q_a^{(-)}(t)\;q_b^{(-)}(t)$.

According to whether the boundary condition of Ψ_{ab} is specified in the remote past or future, we have

$$\chi_{ab}^{(+)}(t)=-\int_{-\infty}^{t}\frac{\partial \boldsymbol{q}_{ab}^{(-)}(t)}{\partial t}\;dt\;\Psi_0, \tag{3·2}$$

$$\chi_{ab}^{(-)}(t)=\int_{t}^{\infty}\frac{\partial \boldsymbol{q}_{ab}^{(-)}(t)}{\partial t}\;dt\;\Psi_0. \tag{3·3}$$

So that

$$\Psi_{ab}^{+}=\boldsymbol{q}_{ab}^{(-)}(t)\,\Psi_0-\int_{-\infty}^{t}\frac{\partial \boldsymbol{q}_{ab}^{(-)}(t)}{\partial t}\;dt\;\Psi_0=\boldsymbol{q}_{ab}^{(-)}(-\infty)\,\Psi_0, \tag{3·4}$$

$$\Psi_{ab}^{-}=\boldsymbol{q}_{ab}^{(-)}(t)\,\Psi_0+\int_{t}^{\infty}\frac{\partial \boldsymbol{q}_{ab}^{(-)}(t)}{\partial t}dt\;\Psi_0=\boldsymbol{q}_{ab}^{(-)}(+\infty)\,\Psi_0. \tag{3·5}$$

Using (2·7) and (2·11), (3·4) becomes

$$\Psi_{ab}^{+}=\boldsymbol{q}_{ab}^{(-)}(t)\,\Psi_0-i\int_{-\infty}^{t}\boldsymbol{V}_{ab}^{(-)}(t)\;dt\,\Psi_0$$

$$=\boldsymbol{q}_{ab}^{(-)}(t)\,\Psi_0+\frac{e^{\varepsilon t\,(H-E_a-E_b)}}{E_a+E_b-H+i\epsilon}\boldsymbol{V}_{ab}^{(-)}(0)\;\Psi_0, \tag{3·6}$$

here $V_{ab}^{(-)}(t)=V_a^{(-)}(t)q_b^{(-)}(t)+q_b^{(-)}(t)V_b^{(-)}(t)$ and ϵ is an infinitesimal real positive number. Also we get for Ψ_{ab}^{-} an expression where the sign of ϵ is reversed. In the expressions (3·4) and (3·6) t is quite arbitrary, we may take t as $+\infty$. So we get

$$\Psi_{ab}^{+}=\boldsymbol{q}_{ab}^{(-)}(\infty)\Psi_0-i\int_{-\infty}^{\infty}V_{ab}^{(-)}(t)\,dt\Psi_0$$
$$=\Psi_{ab}^{-}-2\pi i\delta(H-E_a-E_b)V_{ab}^{(-)}(0)\Psi_0, \tag{3·7}$$

here we have used (3·5).

The next problem is to correlate the above results to the S-matrix. We define the S-matrix in the usual way as

$$S_{M,N}=(\Psi_M^{-}|\Psi_N^{+}). \tag{3·8}$$

This can be written as

$$S_{M,N}=\lim_{t\to-\infty}(\Psi_M^{-}|\boldsymbol{q}_M^{(-)}(t)|\Psi_0)=\lim_{t\to+\infty}(\Psi_0|\boldsymbol{q}_M^{(-)}(t)|\Psi_N^{+}), \tag{3·9}$$

here $N=(a^{n_a},\ b^{n_b},\cdots)$ and $\boldsymbol{q}_N^{(-)}(t)=\dfrac{1}{\sqrt{n_a!\,n_b!\cdots}}[q_a^{(-)}(t)]^{n_a}[q_b^{(-)}(t)]^{n_b}\cdots$.

Substituting the precise expression of $q_\lambda^{(\pm)}(t)$, it can easily be shown that the expression (3·9) for the S-matrix coincides with that of Nishijima[6] even when the initial and final states include bound particles. Next substituting (3·7) in (3·8), we get

$$S_{M,ab}=(\Psi_M^{-}|\Psi_{ab}^{+})=(\Psi_M^{-}|\Psi_{ab}^{-})-2\pi i\delta(E_M-E_a-E_b)(\Psi_M^{-}|V_{ab}^{(-)}(0)|\Psi_0)$$
$$=\delta_{M,ab}-2\pi i\delta(E_M-E_a-E_b)(\Psi_M^{-}|V_{ab}^{(-)}(0)|\Psi_0). \tag{3·10}$$

Thus we have got the expression of the S-matrix for the case i).

For the other cases we get the following results in the similar manner.
For the case ii)

$$\Psi_{ab}^{+}=q_b^{(-)}(t)\Psi_a+\frac{e^{i\epsilon t(H-E_a-E_b)}}{E_a+E_b-H+i\epsilon}V_b^{(-)}(0)\Psi_a \tag{3·11}$$
$$=\Psi_{ab}^{-}-2\pi i\delta(H-E_a-E_b)V_b^{(-)}(0)\Psi_a,$$
$$S_{M,ab}=\delta_{M,ab}-2\pi i\delta(E_M-E_a-E_b)(\Psi_M^{-}|V_b^{(-)}(0)|\Psi_a). \tag{3·12}$$

For the case iii) "a" and "b" are exchanged in (3·11) and (3·12).

The equivalence of (3·10) and (3·12) is shown in the appendix. Though the various expressions of Ψ_{ab} have the same value, their scattered parts by themselves do not.

§ 4. Derivation of the Low equation and symmetry properties of the S-matrix

We begin by defining the following quantities:

$$U_{M,ab}(t)=(\Psi_M^{-}|\boldsymbol{q}_{ab}^{(-)}(t)|\Psi_0), \tag{4·1}$$

$$W_{M,ab}(t) = (\Psi_M^- | q_{ab}^{(+)}(t) | \Psi_0). \tag{4·2}$$

In the limits $t \to \pm\infty$ they become

$$U_{M,ab}(+\infty) = \delta_{M,ab}, \quad W_{M,ab}(+\infty) = 0, \tag{4·3}$$

$$U_{M,ab}(-\infty) = S_{M,ab}, \quad W_{M,ab}(-\infty) = 0, \tag{4·4}$$

where we have used (2·8) and (3·8). Using the relation

$$q_{ab}^{(-)}(t)\Psi_0 = \Psi_{ab}^- - \frac{e^{it(H-E_a-E_b)}}{E_a+E_b-H-i\epsilon} V_{ab}^{(-)}(0)\Psi_0,$$

the time dependence of $U_{M,ab}(t)$ can be separated as follows:

$$U_{M,ab}(t) = \delta_{M,ab} - \frac{e^{it(E_M-E_a-E_b)}}{E_a+E_b-E_M-i\epsilon} T_{M,ab}, \tag{4·5}$$

with $T_{M,ab} = (\Psi_M^- | V_{ab}^{(-)}(0) | \Psi_0)$ For $W_{M,ab}(t)$, subtracting $(\Psi_M^- | q_{ab}^{(+)}(-\infty) | \Psi_0) = 0$ from the right side of (4·2), we get

$$\begin{aligned} W_{M,ab}(t) &= (\Psi_M^- | (q_{ab}^{(+)}(t) - q_{ab}^{(+)}(-\infty)) | \Psi_0) \qquad (4·6)\\ &= (\Psi_M^- | \int_{-\infty}^{t} \frac{\partial q_{ab}^{(+)}(t)}{\partial t} dt | \Psi_0) = -i(\Psi_M^- | \int_{-\infty}^{t} V_{ab}^{(+)}(t) dt | \Psi_0) \\ &= -\frac{e^{it(E_M-E_a-E_b)}}{E_a+E_b+E_M} \mathcal{J}_{M,ab}, \end{aligned}$$

with $\mathcal{J}_{M,ab} = (\Psi_M^- | V_{ab}^{(+)}(0) | \Psi_0)$.

Next let us derive the Low equation. We consider the following quantity and inserting the orthonormal complete set Ψ_M^- into it, we get

$$\begin{aligned} &(\Psi_0 | [q_{a'b'}^{(+)}(t),\ q_{ab}^{(-)}(t)]_+ | \Psi_0) = i(\Psi_0 | [q_{a'b'}^{(+)}(t),\ V_{ab}^{(-)}(t)]_+ | \Psi_0) \\ &= \sum_M (\Psi_0 | q_{a'b'}^{(+)}(t) | \Psi_M^-)(\Psi_M^- | q_{ab}^{(-)}(t) | \Psi_0) + \sum_M (\Psi_0 | q_{ab}^{(-)}(t) | \Psi_M^-)(\Psi_M^- | q_{a'b'}^{(+)}(t) | \Psi_0) \\ &= \sum_M U_{M,a'b'}^*(t) \dot{U}_{M,ab} + \sum_M \dot{W}_{M,ab}^*(t) W_{M\,a'b'}(t), \qquad (4·7) \end{aligned}$$

where the dot represents the time derivative.

In the same way we get also

$$\begin{aligned} &-i(\Psi_0 | [q_{a'b'}^{(+)}(t),\ V_{ab}^{(-)}(t)]_+ | \Psi_0) \\ &= \sum_M U_{M,a'b'}^*(t) \dot{W}_{M,ab}(t) + \sum_M \dot{U}_{M,ab}^*(t) W_{M,a'b'}(t). \qquad (4·8) \end{aligned}$$

These are the differential form of the Low equation. The covariant form can easily be obtained by substituting the time derivative with the functional derivative with respect to σ. The boundary conditions of these equations are given by (4·3), and the S-matrix can be obtained by (4·4). The equations (4·7) and (4·8) are the identities of course, but we can interprete them as they represent the equations, which connect $U(t)$ or $W(t)$ — the matrix elements of the q-operators between arbitrary states and the vacuum state

— with the vacuum expectation values of the commutators of the q-and V-operators. To solve (4·7) and (4·8) we have already to know such vacuum expectation values, otherwise they become only identities. Through these terms (or through V operators appearing in them) the structures of the interactions enter into the Low equations. In this respect Low's theory differs from that of Lehmann et al.; the theory of Lehmann et al. does not concerned with the structures of the interactions at all, but Low's theory includes them through these terms. For the other cases such quantities we have to know are the expectation values of the q-and V-operators for the one nucleon or one-meson state. Such term could have been evaluated easily for the static nucleon theory (so called the Born approximation term), but for a more complicated theory evaluation of them will be a more difficult task.

These Low equations in the differential form can easily be converted into the form like the second order perturbation formula. Substituting (4.5) and (4·6) in (4·7) and (4·8), we get

$$T_{oa'b',ab}=T_{a'b',ab}-\sum_M \frac{T^*_{M,a'b'}\,T_{M,ab}}{E_{a'}+E_{b'}-E_M+i\epsilon}-\sum_M \frac{\mathcal{J}^*_{M,ab}\,\mathcal{J}_{M,a'b'}}{E_{a'}+E_{b'}+E_M}, \tag{4·9}$$

$$\mathcal{J}_{oa'b',ab}=\mathcal{J}_{a'b',ab}-\sum_M \frac{T^*_{M,a'b'}\,\mathcal{J}_{M,ab}}{E_{a'}+E_{b'}-E_M+i\epsilon}-\sum_M \frac{T^*_{M,ab}\,\mathcal{J}_{M,a'b'}}{E_{a'}+E_{b'}+E_M}, \tag{4·10}$$

here $$T_{oa'b',ab}=(\Psi_0|[\boldsymbol{q}^{(+)}_{a'b'}(0),\ \boldsymbol{V}^{(-)}_{ab}(0)]_+|\Psi_0)$$

and $$\mathcal{J}_{oa'b',ab}=(\Psi_0|[\boldsymbol{q}^{(+)}_{a'b'}(0),\ \boldsymbol{V}^{(+)}_{ab}(0)]_+|\Psi_0).$$

In these form of the Low equation the S-matrix are given by (3·10).

The physical meaning of W and $\mathcal{J}$ is given by the following relations which are proved in the appendix;

$$W_{M,ab}(t)=U_{M,-a-b}(t),\quad W^*_{M,ab}(t)=-U^*_{M,-a-b}(t), \tag{4·11}$$

$$\mathcal{J}_{M,ab}=T_{M,-a-b},\quad \mathcal{J}^*_{M,ab}=-T^*_{M,-a-b}. \tag{4·12}$$

where the meanings of the symbol " $-a$ " and " $-b$ " are as follows; for the meson index the state " b " is time reversed we use (the word " time reversal ") in the sense of simple time reversal plus charge conjugation) and the sign of the energy E_b is reversed For the nucleon index the state " a " is time reversed, the sign of the energy E_a is reversed and the spinor covariant components of the state " a ", which appear in the equation (2·2) and enter into through the equation (4·1), are multiplied by the Dirac matrix $i\rho_2$, i.e., $(\Psi_0|\psi^\nu(0)|\Psi_a)$ and $(\Psi_a|\bar{\psi}^\nu(0)|\Psi_0)$ are substituted with $(\Psi_{\bar{a}}|i\rho_2\psi^\nu(0)|\Psi_0)$ and $(\Psi_0|i\bar{\psi}^\nu(0)\rho^2|\Psi_{\bar{a}})$ where $\Psi_{\bar{a}}$ is the time reversed state of Ψ_a. Note that the operation " $-$ " is not a simple time reversal.

Using (4·11) and (4·12), (4·7)$\sim$(4·10) can be rewritten in the unified and symmetrical form,

$$iT_{oa'b',ab}(t)=\sum_M U^*_{M,a'b'}(t)\dot{U}_{M,ab}(t)-\sum_M \dot{U}^*_{M,-a-b}(t)\,U_{M,-a'-b'}(t), \tag{4·13}$$

$$T_{a'b',ab}=T_{oa'b',ab}+\sum_M \frac{T^*_{M,a'b'}\,T_{M,ab}}{E_{a'}+E_{b'}-E_M+i\epsilon}-\sum_M \frac{T^*_{M,-a-b}\,T_{M,-a'-b'}}{E_{a'}+E_{b'}+E_M}, \tag{4·14}$$

here $$T_{\alpha'\beta',\alpha\beta}(t)=(\Psi_0|[\boldsymbol{q}^{(+)}_{\alpha'\beta'}(t),\ \boldsymbol{V}^{(-)}_{\alpha\beta}(t)]_+|\Psi_0).$$

We shall prove in the appendix that $T_{\alpha'\beta',\alpha\beta}=T_{\alpha-\alpha-\beta,-\alpha'-\beta'}$ when $E_\alpha+E_\beta=E_{\alpha'}+E_{\beta'}$, therefore the substitution $\alpha',\beta'\to-\alpha,-\beta,\ \alpha,\beta\to-\alpha',-\beta'$ and $E_{\alpha'}+E_{\beta'}\to-(E_\alpha+E_\beta)$ in the equation (4·14) does not cause any change in the right side of (4·14) when $E_\alpha+E_\beta=E_{\alpha'}+E_{\beta'}$, so that we get

$$T_{\alpha'\beta',\alpha\beta}=T_{-\alpha-\beta,-\alpha'-\beta'}\ \text{for}\ E_\alpha+E_\beta=E_{\alpha'}+E_{\beta'}. \tag{4·15}$$

This is one of the symmetry properties of the S-matrix.

In the similar fashion we get the following results for the other cases. For the case ii)

$$iT'_{\alpha'\beta',\alpha\beta}(t)=\sum_M U'^{*}_{M,\alpha'\beta'}(t)\ \dot{U}'_{M,\alpha\beta}(t)-\sum_M \dot{U}'^{*}_{M,\alpha'-\beta}(t)\,U'_{M,\alpha-\beta'}, \tag{4·16}$$

$$T'_{\alpha'\beta',\alpha\beta}=T'_{0\alpha'\beta',\alpha\beta}(0)+\sum_M \frac{T'^{*}_{M,\alpha'\beta'}\,T'_{M,\alpha\beta}}{E_{\alpha'}+E_{\beta'}-E_M+i\epsilon}+\sum_M \frac{T'^{*}_{M'\alpha'-\beta}\,T'_{M,\alpha-\beta'}}{E_\alpha-E_{\beta'}-E_M+i\epsilon}, \tag{4·17}$$

where $$U'_{M,\alpha\beta}(t)=(\Psi^-_M|q^{(-)}_\beta(t)\,|\Psi_\alpha),\ T'_{M,\alpha\beta}=(\Psi^-_M|V^{(-)}_\beta(0)\,|\Psi_0),$$

and $$T'_{0\alpha'\beta',\alpha\beta}(t)=(\Psi_{\alpha'}|[q^{(+)}_{\beta'}(t),\ V^{(-)}_\beta(t)]_-|\Psi_\alpha).$$

The symmetry property of T' is

$$T'_{\alpha'\beta',\alpha\beta}=T'_{\alpha'-\beta,\ \alpha-\beta'}\ \text{for}\ E_\alpha+E_\beta=E_{\alpha'}+E_{\beta'}. \tag{4·18}$$

For the case iii), "α" and "β" are exchanged in (4·16)$\sim$(4·18), and in the expression of T_0' the minus commutator must be changed into the plus commutator.

From (4·15) and (4·18) we get the symmetry properties of the S-matrix because T and T' have the same value on the energy shell and are connected to the unique S-matrix by (3·10) and (3·12).

$$\left\{\begin{aligned} R_{\alpha'\beta',\alpha\beta}&=R_{-\alpha-\beta,\ -\alpha'-\beta'}\\ &=R_{\alpha'-\beta,\alpha-\beta'}\\ &=R_{-\alpha\beta',-\alpha'\beta}\,, \end{aligned}\right. \tag{4·19}$$

here $$R_{\alpha'\beta',\alpha\beta}=S_{\alpha'\beta',\alpha\beta}-\delta_{\alpha'\beta',\alpha\beta}\,.$$

The second relation in (4·19) is the familiar Gell-Mann-Goldberger crossing relation.

§ 5. Linearization of the Low equation

One of the most remarkable features of the Low equations is their non-linearity. As we have not yet met such non-linear equation, they raise such questions as whether they have unique solution and whether their non-linearity is an essential one. About the first question Castillejo el al.[5] have recently obtained a negative conclusion using a special model. In this section we will show generally that the Low equation in which the comparison state is the vacuum can be linearized and in the course of its linearization the same arbitrariness that has been found by Castillejo[5] et al. enters into the linearized equation, so that we have infinitely many solutions for the Low equation.

We use the differential form of the Low equation. For the case i) the Low equa-

tion is written down generally as follows:

$$\sum_L U^*_{L,M}(t)\dot{U}_{L,N}(t) \mp \sum_L \dot{W}^*_{L,N}(t)W_{L,M}(t) = A_{M,N}(t), \tag{5·1}$$

$$\sum_L U^*_{L,M}(t)\dot{W}_{L,N}(t) \mp \sum_L \dot{U}^*_{L,N}(t)W_{L,M}(t) = B_{M,N}(t), \tag{5·2}$$

where $U_{L,M}(t) = (\Psi^-_L | \boldsymbol{q}^{(-)}_M(t) | \Psi_0)$, $W_{L,M}(t) = (\Psi^-_L | \boldsymbol{q}^{(+)}_L(t) | \Psi_0)$,

$$A_{M,N}(t) = i(\Psi_0 | [\boldsymbol{q}^{(+)}_M(t),\ \boldsymbol{V}^{(-)}_N(t)]_\mp | \Psi_0),$$

$$B_{M,N}(t) = -i(\Psi_0 | [\boldsymbol{q}^{(+)}_N(t),\ \boldsymbol{V}^{(+)}_N(t)]_\mp | \Psi_0),$$

$$i\boldsymbol{V}^{(-)}_N(t) = \dot{\boldsymbol{q}}^{(-)}_N(t),\ -i\boldsymbol{V}^{(+)}_N(t) = \dot{\boldsymbol{q}}^{(+)}_N(t),$$

and the minus sign corresponds to the case in which M, N states include even numbers of fermion, the plus sign to the case in which they include odd numbers of fermion. (5·1) and (5·2) can be written compactly in the matrix product form,

$$U^+(t)\dot{U}(t) + \varepsilon W^T(t)\dot{W}^*(t) = A(t), \tag{5·3}$$

$$U^+(t)\dot{W}(t) + \varepsilon W^T(t)\dot{U}^*(t) = B(t), \tag{5·4}$$

here ε is the above mentioned sign operator.

The boundary conditions of them are

$$U(+\infty) = 1,\ W(\pm\infty) = 0, \tag{5·5}$$

and the S-matrix is given by

$$S = U(-\infty). \tag{5·6}$$

Let us solve (5·3) and (5·4) when $A(t)$ and $B(t)$ are the known arbitrary operators. It can be easily shown that (5·3) and (5·5) have the following first integrals,

$$U^+(t)U(t) + \varepsilon W^T(t)W^*(t) = 1 - \int_t^\infty (A(t) + A^+(t))dt = I(t), \tag{5·7}$$

$$U^+(t)W(t) + \varepsilon W^T(t)U^*(t) = -\int_t^\infty (B(t) + \varepsilon B^T(t))dt = J(t). \tag{5·8}$$

In our case $I(t)$ and $J(t)$ are

$$I_{M,N}(t) = (\Psi_0 | [\boldsymbol{q}^{(+)}_M(t),\ \boldsymbol{q}^{(-)}_N(t)]_\mp | \Psi_0),$$

$$J_{M,N}(t) = (\Psi_0 | [\boldsymbol{q}^{(+)}_M(t),\ \boldsymbol{q}^{(+)}_N(t)]_\mp | \Psi_0) = 0,$$

so that

$$I_{M,N}(\mp\infty) = \delta_{M,N}.$$

Then the relation (5·7) reduce to the unitarity of the S-matrix when $t \to -\infty$.

Next we introduce a matrix $\rho(t)$ by

$$W(t) = U(t)\rho(t). \tag{5·9}$$

In the following derivations we omit the time index in $U(t)$, etc. for the sake of brevity.

Inserting (5·9) into (5·3) and (5·4), they become

$$U^+\dot{U}+\varepsilon\rho^T U^T\dot{W}^*=A, \tag{5·10}$$

$$U^+\dot{W}+\varepsilon\rho^T U^T\dot{U}^*=B. \tag{5·11}$$

Taking the complex conjugate of (5·11), and inserting it into (5·10) we get

$$(1-\varepsilon\rho^T\varepsilon\rho^+)U^+\dot{U}=A-\varepsilon\rho^T B^*. \tag{5·12}$$

Similarly we have also

$$(1-\varepsilon\rho^T\varepsilon\rho^+)U^+\dot{W}=B-\varepsilon\rho^T A. \tag{5·13}$$

In the same way we get from (5·7), (5·8) and (5·9)

$$(1-\varepsilon\rho^T\varepsilon\rho^+)U^+U=I-\varepsilon\rho^T J^*, \tag{5·14}$$

$$(1-\varepsilon\rho^T\varepsilon\rho^+)U^+W=J-\varepsilon\rho^T I^*. \tag{5·15}$$

Writing

$$(1-\varepsilon\rho^T\varepsilon\rho^+)U^+=X, \tag{5·16}$$

$$-\varepsilon\rho^T=Y, \tag{5·17}$$

(5·12)$\sim$(5·15) become

$$\begin{cases} X\dot{U}=A+YB^*, & X\dot{W}=B+XA^*, \\ XU=I+YJ^*, & XW=J+YI^*. \end{cases} \tag{5·18}$$

(5·18) suggest that $\dot{U}$ and $\dot{W}$ are the linear combination of U and W, so we put

$$\begin{cases} \dot{U}=U\alpha+W\beta+\varepsilon, \\ \dot{U}=W\gamma+U\delta+\eta, \end{cases} \tag{5·19}$$

here α, β, etc., are the matrices to be determined later. Insertion of (5·19) into (5·18) shows that (5·19) statisfies (5·18) when

$$\gamma=\alpha^*,\ \delta=\beta^*, \tag{5·20}$$

$$\begin{cases} I\alpha+J\beta+\zeta=A, \\ J\alpha^*+I\beta^*+\eta=B. \end{cases} \tag{5·21}$$

and

$$\begin{cases} X\zeta-X\eta^*-\zeta=0, \\ X\eta-Y\zeta^*-\eta=0. \end{cases} \tag{5·22}$$

Taking the complex conjugate of the second relation of (5·22) and inserting it into the first one, (5·22) become

$$\begin{cases} X\zeta-YX^*\eta^*-(1-YY^*)\zeta=0, \\ X\eta-YX^*\zeta^*-(1-YY^*)\eta=0. \end{cases} \tag{5·23}$$

Remembering that

$$X=(1-YY^*)U^+,$$
$$\varepsilon W^T=YU^T,$$

(5·23) become

$$\begin{cases} U^+\zeta+\varepsilon W^T\eta^*-\zeta=0, \\ U^+\eta+\varepsilon W^T\zeta^*-\eta=0. \end{cases} \tag{5·24}$$

Thus the final result is

$$\begin{cases} \dot{U}=U\alpha+W\beta+\zeta, \\ \dot{W}=W\alpha^*+U\beta^*+\eta, \end{cases} \tag{5·25}$$

where α, β, ζ and η are the matrices satisfying (5·21) and (5·24). It can easily be shown that the U and W, satisfying (5·25) with (5·21), (5·24) and the boundary condition (5·5), satisfy the original Low equations (5·3) and (5·4). The equations (5·25) are the homogeneous equations for ζ and η, therefore they have solutions only when the determinant

$$\begin{vmatrix} U^+-1, & \varepsilon W^T \\ \varepsilon W^+, & U^T-1 \end{vmatrix}$$

is zero. (In order to explain this, we refer to the results of the theory of finite dimensional linear equation. Of course, for the continuous spectrum cases the direct usage of its terminologies is not correct, but the essential points are same also for the continuous spectrum cases.) As the rank of the determinant can be taken arbitrary, ζ and η can contain many arbitrary pramaters. Turning t to $-\infty$ (5·24) become

$$S^+\zeta(-\infty)=\zeta(-\infty),\ S^+\eta(-\infty)=\eta(-\infty),$$

therefore $\zeta(-\infty)$ and $\eta(-\infty)$ have non vanishing value only at the points where $S=1$ or phase shifts pass $n\pi$. Thus this arbitrariness of ζ and η is the same one as what Castillejo et al. have found.

The above method can not be applied to the other types of the Low equation, but as the above result is so general we can conclude that the Low equation does not fix the solution of the scattering problems. The essential simplicity of the case i) type Low equation consists in the fact that when they are decomposed into partial waves they are separated from each other and the mixing of the partial waves does not occur, but for the other types of the Low equation, as is well known, mixing of the partial waves occurs so that it becomes complicated to solve the equation.

§ 6. Example

For practical calculations it is convenient to rewrite the equations (5·25), (5·24) into the equations for T and $\mathcal{T}$. Using (4·5) and (4·6), they become

$$\begin{cases} iT_{M,N}=\alpha_{M,N}(0)+\sum_L \dfrac{T_{M,L}\,\alpha_{L,N}(0)}{E_M-E_L+i\epsilon}-\sum_L \dfrac{\mathcal{J}_{M,L}\,\beta_{L,N}(0)}{E_M+E_L}+\zeta_{M,N}(0), \\ -i\mathcal{J}_{M,N}=\beta^*{}_{M,N}(0)+\sum_L \dfrac{T_{M,N}\,\beta^*_{L,N}(0)}{E_M-E_L+i\epsilon}-\sum_L \dfrac{\mathcal{J}_{M,L}\,\alpha^*_{L,N}(0)}{E_M+E_L}+\eta_{M,N}(0), \end{cases} \tag{6·1}$$

$$\begin{cases} \sum_L \dfrac{T^*_{L,M}\,\zeta_{L,N}(0)}{E_L-E_M-i\epsilon}+\sum_L \dfrac{\varepsilon_M\,\mathcal{J}_{L,M}\,\eta^*_{L,N}(0)}{E_L+E_M}=0, \\ \sum_L \dfrac{T^*_{L,M}\,\eta_{L,N}(0)}{E_L-E_M-i\epsilon}+\sum_L \dfrac{\varepsilon_M\mathcal{J}_{L,M}\,\zeta^*_{L,N}(0)}{E_L+E_M}=0. \end{cases} \tag{6·2}$$

From (5·7) and (5·8), I and J are given by

$$\begin{cases} I_{M,N}=\delta_{M,N}-\dfrac{i}{E_M-E_N}\,(A_{M,N}+A^*_{N,M}), \\ J_{M,E}=-\dfrac{i}{E_M+E_N}\,(B_{M,N}+\varepsilon_M\,B_{N,M}). \end{cases} \tag{6·3}$$

As an example of our method we treat here the case of Castillejo et al., i.e. the Low equation for the neutral scalar meson interacting with the static extended nucleon and in the one meson approximation. Although in this example the comparison state is the nucleon state, not the vacuum state, the nucleon indices drop out of the Low equation and it becomes the form which we have treated in section 5, hence we can apply our method to it.

In this example the Low equation can be written in the form of (5·3) and (5·4) and the matrices A and B are given by

$$A_{p,q}(0)=-B_{p,q}(0)=\frac{i}{4\pi^2}\,\frac{v_p}{\sqrt{\omega_p}}\Big[\sum\Lambda_i\Big(\frac{1}{\omega_i-\omega_p}+\frac{1}{\omega_i+\omega_p}\Big)\Big]\frac{v_q}{\sqrt{\omega_q}}, \tag{6·4}$$

where p, q are the momenta of the meson, v_p is the form factor of nucleon, ω_i are the energies of the bound states $(0<\omega_i<\mu)$ and Λ_i are the coupling strengths of the bound states. Also from (6·3) and (6·4) we have

$$I_{p,q}(0)=\delta_{p,q}+\frac{1}{4\pi^2}\,\frac{v_p}{\sqrt{\omega_p}}\Big[\sum_i\frac{2\omega_i\,\Lambda_i(\omega_p+\omega_q)}{(\omega_i^2-\omega_p^2)\,(\omega_i^2-\omega_q^2)}\Big]\frac{v_q}{\sqrt{\omega_q}}, \tag{6·5}$$

$$J_{p,q}(0)=-\frac{1}{4\pi^2}\,\frac{v_p}{\sqrt{\omega_p}}\Big[\sum_i\frac{2\omega_i\Lambda_i(\omega_p-\omega_q)}{(\omega_i^2-\omega_p^2)\,(\omega_i^2-\omega_q^2)}\Big]\frac{v_q}{\sqrt{\omega_q}}. \tag{6·6}$$

In this example the difference of T and $\mathcal{J}$ disappears and the two equations in (6·1), (6·2) and (5·21) reduce to the same one. So we put

$$T_{pq}=\mathcal{J}_{pq}=\frac{1}{4\pi^2}\,\frac{v_p}{\sqrt{\omega_p}}\,h(\omega_p)\,\frac{v_q}{\sqrt{\omega_q}}, \text{ *} \tag{6·7}$$

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*) Our $h(\omega)$ is connected with Castillejo et al's $h^c(\omega)$ by $h^c(\omega)=h^c(\omega)/\omega$.[10]

$$\alpha_{p,q}(0)=\beta_{p,q}(0)=\frac{i}{4\pi^2}\frac{v_p}{\sqrt{\omega_p}}\gamma(\omega_p)\frac{v_q}{\sqrt{\omega_q}}, \tag{6·8}$$

$$\zeta_{p,q}(0)=-\eta_{p,q}(0)=\frac{i}{4\pi^2}\frac{v_p}{\sqrt{\omega_p}}z(\omega_p)\frac{v_q}{\sqrt{\omega_p}}, \tag{6·9}$$

where $\gamma(\omega)$ and $z(\omega)$ are the real functions of ω. Using $(6\cdot4)\sim(6\cdot9)$, the matrix equations (6·1), (6·2) and (5·21) reduce to the equation for $h(\omega)$, $\gamma(\omega)$ and $z(\omega)$,

$$h(\omega)=\gamma(\omega)+z(\omega)-\frac{h(\omega)}{\pi}\int_\mu^\infty kv_k^2\,\gamma(\omega_k)\left(\frac{1}{\omega_k-\omega-i\epsilon}+\frac{1}{\omega_k+\omega}\right)d\omega_k, \tag{6·10}$$

$$\int_\mu^\infty\left(\frac{h^*(\omega_k)}{\omega_k-\omega-i\epsilon}+\frac{h(\omega_k)}{\omega_k+\omega}\right)z(\omega_k)kv_k^2\,d\omega_k=0, \tag{6·11}$$

$$\gamma(\omega)+z(\omega)+\frac{1}{\pi}\int_\mu^\infty\left[\sum_i\frac{4\omega_i\Lambda_i\omega_k}{(\omega_i^2-\omega^2)(\omega_i^2-\omega_k^2)}\right]\gamma(\omega_k)kv_k^2\,d\omega_k \tag{6·12}$$

$$=\sum_i\frac{2\omega_i\Lambda_i}{\omega_i^2-\omega^2}.$$

The equation (6·12) can easily be solved by putting

$$\gamma(\omega)+z(\omega)=\sum_i\frac{2\omega_i\lambda_i}{\omega_i^2-\omega^2}. \tag{6·13}$$

Inserting (6·13) into (6·12) we get

$$\lambda_i+\frac{\Lambda_i}{\pi}\sum_j\int_\mu^\infty\frac{2\lambda_i R\omega_k v_k^2}{(\omega_i^2-\omega_k^2)(\omega_j^2-\omega_k^2)}d\omega_k-\frac{\Lambda_i}{\pi}\int_\mu^\infty\frac{2\omega_k}{\omega_i^2-\omega_k^2}z(\omega_k)kv_k^2d\omega_k \tag{6·14}$$

$$=\Lambda_i.$$

Also we have

$$\left(\sum_i\frac{2\omega_i\Lambda_i}{\omega_i^2-\omega^2}\right)/h(\omega)=1+\frac{1}{\pi}\int_\mu^\infty\left(\sum_i\frac{2\omega_i\lambda_i}{\omega_i^2-\omega_k^2}\right)\left(\frac{1}{\omega_k-\omega-i\epsilon}+\frac{1}{\omega_k+\omega}\right)kv_k^2d\omega_k \tag{6·15}$$

$$-\frac{1}{\pi}\int_\mu^\infty\frac{2\omega_k}{\omega_k^2-\omega^2}z(\omega_k)kv_k^2\,d\omega_k.$$

Putting $z(\omega)=\sum_n R_n\,\delta(\omega-\Omega_n)$ with arbitrary real numbers R_n, and $\Omega_n>\mu$, we can show that $h(\omega)$ given by (6·15) satisfies (6·11). Thus we get the solution for the Low equation by

$$h(\omega)=\left(\sum_i\frac{2\omega_i\lambda_i}{\omega_i^2-\omega^2}\right)\Big/\left[1+\frac{1}{\pi}\int_\mu^\infty\left(\sum_i\frac{2\omega_i\lambda_i}{\omega_i^2-\omega_k^2}\right)\left(\frac{1}{\omega_k-\omega-i\epsilon}+\frac{1}{\omega_k+\omega}\right)kv_k^2d\omega_k\right. \tag{6·16}$$

$$\left.-R(\omega)\right]$$

with $R(\omega)=\sum_n R_n'\left(\frac{1}{\Omega_n-\omega}+\frac{1}{\Omega_n+\omega}\right)$ and λ_i given by (6·14). The form of this

solution differs slightly from that of Castillejo et al., but it can easily be brought into the form obtained by Castillejo et al. For example when then there is only one bound state, it becomes

$$[h(\omega)]^{-1}=\frac{1}{2\lambda_0}+\frac{2}{\pi}\int\frac{k\omega_k v_k^2}{\omega_0^2-\omega_k^2}d\omega_k-\frac{1}{\omega_0\lambda_0}\sum_n R_n'\Omega_n-\frac{\omega^2}{2\lambda_0\omega_0} \qquad (6\cdot17)$$

$$+\frac{1}{\pi}\int_\mu^\infty kv_k^2\left(\frac{1}{\omega_k-\omega-i\epsilon}+\frac{1}{\omega_k+\omega}\right)d\omega_k-\frac{1}{\lambda_0\omega_0}\sum_n\frac{R_n'\Omega_n(\omega_0^2-\Omega_n^2)}{\Omega_n^2-\omega^2},$$

and $$\lambda_0=\Lambda_0(1+R(\omega_0))/\left(1+\frac{2\Lambda_0}{\pi}\int_\mu^\infty\frac{k\omega_k v_k^2}{(\omega_0^2-\omega_k^2)^2}d\omega_k\right).$$

In (6·16) the conditions for $h(\omega)$ that have been used by Castillejo et al. to get the solution of the Low equation are satisfied explicitly and R_n' and Ω_n are quite arbitrary except that $R(\omega)$ has a convergent value. On the other hand the form of the solution of Castillejo needs some restriction for the parameters appearing in the solution and is unclear to see to what extent the form of the interaction fixes the solution of the Low equation.

§ 7. Conclusion

Summarizing the above results we can conclude that the Low equations have a fairly general character independent of the detailed structures of the interactions. However, differing from that of Lehmann et al. they contain some respects of the interactions through the term T_0 in the equations (4·13), (4·14), but as has been shown in § 5 they are not sufficient to fix the solution of the scattering problems uniquely. Our method employed in § 5 is the rigorous one, and independent of any approximation like one meson approximation, therefore the conclusion of Castillejo et al. holds in general. The Low equations are the frames for the structures of the S-matrix, and, as have been stressed by Castillejo et al., they resemble in many respects the Wigner-Eisenbud dispersion formula. They need some knowledge about the outer parts of the system, but do not concern the inner structures of the system. The general structures of the S-matrix are determined by them, but they leave infinitely many parameters.

In conclusion the auther would like to express his sincere thanks to Messers. Y. Munakata and H. Tanaka for their kind discussions and encouragements.

Appendix A

We give here the proof of the equivalence of (3·10) and (3·12). In the same manner as has been given in section 3, Ψ_a can be written as

$$\Psi_a=q_a^{(-)}(t)\Psi_0+\frac{1}{E_a-H}V_a^{(-)}(t)\Psi_0. \qquad (A\cdot1)$$

Then

$$(\Psi_{\bar{M}}|V_b^{(-)}(t)|\Psi_a)=(\Psi_{\bar{M}}|V_b^{(-)}(t)q_a^{(-)}|\Psi_0)+(\Psi_{\bar{M}}|V_b^{(-)}(t)\frac{1}{E_a-H}V_a^{(-)}(t)|\Psi_0). \quad (A\cdot 2)$$

Using (2·11) the second term of the right side of (A·2) can be rewritten as

$$(\Psi_{\bar{M}}|V_b^{(-)}(t)\frac{1}{E_a-H}V_a^{(-)}(t)|\Psi_0) \quad (A\cdot 3)$$

$$=(\Psi_{\bar{M}}|\{[H,q_b^{(-)}(t)]-E_bq_b^{(-)}(t)\}\frac{1}{E_a-H}V_a^{(-)}(t)|\Psi_0)$$

$$=(\Psi_{\bar{M}}|q_b^{(-)}(t)(E_M-E_b-H)\frac{1}{E_a-H}V_a^{(-)}(t)|\Psi_0)$$

$$=(\Psi_{\bar{M}}|q_b^{(-)}(t)V_a^{(-)}(t)|\Psi_0)$$

$$+(E_M-E_a-E_b)(\Psi_{\bar{M}}|q_b^{(-)}(t)\frac{1}{E_a-H}V_a^{(-)}(t)|\Psi_0).$$

Therefore

$$(\Psi_{\bar{M}}|V_b^{(-)}|\Psi_a)=(\Psi_{\bar{M}}|V_{ab}^{(-)}(t)|\Psi_0) \text{ for } E_M=E_a+E_b. \quad (A\cdot 4)$$

Q.E.D.

Appendix B

(Proof of (4·11) and (4·12))[11]

The charge conjugate operators of $q_a^{(\pm)}(\sigma)$, etc., are given by

$$q_a^{(-)}(\sigma)=N_a^{-1}\int_\sigma(\Psi_0|\bar{\psi}(x)|\Psi_{\underline{a}})\gamma_\nu\psi(x)\,d\sigma_\nu, \quad (B\cdot 1)$$

$$q_a^{(+)}(\sigma)=N_a^{-1}\int\bar{\psi}(x)\gamma_\nu(\Psi_{\underline{a}}|\psi(x)|\Psi_0)\,d\sigma_\nu, \quad (B\cdot 2)$$

$$q_{\underline{b}}^{(-)}(\sigma)=-N_b^{-1}\int_\sigma(\Psi_0|A(x)|\Psi_b)\overleftrightarrow{\partial}_\nu A(x)\,d\sigma_\nu, \quad (B\cdot 3)$$

$$q_{\underline{b}}^{(+)}(\sigma)=-N_b^{-1}\int_\sigma A(x)\overleftrightarrow{\partial}_\nu(\Psi_{\underline{b}}|A(x)|\Psi_0)\,d\sigma_\nu, \quad \text{etc.,} \quad (B\cdot 4)$$

where $\Psi_{\underline{a}}$, $\Psi_{\underline{b}}$ are the charge conjugate states of Ψ_a, Ψ_b. The connection between $\Psi_{\underline{M}}$ and Ψ_M is given by

$$\Psi_{\underline{M}}=C\Psi_M, \quad (B\cdot 5)$$

here C is a unitary operator.

The simple time reversed state $\Psi_{\bar{M}}$ of Ψ_M is given by

$$\Psi_{\bar{M}}=\Psi_M^*R,\ \Psi_M=R\Psi_{\bar{M}}^*, \quad (B\cdot 6)$$

here R is a unitary operator.

The time reversed (simple time reversal plus charge conjugation) state Ψ_{rM} of Ψ_M is given by

$$\Psi_{rM}=\Psi_M^* RC^T,\ \Psi_M=RC^T\Psi_{rM}^*. \tag{B·7}$$

By R and C the fields operators are transformed as follows;

$$(C^{-1T}R^{-1}\psi(\boldsymbol{r},\ t)RC^T)^T=i\rho_2\psi(\boldsymbol{r},-t), \tag{B·8}$$

$$(C^{-1T}R^{-1}\bar{\psi}(\boldsymbol{r},\ t)RC^T)^T=-i\bar{\psi}(\boldsymbol{r},\ -t)\rho_2, \tag{B·9}$$

$$(C^{-1T}R^{-1}A(\boldsymbol{r},\ t)RC^T)^T=A(\boldsymbol{r},\ -t), \tag{B·10}$$

$$(C^{-1T}R^{-1}A^+(\boldsymbol{r},\ t)RC^T)^T=A^+(\boldsymbol{r},\ -t). \tag{B·11}$$

From (B·7) and (B·8) we get

$$\begin{aligned}(\Psi_0|\psi(x)|\Psi_a)&=(RC^T\Psi_0|\psi(x)|RC^T\Psi_{ra}^*) \qquad\qquad (B\cdot12)\\ &=(\Psi_{ra}|(C^{-1T}R^{-1}\psi(x)RC^T)^T|\Psi_0)=(\Psi_{ra}|i\rho_2\psi(\boldsymbol{r},\ -t)|\Psi_0),\end{aligned}$$

here we have supposed for the vacuum state

$$RC^T\Psi_0=\Psi_0 \text{ and } \Psi_0{}^*=\Psi_0.$$

In the same manner

$$(\Psi_a|\bar{\psi}(x)|\Psi_0)=-(\Psi_0|i\bar{\psi}(\boldsymbol{r},\ -t)\rho_2|\Psi_{ra}), \tag{B·13}$$

$$(\Psi_0|A(x)|\Psi_b)=(\Psi_{rb}|A(\boldsymbol{r},\ -t)|\Psi_0), \tag{B·14}$$

$$(\Psi_0|A(x)|\Psi_b)=(\Psi_{rb}|A(\boldsymbol{r},\ -t)|\Psi_0). \tag{B·15}$$

Using (B·12) we can rewrite $q_a^{(-)}(\sigma)$ as follows:

$$\begin{aligned}q_a^{(-)}(\sigma)&=N_a^{-1}\int_\sigma\bar{\psi}(x)\gamma_\nu(\Psi_0|\psi(x)|\Psi_a)d\sigma_\nu \qquad\qquad (B\cdot16)\\ &=N_a^{-1}\int_\sigma\bar{\psi}(x)\gamma_\nu(\Psi_{ra}|i\rho_2\psi(\boldsymbol{r},\ -t)|\Psi_0)d\sigma_\nu.\end{aligned}$$

Comparing (B·16) with (B·2) we get

$$q_a^{(-)}(\sigma)=q_{-a}^{(+)}(\sigma). \tag{B·17}$$

The meaning of the symbol $(-a)$ has been stated below the relations (4·11) and (4·12) in section 4.

In the similar manner we have

$$q_a^{(+)}(\sigma)=-q_{-a}^{(-)}(\sigma), \tag{B·18}$$

$$q_b^{(-)}(\sigma)=-q_{-b}^{(+)}(\sigma), \tag{B·19}$$

$$q_b^{(+)}(\sigma)=-q_{-b}^{(-)}(\sigma). \tag{B·20}$$

The difference in sign in (B·17) and (B·18) might seem contradictory to the relation $q_a^{(+)}(\sigma)=q_a^{(-)\dagger}(\sigma)$, but it is the consequence of the two valuedness of the spinor with respect to time reversal, and from this two valuedness we have been forced to take the plus commutator in the equations (4·7), (4·8). From the relations (B·17) to (B·20)

we get immediately (4·11) and (4·12).

Appendix C

Let us prove the relation

$$T_{oa'b',ab}(t)=T_{o-a-b,-a'-b'}(t) \quad \text{for } E_a+E_b=E_{a'}+E_{b'}, \tag{C·1}$$

here $$T_{oa'b',ab}(t)=-i(\Psi_0|\boldsymbol{q}^{(+)}_{a'b'}(t),\ \boldsymbol{q}^{(-)}_{ab}(t)]_+|\Psi_0).$$

Using (B·17)∼(B·20)

$$\begin{aligned}T_{o-a-b,\,-a'-b'}(t)&=-i(\Psi_0|\boldsymbol{q}^{(+)}_{-a-b}(t),\ \boldsymbol{q}^{(-)}_{-a'-b'}(t)]_+|\Psi_0)\\&=i(\Psi_0|[\boldsymbol{q}^{(-)}_{a'b'}(t),\ \boldsymbol{q}^{(+)}_{a'b'}(t)]_+|\Psi_0)\\&=i\frac{\partial}{\partial t}(\Psi_0|[\boldsymbol{q}^{(-)}_{ab}(t),\ \boldsymbol{q}^{(+)}_{a'b'}(t)]_+|\Psi_0)-i(\Psi_0|[\boldsymbol{q}^{(+)}_{a'b'}(t),\ \boldsymbol{q}^{(-)}_{ab}(t)]_+|\Psi_0)\\&=i\frac{\partial}{\partial t}e^{it(E_a+E_b-E_{a'}-E_{b'})}(\Psi_0|[\boldsymbol{q}^{(+)}_{a'b'}(0),\ \boldsymbol{q}^{(-)}_{ab}(0)]_+|\Psi_0)+T_{oa'b',\,ab}(t).\end{aligned} \tag{C·2}$$

The first term of (C·2) is zero when $E_a+E_b=E_{a'}+E_{b'}$, (C·1) has been proved.

References

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General Integral Formula of Perturbation Term in the Quantized Field Theory

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An arbitrary perturbation term in the quantized field theory is expressed explicitly in terms of generalized Feynman parametric integral, after all integrations of momenta are carried out. By means of this general integral formula Dyson's power-counting theorem as well as the absence of odd order divergences are proved rigorously, and Lehmann's spectral function $\rho(u)$ is given explicitly in perturbation expansion. Some remarks are made on the convergence of perturbation series.

§ 1. Introduction

Although the quantized field theory has advanced remarkably since the appearance of the Feynman-Dyson theory, it is recently believed generally that the renormalized field theory may be self-inconsistent. It seems, however, that in some of recent works concerning such points somewhat doubtful arguments are found. Therefore it is desirable to review the recent works on a more reliable foundation. This paper is an attempt for such purpose.

It seems quite queer to the author that many great investigators believed any perturbation term would be *absolutely* convergent provided that the usual divergences were absent, despite the fact that it is *never* so. For instance, in Dyson's proof[1] of the power-counting theorem (see § 4) the absolute convergence is assumed implicitly and hence this proof is inadequate. This point is not a merely mathematical problem.

Though Feynman's calculation method[2] is very powerful and beautiful, it is in its original form still inconvenient mathematically for higher order terms—especially for the investigation on the general properties of perturbation terms. Hence we derive a more convenient general formula in a compact and explicit parametric integral form for any perturbation term, by means of which the power-counting theorem is proved reasonably.

Since the physical meanings on the formula is clear, it is also applicable to the spectral functions in Nambu's terminology.[3] As the result, we obtain the explicit perturbation expansion of Lehmann's spectral function $\rho(u)$,[4] and at the same time the general spectral representation of many-body Green functions, but the latter does not coincide with Nambu's postulate.[3]

Further the formula is applied to the problem on the convergence of the perturbation series. And it proves that what one can prove to be divergent is merely Z-factor but not the renormalized propagators.

§ 2. Derivation of the general integral formula

We change a given Feynman diagram into the following form for convenience sake :

1) Without regard to the kinds of elementary particles, every internal line by the cutting of which the number of disconnected graphs does not increase is denoted by a solid line, and the other internal line by a dotted line.
2) External line is extinguished and put the mark $\times$ at the vertex. Mass term, if we consider it, is similarly marked.
3) Every line is given a direction. But it should be same with the direction indicated in the original Feynman diagram for fermions.

Let us call such diagram " Q-diagram." In a Q-diagram any closed circuit is composed of solid lines only. Each of solid lines and dotted ones corresponds to a propagator. These are marked by numbers $i=1, 2,\ldots, \alpha$ (α is the total number of propagators).

A boson propagator denoted by a solid line is

$$1/\{(p_i+q_i)^2+\bar{M}_i\}, \tag{2·1}$$

where p_i is an integration momentum, q_i a constant one and M_i the square of mass m_i, and the quantity with a bar means that it contains an infinitesimal negative imaginary part $-i\varepsilon$. Similarly, a fermion propagator denoted by a solid line is

$$(-i\gamma D_i+m_i)/\{(p_i+q_i)^2+\bar{M}_i\}, \tag{2·2}$$

where D_i is the following formal operator (cf. Appendix A) :

$$D_i \equiv \frac{1}{2}\int_{\infty}^{M_i} dM_i \frac{\partial}{\partial q_i}. \tag{2·3}$$

The dotted line propagators contain no integration variables.

In order to perform the integrations of p_i, we make use of following generalized Feynman identity or poly-beta function formula :

$$\frac{1}{\prod_{i=1}^{l} A_i^{n_i}} = \frac{(n-1)!}{\prod_{i=1}^{l}(n_i-1)!}\int \frac{\prod_{i=1}^{l} x_i^{n_i-1}dx}{(\sum_{i=1}^{l} x_i A_i)^n} \tag{2·4}$$

with

$$n=\sum_{i=1}^{l} n_i,$$

$$dx \equiv \prod_{i=1}^{l} dx_i \cdot \delta(\sum_{i=1}^{l} x_i - 1), \quad (x_i \geqq 0).$$

By means of (2·4), the integration part of the given perturbation term can be written

$$\int \frac{\prod_{j=1}^{n} d^4p_j}{\prod_{i=1}^{\alpha}\{(p_i+q_i)^2+\bar{M}_i\}} = (\alpha-1)!\int dx \int \frac{\prod_{j=1}^{n} d^4p_j}{\{\bar{\varphi}(p)\}^{\alpha}}, \tag{2·5}$$

where

$$\left.\begin{aligned} &\varphi(p) = \sum_{i=1}^{\alpha} x_i p_i^2 + 2\sum_{i=1}^{\alpha} q_i' p_i + L, \\ &q_i' \equiv x_i q_i, \\ &L \equiv \sum_{i=1}^{\alpha} x_i (M_i + q_i^2), \\ &dx \equiv \prod_{i=1}^{\alpha} dx_i \cdot \delta\Big(\sum_{i=1}^{\alpha} x_i - 1\Big), \quad (x_i \geqq 0) \end{aligned}\right\} \tag{2·6}$$

and n is the number of independent variables p_j, which is given by[1)]

$$n = \alpha - m + r \tag{2·7}$$

where m is the order of the perturbation term (i.e. the number of vertices) and r the number of disconnected graphs.

Theorem

If $\alpha > 2n$,

$$\int \frac{\prod_{j=1}^{n} d^4 p_j}{\{\varphi(p)\}^{\alpha}} = (\pi^2 i)^n \frac{(\alpha - 2n - 1)!}{(\alpha - 1)!} \cdot \frac{1}{U^2(x)\{V(q, x)\}^{\alpha - 2n}}. \tag{2·8}$$

The meaning of the used symbols is as follows:

$$U(x) \equiv \sum x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n} \tag{2·9}$$

where $\sum$ covers all such possible combinations that $p_{\nu_1}, p_{\nu_2}, \ldots, p_{\nu_n}$ are linearly independent.

$$V \equiv L - (Q/U), \tag{2·10}$$

$$Q \equiv \sum_{C} U_C \cdot (\pm q_{C_1}' \pm q_{C_2}' \pm \ldots \pm q_{C_l}')^2. \tag{2·11}$$

C is a closed circuit belonging to the Q-diagram, and $C_1, C_2, \ldots, C_l$ are the lines belonging to C. The double signs should be taken as plus or minus corresponding to whether their relative directions on the round of C are same or not. U_C is a homogeneous expression of $(n-1)$-th order defined similarly to U with the remaining x_i except $x_{C_1}, x_{C_2}, \ldots, x_{C_l}$. $\sum_C$ covers all possible closed circuits. For example, for the Q-diagram of Fig. 1,

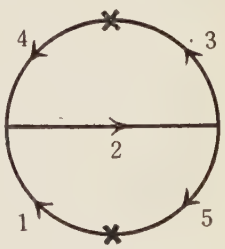

Fig. 1.

$$U = (x_1 + x_5)(x_3 + x_4) + (x_3 + x_4)x_2 + x_2(x_1 + x_5), \tag{2·12}$$

$$Q = x_2(q_1' - q_4' - q_3' + q_5')^2 + (x_3 + x_4)(q_1' + q_2' + q_5')^2 + (x_1 + x_5)(q_2' + q_3' + q_4')^2. \tag{2·13}$$

Proof

If $p_1, p_2, \ldots, p_n$ are independent, we can write

$$\varphi(p) = \sum_{i,j=1}^{n} a_{ij} p_i p_j + 2\sum_{i=1}^{n} b_i p_i + L, \tag{2·14}$$

where $a_{ij}=a_{ji}$ and b_i are linear expressions of x_i and of q_i', respectively. We introduce a matrix notation :

$$A=(a_{ij}),$$
$$(A^{-1})_{ij}=\mathfrak{A}_{ij}/\det A, \tag{2·15}$$

where $\mathfrak{A}_{ij}$ are the co-factors of a_{ij}. In (2·14), if a linear transformation

$$p_i=p_i'-\sum_j (A^{-1})_{ij}b_j \tag{2·16}$$

is performed,

$$\varphi(p')=\sum_{i,j} a_{ij}p_i'p_j'-\sum_{i,j}(A^{-1})_{ij}b_ib_j+L. \tag{2·17}$$

Furthermore, we transform it into a diagonal form :

$$\varphi(p'')=\sum_i c_i p_i''^2+L-\sum_{i,j}\mathfrak{A}_{ij}b_ib_j/\det A, \tag{2·18}$$

where c_i are the eigenvalues of A. In the form of (2·18), the integrations can be easily carried out by means of the integral formula[2],

$$\int\frac{d^4p}{(cp^2+M)^m}=\frac{\pi^2 i}{(m-1)(m-2)}\cdot\frac{1}{c^2M^{m-2}},\quad (m\geqq 3). \tag{2·19}$$

The numerical coefficient of (2·8) is obvious from (2·18) and (2·19). The denominator of the last factor becomes

$$(\det A)^2(\bar{L}-\sum_{i,j}\mathfrak{A}_{ij}b_ib_j/\det A)^{\alpha-2n} \tag{2·20}$$

using $\prod_{i=1}^{n} c_i=\det A$.

Now, $\det A$ is obviously invariant under unimodular transformation, and hence it is irrelevant to the choice of basic momenta. Since if $p_{\nu_1}, p_{\nu_2},\ldots, p_{\nu_n}$ are chosen as basic momenta the term $+x_{\nu_1}x_{\nu_2}\ldots x_{\nu_n}$ comes out from the main diagonal line of $\det A$, $\det A$ must contain U given by (2·9). Then we consider a quantity $R=\det A-U$, which is evidently an n-th order homogeneous function of x and a sum of the terms indicated by such combinations of suffices $(\nu_1', \nu_2', \ldots, \nu_n')$ that $p_{\nu_1'}, p_{\nu_2'},\ldots, p_{\nu_n'}$ are not independent. We assume that such terms exist. Then since $p_{\nu_1'}, p_{\nu_2'},\ldots, p_{\nu_n'}$ are not independent, there exist the momenta independent of these, one of which we denote by p_k. If a base containing p_k is adopted, we can regard that p_k flows along a closed circuit in the Q-diagram. If all x_j corresponding to the lines that compose the circuit are put equal to zero, then since

$$a_{jk}=a_{kj}=0,\quad (j=1, 2,\ldots, n), \tag{2·21}$$

$\det A=0$, and of course $U=0$, while $R\neq 0$. This is a contradiction. Hence it must be $R\equiv 0$, i.e.

$$\det A\equiv U. \tag{2·22}$$

Next, we prove that

$$Q \equiv \sum_{i,j} \mathfrak{A}_{ij} b_i b_j \tag{2·23}$$

can be written in the form as (2·11). Similarly to the above case, we consider p_k flowing along a closed circuit C, and choose it as one of basic momenta. When all x_i belonging to C are equal to zero, from (2·21) $\mathfrak{A}_{ij}=0$ except $i=j=k$, hence $Q=Q_C$, where

$$Q_C \equiv \mathfrak{A}_{kk} b_k^2 \equiv U_C \cdot (\pm q'_{C_1} \pm \cdots \pm q'_{C_l})^2. \tag{2·24}$$

Since this holds for every circuit exclusively, a quantity $R' \equiv Q - \sum_C Q_C$ is zero whenever all x_i of every circuit are equal to zero. But since any term of R' is of $(n-1)$-th order, the circuits which contain no x_i belonging to the term always exist. In order that these are consistent, it must be $R' \equiv 0$, i.e.

$$Q \equiv \sum_C Q_C \,. \tag{q.e.d.}$$

From (2·5) and (2·8) we get as the final result*

$$(\pi^2 i)^n (\alpha - 2n - 1)! \int \frac{f(D)\,dx}{U^2(x)\,\{V(q,x)\}^{\alpha-2n}}\,, \tag{2·25}$$

where $f(D)$ is a (matrix) function of D-operators yielded from fermion propagators (2·2) and derivative couplings. The effect of D-operators is calculated in the following way. To operate D_i firstly is equivalent to multiply by a factor

$$Y_i(q,x) \equiv (2x_i)^{-1} (\partial V/\partial q_i) = q_i - U^{-1} \sum_{C(i)} U_{C(i)} (x_i q_i \pm \ldots) \tag{2·26}$$

to the integrand, where $C(i)$'s mean circuits containing line i. When D_j operates further, we get a term, in which the power of V diminishes by one owing to the integration of M_j in (2·3), multiplied by a factor

$$X_{ij}(x) \equiv -x_j^{-1} \cdot \partial Y_i/\partial q_j = U^{-1} \cdot \sum_{C(i,j)} \pm U_{C(i,j)} \cdot \delta = X_{ji}(x) \tag{2·27}$$

with a numerical coefficient, besides the term of $Y_i Y_j$. In (2·27), $C(i,j)$'s denote circuits containing both of lines i and j, and δ is a usual 4-dimensional Kronecker's delta. The double sign should be taken as plus or minus corresponding to whether the directions of lines i and j on the round of C are same or not. Since X_{ij} does not contain q or M, we need no more new kinds of factors if D-operators operate furthermore. Therefore we can easily write down the result that $f(D)$ operated, using Y_i and X_{ij}.

§ 3. Expression by external momenta

In the last section we expressed explicitly any perturbation term in the parametric integral form. But it is somewhat inconvenient that we must use the quantities q_i which

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* Similar parametric representations have been used by Hurst,[5] Thirring[6] and Petermann[7], but no one of them gave the explicit form or treated fermion propagators.

are undetermined uniquely by external momenta k_j. Therefore in this section we consider to express V and Y_i as functions of k_j.

First, we find easily for vacuum polarization

$$V(x)=M(x)\equiv\sum_{i=1}^{\alpha}x_iM_i\,. \tag{3·1}$$

There is no problem. Then, we consider self-energy (Feynman) diagram (Fig. 2). Since external momentum is k only, we can write

$$V(k,x)=M(x)+\eta(x)k^2, \tag{3·2}$$

with

$$\eta(x)\equiv W(x)/U(x), \tag{3·3}$$

where $W(x)$ is an expression of $(n+1)$-th order which is determined in the following way. Introducing a matrix of $(n+1)$-th order

Fig. 2.

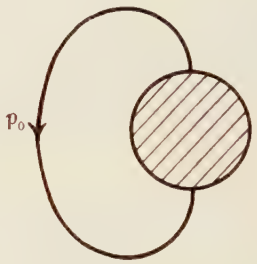

Fig. 3.

$$B(q,x)\equiv\begin{pmatrix}\sum_{j=1}^{\alpha}x_jq_j^2 & b_i\\ b_i & A\end{pmatrix}, \tag{3·4}$$

we get from (2·20) and (2·6)

$$V(q,x)=M(x)+\det B(q,x)/U(x) \tag{3·5}$$

for general V. Comparing (3·2) and (3·5), it follows

$$W(x)\cdot k^2=\det B(k,x), \tag{3·6}$$

in which the actual values of q_i in Fig. 2 are used. If we suppose the vacuum polarization graph (Fig. 3) which is obtained by the conjunction of two external lines in Fig. 2, the matrix A of Fig. 3 is just $B(k,x)|_{k=1}$, that is, W equals to U of Fig. 3. Since the connected line denoted by 0, however, does not really exist, $x_0=0$. Thus the general form of $W(x)$ has been found explicitly. For example, for Fig. 1,

$$W(x)=x_1x_2x_3+x_1x_2x_5+x_1x_3x_4+x_1x_3x_5+x_1x_4x_5+x_2x_3x_4+x_2x_4x_5+x_3x_4x_5. \tag{3·7}$$

Since the supposed momentum p_0 could be regarded as the $(n+1)$-th independent variable, every term of $W(x)$ must contain x_i which belongs to the circuit along which p_0 flows. Therefore we obtain an inequality as

$$0\leqq\eta(x)\leqq\sum_i{}'x_i\leqq 1, \tag{3·8}$$

where $\sum'$ denotes a sum over an arbitrary set of lines connecting two points of intersection with the external lines in Fig. 2. As such sets with no overlaps exist at least two for a proper graph, an upper bound of η is $1/2$.*)

Now we can easily generalize the result to the diagram Fig. 4 with $l+1$ external momenta, between which the conservation law

$$\sum_{i=1}^{l} k_i = k_t \tag{3·9}$$

holds. We write

$$V(k, x) = M(x) + \sum_{i=1}^{l} \eta_i(x) k_i^2 + \sum_{i>j} \eta_{ij}(x) k_i k_j . \tag{3·10}$$

If $l-1$ external momenta vanished, Fig. 4 would reduce to a graph such as Fig. 2. We discriminate η of the reduced graphs by the same superscripts as the suffices of the remaining k_j. Then

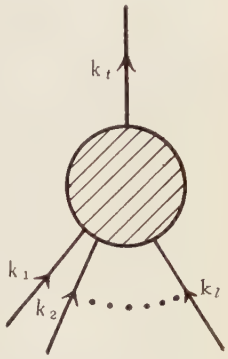

Fig. 4.

and

$$\left.\begin{aligned} &\eta_i = \eta^{(it)} \\ &\eta_i + \eta_j - \eta_{ij} = \eta^{(ij)}, \end{aligned}\right\} \tag{3·11}$$

the right-hand sides of which are known. Hence from (3·10) and (3·11),

$$V(k, x) = M(x) + \sum_{i=1}^{l} \eta^{(it)}(x) k_i^2 + \sum_{i>j} \{\eta^{(it)}(x) + \eta^{(jt)}(x) - \eta^{(ij)}(x)\} k_i k_j . \tag{3·12}$$

Finally, we consider the expression of Y_i by k_j. In Appendix B it will be proved that

$$Y_i(k, x) = (U(x))^{-1} \sum_{C(0,i)} \pm W_{C(0,i)}(x) \cdot k \tag{3·13}$$

for the self-energy graph (Fig. 2). The notations are analogous to those used in (2·27) in relation to Fig. 3. For the example for Fig. 1

$$\begin{aligned} Y_1 &= U^{-1} \cdot (x_2 x_3 + x_2 x_5 + x_3 x_5 + x_4 x_5) k, \\ Y_2 &= U^{-1} \cdot (x_4 x_5 - x_1 x_3) k, \quad \text{etc.} \end{aligned} \tag{3·14}$$

For Fig. 4, we have

$$Y_i(k, x) = (U(x))^{-1} \sum_{j=1}^{l} \Big(\sum_{C_j(0,i)} \pm W^{(jt)}_{C_j(0,i)} \Big) k_j . \tag{3·15}$$

Thus with ignorance of the explicit expression of $V(q, x)$ ((2·10) and (2·11)), we can write down the effect of D-operators using (3·15) and (2·27). Hence we may use D-operators for functions of V expressed by k_j.

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* Though the physically requested upper limit is 1/4, that is not yet proved mathematically.

§ 4. Proof of power-counting theorem*)

From the rigorous standpoint the integral of perturbation term cannot be regarded as the well-defined one *a priori*. Because since the propagators D_F, Δ_F and S_F are not usual functions but pseudo-functions[8] which have continuous singularity, the integral may be meaningful only when an integration method is actually indicated. For instance, the derivation of Feynman's formula (2·19) is not covariant under Lorentz transformation of the integration variable p, although the right-hand side of (2·19) is of course Lorentz invariant. Therefore from such point of view the order exchange of momentum integrations and parametric ones in § 2 should be regarded as the definition of the general perturbation term, which is the most natural generalization of Feynman's method. This method satisfies the following requirements, as is easily verified from the general integral formula :

1) It is applicable to any perturbation term.
2) The result of integration is unique.
3) It is formally covariant.

It will be verified in Appendix A that the use of D-operators does not give rise to any ambiguity.

Now, Dyson[1] has used a non-linear transformation of energies, (47) in his paper, in order to prove the power-counting theorem. But this integration method has very unsatisfactory features, namely it destroys the conditions 2) and 3).** Therefore since his method should not be applied to the actual calculations except the case of $n=1$, we can by no means assert that the theorem has been proved (his proof is incomplete also in other respects). Hence we must prove it by means of the general integral formula reasonably. In order to prove the power-counting theorem more elegantly, we make use of a more general formula which is applicable also in the case of $\alpha \geqq 2n+[\zeta/2]$ (ζ is the power of D's in $f(D)$ and $[\ ]$ is Gauss' symbol),

$$(\pi^2 i)^n(\alpha-2n+s-1)!\cdot\lambda^s\int\frac{f(D)\,d(x,y)}{U^2(x)\{\bar{V}(k,x,y)\}^{\alpha-2n+s}} \tag{4·1}$$

with

$$d(x,y)\equiv(y^{s-1}/(s-1)!)\,dy\prod_{i=1}^{\alpha}dx_i\cdot\delta(y+\sum_{i=1}^{\alpha}x_i-1),\quad (x_i,\,y\geqq 0) \tag{4·2}$$

$$V(k,x,y)\equiv y\lambda+V(k,x) \tag{4·3}$$

instead of (2·25). (4·1) is derived by multiplication by a factor $\lambda^s/\bar{\lambda}^s$, where $\alpha-2n-|\zeta/2|+s\geqq 0$, before (2·4) is applied. λ is an arbitrary constant but must be

* By the terminology of "power-counting theorem" we mean the theorem that in order to judge the ultra-violet divergence of any perturbation term we have only to count powers of the denominator and the numerator of the integrand.

** In fact, we reach to different results corresponding to the choices of basic momenta or his variable α.[9]

positive due to a physical requirement.*)

We first examine the divergences of (4·1) without $f(D)$. The divergences may happen from zero points of U and $\bar{V}$, but the latters would not exist if $\mathcal{E}$ were finite. Hence for the ultra-violet divergences we have only to examine zeros of U. Since every term of U is non-negative, $U=0$ requires some of x_i are zero. Then, if the number of the independent closed circuit contained in the part of $x_i=0$ is m, U has a zero point of just m-th order. Therefore the condition that the divergence does not occur is

$$\alpha' > 2m \quad (m=1, 2,\ldots, n) \tag{4·4}$$

for any choice of independent circuits (α' is the number of lines $x_i=0$). Because if (4·4), i.e. $\alpha'-\alpha'/\alpha \geqq 2m$, holds,

$$1/U^2(x) \leqq \text{const.}/\prod_{i=1}^{\alpha} x_i^{1-1/\alpha}, \tag{4·5}$$

and hence the integral (4·1) (but not (2·5)) is absolutely convergent when $\mathcal{E}$ is finite. Thus we have proved the power-counting theorem, which is stated in the following manner :

Theorem

When we regard as if only $m (m=1, 2, \ldots, n)$ independent variables p_i were variables and the other $n-m$ independent ones were constants, there are no ultra-violet divergences if the power of the denominator (i.e. $2\alpha'$) is always higher than that of the numerator (i.e. $4m$).

Next, we must examine the effect of $f(D)$ on divergences. From (2·26) and (2·27), it proves that Y_i have no singularities and the poles of X_{ij} occur in first order only when all x_k belonging to a $C(i, j)$ are zero. Therefore since the order of the new poles induced by a product of D-operators is the maximum number of their pairs contained in the part of $x_i=0$, we obtain, instead of (4·4), as the convergence condition

$$\alpha' - [\zeta'/2] > 2m \quad (m=1, 2,\ldots,n), \tag{4·6}$$

where ζ' is the number of D-operators contained in the above part. By (4·6) the power-counting theorem has been proved for the general case.

Finally we remark the fact that Y_i have no poles. This means that odd D cannot induce any new divergence, namely any odd order divergence never appears. This fact is already pointed out for the electrodynamics.[10]

§ 5. Relation with spectral functions

The zero points of V are classified into two kinds.

i) The case in which they occur at the limits of integration (i.e. $x_i=\delta_{ij}$). Then the

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* Negative $1/\bar{\lambda}$ corresponds to a propagator of imaginary mass, therefore physically unallowable. But, of course, $\lambda \geqq 0$ is indifferent for the case of $\alpha > 2n+[\zeta/2]$.

integral may give an infinity for $\varepsilon \to +0$. But since in this case $V=M_j$, it can happen only when $M_j=0$. Hence it is an infra-red divergence.

ii) The other case. From the formula

$$1/(u-i\varepsilon)^m = \mathrm{Pf}\ 1/u^m + (\pi i/(m-1)!)\,\delta^{(m-1)}(-u), \tag{5·1}$$

where Pf denotes Hadamard's pseudo-function,[8] it proves that instead of divergence there appears an imaginary part, on which we shall study below in detail.

We consider a connected graph G ($r=1$ in (2·7)). The matrix element corresponding to G is $ic\cdot\Xi_G$, where c is a positive constant and Ξ_G is the expression (4·1) without the coefficient $(\pi^2 i)^n$. The imaginary part of Ξ_G, which corresponds to real processes, appears from only $\bar{V}$, because $i\gamma$ in $f(D)$ is not imaginary in the true sense. Hence from (4·1), (4·2), (4·3) and (5·1) by the transformation $y=1-t$, $x_i=tx_i'$, we have

$$\mathrm{Im}\,\Xi_G = \frac{\pi\lambda^s}{(s-1)!}\int \frac{f(D)}{U^2(x')}\,dx' \int t^{\alpha-2n-1}(1-t)^{s-1}\cdot\delta^{(\alpha-2n-s-1)}(-tV(k,x')-(1-t)\lambda)\,dt. \tag{5·2}$$

Since the formula

$$\delta^{(m-1)}(u) = \frac{\lambda^l}{(l-1)!}\int_0^1 t^{m-1}(1-t)^{l-1}\cdot\delta^{(m+l-1)}(tu-(1-t)\lambda)\,dt^{*)} \tag{5·3}$$

holds even for $m \leqq 0$ in the sense of

$$\delta^{(m-1)}(u) \equiv (u^{|m|}/|m|!)\,\theta(u),$$

provided $\lambda>0$, (5·2) becomes

$$\mathrm{Im}\,\Xi_G = \pi\int \frac{f(D)}{U^2(x)}\,\delta^{(\alpha-2n-1)}(-V(k,x))\,dx, \tag{5·4}$$

which is always meaningful.

After the above preparation, we consider the relation between the general integral formula and spectral functions. From (4·1) and (3·2), the contribution from G, which consists of l proper self-energy parts, to a modified boson (scalar) propagator $\Delta_F'(k)$ is given by

$$|\Delta_F'(k)|_G = -2i\,\frac{1}{(k^2+\bar{m}^2)^{l+1}}\cdot\frac{(\beta+s-1)!}{(4\pi)^{2n}}\lambda^s\int \frac{F(k^2,x,y)\,d(x,y)}{U^2(x)\,\{\eta(x)k^2+\bar{M}(x,y)\}^{\beta+s}}. \tag{5·5}$$

In (5·5), the dotted line propagators in the Q-diagram are excluded from V for convenience sake, hence $\beta=\alpha-2n-l-1$, and correspondingly the definitions of η and M are slightly modified; m is the mass of particle under consideration, and $F(k^2,x,y)$ is a function caused by $f(D)$. Introducing a parametric integral (2·4) and a new spectral variable u, we have

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* This is obtained by integrating the imaginary part of Feynman identity.

$$[\Delta'_F(k)]_G = -2i\frac{(\beta+s-1)!}{(4\pi)^{2u}}\lambda^s\int\frac{F(k^2,x,y)\,d(x,y)}{U^2(x)\eta_j^{\beta+s}(x)}\int_0^1 t^l(1-t)^{\beta+s-1}dt$$
$$\cdot\int_{-\infty}^{\infty}\frac{\delta(u-tm^2-(1-t)M(x,y)/\eta(x))}{(k^2+\bar{u})^{\beta+s+l+1}}du. \qquad (5\cdot6)$$

Further expanding $F(k^2, x, y)$ into a (finite) power series of (k^2+u) and integrating by parts on each term, we can write

$$[\Delta'_F(k)]_G = -2i\int_0^\infty\frac{\tilde{\sigma}_G(u)}{k^2+\bar{u}}du, \qquad (5\cdot7)$$

using some real function $\tilde{\sigma}_G(u)$. The lower limit of integration in (5·7) is zero owing to $\tilde{\sigma}_G(u)=0$ for $u<0$ from (3·8) and (5·6). The form of $\tilde{\sigma}_G(u)$ is decided from the comparison of real parts of (5·7) and (5·5) :

$$\tilde{\sigma}_G(u)=\sigma_G(u)+\sum_{j=0}^{l}A_{Gj}\cdot\delta^{(j)}(u-m^2) \qquad (5\cdot8)$$

with

$$\sigma_G(u)\equiv\frac{1}{(-u+m^2)^{l+1}}\cdot\frac{1}{(4\pi)^{2n}}\int\frac{f(D)}{U^2(x)}\delta^{(\beta-1)}(\eta(x)u-M(x))\,dx, \qquad (5\cdot9)$$

making use of (5·4). $\sigma_G(u)$ has no spectrum at $u=m^2$ for a stable particle. A_{Gj} are some (usually divergent) constants, which are removed by mass and coupling renormalizations.

Thus the existence of Lehmann's spectral function $\rho(u)$[4] has been proved by the perturbation method, and at the same time we obtain the *explicit* perturbation expansion of $\rho(u)$:

$$\left.\begin{aligned}&\rho(u)=\delta(u-m^2)+\sum_{m=1}^{\infty}g^{2m}\rho_m(u)\\ \text{with}\quad&\rho_m(u)=\sum_{i=1}^{m}z_{m-i}\cdot\sigma_i(u),\\ &z_0=1,\quad z_m=\int_0^\infty\rho_m(u)\,du,\end{aligned}\right\} \qquad (5\cdot10)$$

where σ_m is the total sum of σ_G of order $2m$ except irreducible mass terms.*) In (5·10) the relation between $\rho(u)$ and the renormalization factor Z

$$Z^{-1}=\int_0^\infty\rho(u)\,du \qquad (5\cdot11)$$

was utilized.

The Green function of fermion is similar. For example, the expression corresponding to (5·9) is (for proper graph)

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* In the actual calculation it is convenient to use the above method also for general mass renormalizations.

$$i\gamma k\sigma_{\alpha}^{(l)}(u)+\sigma_{\alpha}^{(l)}(u)=\frac{1}{i\gamma k+m}\cdot\frac{1}{(4\pi)^{2\mu}}\int\frac{f(D)}{U^{\mu}(x)}\delta^{(\beta-1)}(\eta(x)u-M(x))\,dx\cdot\frac{1}{i\gamma k+m}\,. \tag{5·12}$$

Particularly, for g^2-term of the $ps(ps)$ meson theory this becomes

$$(i\gamma k-\sqrt{u})\,\sigma_1^{(1)}+\sigma_1^{(2)}=\frac{1}{i\gamma k+m}\cdot\frac{1}{(4\pi)^2}\int_0^1(i\gamma kx+m)$$

$$\cdot\theta(x(1-x)u-(1-x)m^2-x\mu^2)\,dx\frac{1}{i\gamma k+m}\,, \tag{5·13}$$

where $\sigma^{(2)}=\sigma^{(0)}+\sqrt{u}\,\sigma^{(1)}$, and m and μ are the masses of nucleon and meson, respectively. $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ are calculated very easily and naturally coincide with Lehmann's results.[4]

The above considerations are immediately generalized to many-body Green functions. For simplicity we take up the three-body Green function of the meson theory:

$$G(k_1,k_2,k_3)=i\sum_{\lambda,\lambda'=0,1}(i\gamma k_1)^{\lambda}\gamma_5(i\gamma k_2)^{\lambda'}G_{\lambda\lambda'}(k_1^2,k_2^2,k_3^2). \tag{5·14}$$

$$(k_3=k_2-k_1\,.)$$

Though V in this case contains three momenta k_1^2, k_2^2 and k_3^2 the above method is applicable if we regard k_2^2 and k_3^2 as mere parameters. Thus we have

$$G_{\lambda\lambda'}(k_1^2,k_2^2,k_3^2)=\int_{-\infty}^{\infty}\frac{\rho_{\lambda\lambda'}(u_1,-k_2^2,-k_3^2)}{k_1^2+\bar{u}_1}du_1 \tag{5·15}$$

where $\rho_{\lambda\lambda'}$ are real. Hence

$$G_{\lambda\lambda'}(k_1^2,k_2^2,k_3^2)=\int\frac{\rho_{\lambda\lambda'}(u_1,u_2,u_3)}{k_1^2+\bar{u}_1}\delta(k_2^2+u_2)\,\delta(k_3^2+u_3)\,du_1du_2du_3, \tag{5·16}$$

which is the spectral representation of the three-body Green function. Taking imaginary part of (5·16), we obtain

$$\rho_{\lambda\lambda'}(u_1,u_2,u_3)=\pi^{-1}\,\mathrm{Im}\,G_{\lambda\lambda'}(-u_1,-u_2,-u_3). \tag{5·17}$$

Therefore the asymmetry of (5·16) on the suffices is merely apparent and they are mutually exchangeable. The expression (5·16) differs from Nambu's one[3],

$$\int\frac{\nu_{\lambda\lambda'}(v_1,v_2,v_3)}{(k_1^2+\bar{v}_1)(k_2^2+\bar{v}_2)(k_3^2+\bar{v}_3)}dv_1dv_2dv_3\,, \tag{5·18}$$

which was postulated without any proof. But it is easily proved that (5·16) is more general than (5·18). Of course v_j do not correspond to u_j.

§ 6. The convergency of perturbation series

On the convergency of perturbation series it has been already investigated by Hurst,[5] Utiyama and Imamura,[11] Thirring,[6] Petermann[7] and many others, and some of them

proved the series is actually divergent in the case of φ^3-interaction. In this section we shall study this problem more elegantly and critically by means of the general integral formula.

First we rigorously review the proof of divergence for arbitrary boson-boson direct interaction. We adopt the following Feynman cut-off. Introducing a universal cut-off parameter Λ, we define the cut-off propagator

$$\frac{1}{(p_i+q_i)^2+M_i}\cdot\frac{\Lambda-M_i}{(p_i+q_i)^2+\Lambda}=\int_{M_i}^{\Lambda}\frac{dM_i}{\{(p_i+q_i)^2+M_i\}^2} \tag{6·1}$$

instead of (2·1). This cut-off method does not destroy the unitarity of S-matrix. The general integral formula then becomes for (6·1)

$$\Xi_G=(2\alpha-2n-1)!\int\prod_{i=1}^{\alpha}dM_i\int\frac{\prod_{i=1}^{\alpha}x_i\cdot dx}{U^2\cdot V^{2\alpha-2n}}, \tag{6·2}$$

which is always convergent.

As mentioned in the last section, when V has zero points Ξ_G has an imaginary part. Therefore we assume

$$V(k,x)>0, \tag{6·3}$$

namely we consider the case that any real process does not take place. Then Ξ_G is positive definite.

Now it should be noted that

$$U\leqq\sum_{\nu}\prod_{i=1}^{n}x_{\nu_i}, \tag{6·4}$$

where $\sum_{\nu}$ means the sum over all combinations of $(\nu_1,\nu_2,\ldots,\nu_n)$. The right-hand side of (6·4) obviously has the maximum at $x_i=1/\alpha$ $(i=1,2,\ldots,\alpha)$ under the condition $\sum_{i=1}^{\alpha}x_i=1$. Hence

$$U\leqq{}_{\alpha}C_n(1/\alpha)^n<2^{\alpha}/\alpha^n. \tag{6·5}$$

From (6·2), (6·5) and (2·4), we obtain

$$\Xi_G>\frac{\alpha^{2n}\cdot(2\alpha-2n-1)!}{2^{2\alpha}\cdot(2\alpha-1)!}\cdot\frac{\prod_{i=1}^{\alpha}(\Lambda-M_i)}{(V_{\max}(\Lambda))^{2\alpha-2n}}, \tag{6·6}$$

alternatively a lower bound can be given also in the form independent of Λ. (6·6) has the form $(\text{const.})^m$ asymptotically, where m means the order of G. Therefore if N_m, the number of graphs of order m, increases so that $(1/m)\log N_m\to\infty$, the series is divergent.

Now, we must remark about the vital restriction, (6·3). If it is removed, any

estimation by inequalities is impossible, because the integrand becomes a pseudo-function*). In fact, we can show that a very slight variation of an approximation for the integrand of (6·2) can change the sign of Re Ξ_G. This suggests when real processes take place the sign of Re Ξ_G depends sensitively on the detailed form of the graph. Hence it is very probable that the series will converge in this case without unreliable hope on γ-matrices.

The above proof takes no account of the renormalization. Petermann[7] has shown the mass renormalization cannot change the situation for φ^3-interaction, though his estimation is rather rough. But he did not consider the coupling renormalization because of its arbitrariness. From Lehmann's standpoint mentioned in the last section, however, the renormalized propagator is represented by the imaginary part and what comes out from the real part of self-energy is $Z=\sum_G f^{2m} A_{G_0}$(cf. (5·8)). Thus what we can prove to be divergent is nothing but Z-factor,**) and hence we have no definite information on the renormalized propagators.

Finally we remark on the proof of convergence by Salam and Matthews,[12] and Edwards,[13] who have made use of Fredholm theory. But their proof is very doubtful, because in the applicability condition of Fredholm theory they use the absolute value of a pseudo-function, which is meaningless and for which, of course, the power-counting theorem breaks down.

As a conclusion, it is perhaps impossible to prove the convergence or divergence of the perturbation series without *exact* calculation of integrations. Therefore it is more preferable to consider the series of $\rho(u)$.

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Appendix A

We shall discuss D-operator introduced in § 2 in more detail. The exchange of D-operators and the p_j-integrations should be regarded as the definition as mentioned in § 4 because of their improper integrals. But they will commute in the ordinary sense if the lower limit of M_i-integration in (2·3) is cut off at Λ, when the propagator has such a form as (6·1). Hence the exchange of them means automatically that Λ tends to infinity after cut-off calculation has been carried out. The absence of odd order divergences mentioned in § 4 is due to the use of D-operator, or Lorentz invariance in physical terminology.

We remark on higher powers of D such as D_i^2 that appears in the case of derivative couplings. This is defined conveniently as if the second D_i differs from the first D_i, so that

* Thirring[6] asserted the proof is valid for arbitrary external momenta if the power of V is always even. But this is clearly a mistaken idea, because, for example, $\int_{-1}^{1} du/(u-i\varepsilon)^2=-2$.

** What can be proved rigorously is Z^{-1} (cf. Appendix C).

$$\frac{D_{i\mu}D_{i\nu}}{(p_i+q_i)^2+\bar{M}_i}=\frac{(p_i+q_i)_\mu(p_i+q_i)_\nu}{(p_i+q_i)^2+\bar{M}_i}\,. \tag{A·1}$$

Then the expression corresponding to the identity

$$(D_i^2+M_i)/\{(p_i+q_i)^2+\bar{M}_i\}=1 \tag{A·2}$$

can be proved from the general integral formula directly. We easily get the identity

$$Y_i^2+M_i=\partial V/\partial x_i \tag{A·3}$$

from (3·5) by means of Jacobi's theorem of determinants, and also

$$\sum_{C(i)} U_{C(i)}=\partial U/\partial x_i\,. \tag{A·4}$$

Therefore for an arbitrary $\beta(\neq 0)$

$$\begin{aligned}(D_i^2+M_i)\frac{1}{U^2\bar{V}^{\beta+1}}&=\frac{1}{U^2}\left\{\frac{\partial V/\partial x_i}{\bar{V}^{\beta+1}}+\frac{4\partial U/\partial x_i}{2\beta\cdot U\bar{V}^\beta}\right\}\\&=-\beta^{-1}(\partial/\partial x_i)\,(1/U^2\bar{V}^\beta).\end{aligned} \tag{A·5}$$

Further we note

$$\int\frac{d'x}{U^2\bar{V}^\beta}=(\beta-\alpha+2n+1)\int\frac{dx}{U^2\bar{V}^\beta}\quad{}^{*)} \tag{A·6}$$

with

$$d'x\equiv \mathop{\Pi}_i dx_i\,\delta'\Big(\sum_i x_i-1\Big),$$

because, by means of Euler's equation for a homogeneous function, integrations by parts and $u\cdot\delta'(u)=-\delta(u)$,

$$\begin{aligned}(\beta+2n)\int\frac{dx}{U^2\bar{V}^\beta}&=-\sum_i\int x_i\frac{\partial}{\partial x_i}\Big(\frac{1}{U^2\bar{V}^\beta}\Big)dx\\&=\sum_i\left[\int\frac{dx}{U^2\bar{V}^\beta}+\int\frac{x_i d'x}{U^2\bar{V}^\beta}\right]\\&=(\alpha-1)\int\frac{dx}{U^2\bar{V}^\beta}+\int\frac{d'x}{U^2\bar{V}^\beta}\,.\end{aligned}$$

Thus we obtain by an integration by parts from (A·5) and (A·6) (putting $\beta=\alpha-2n-1$)

$$(\alpha-2n-1)!(D_i^2+M_i)\int\frac{dx}{U^2\bar{V}^{\alpha-2n}}=(\alpha-2n-2)!\int\Big(\frac{1}{U^2\bar{V}^{\alpha-2n-1}}\Big)_{x_i=0}dx^{(i)}, \tag{A·7}$$

as was expected (the superscript (i) means the exclusion of x_i).

---

* Differentiating by M_i, we obtain

$$\int\frac{\Pi x_i{}^{\alpha_i}d'x}{U^2\bar{V}^{\alpha+\sum\alpha_i-2n-1}}=0,$$

which is useful to prove the expression corresponding to (A·7) in the case of cut-off propagators.

Appendix B

We verify the formula (3·13) (for simplicity we put $i=1$). From (2·26) and (3·5)

$$Y_1=\frac{1}{U}\cdot\frac{1}{2x_1}\left(\frac{\partial}{\partial q_1}\det B\right)_{q=k}, \tag{B·1}$$

where $q=k$ denotes to use the actual values of q_i in Fig. 2.

$$\frac{1}{2x_1}\frac{\partial}{\partial q_1}\det B=q_1\begin{vmatrix} a_{11}a_{12}\cdots a_{1n}\\ a_{21}a_{22}\cdots a_{2n}\\ \cdots\cdots\cdots\\ a_{n1}a_{n2}\cdots a_{nn}\end{vmatrix}-\begin{vmatrix} b_1\,b_2\,\cdots\,b_n\\ a_{21}a_{22}\cdots a_{2n}\\ \cdots\cdots\cdots\\ a_{n1}a_{n2}\cdots a_{nn}\end{vmatrix} \tag{B·2}$$

If the matrix of Fig. 3

$$B'\equiv\begin{pmatrix} \sum_{j=0}^{\alpha}x_jq_j^{\nu} & b_0 & b_i\\ b_0 & a_{00} & a_{0i}\\ b_i & a_{i0} & A\end{pmatrix} \tag{B·3}$$

is introduced, from (2·27) and (2·26)

$$\sum_{C(0,1)}\pm W_{C(0,1)}\cdot\delta=-(1/x_1)(\partial/\partial q_1)(1/2x_0)(\partial/\partial q_0)\det B'$$

$$=-\begin{vmatrix} a_{01}a_{02}\cdots a_{0n}\\ a_{21}a_{22}\cdots a_{2n}\\ \cdots\cdots\cdots\\ a_{n1}a_{n2}\cdots a_{nn}\end{vmatrix}\cdot\delta. \tag{B·4}$$

Hence we have only to prove

$$(q_1a_{1j}-b_j)_{q=k}=-a_{0j}k,\quad (j=1,\,2,\,\ldots,\,n). \tag{B·5}$$

Now, we denote by C one of closed circuits which pass two points of intersection with the external lines in Fig. 2. (On dotted lines C may overlap with itself.)

1) The case in which C does not include line 1. We can regard that both p_0 and k flow along the circuit which consists of a branch of C and line 0 in Fig. 3. Then $q_1=0$ and $(b_j)_{q=k}=a_{0j}\cdot k$.

2) The case in which a branch K of C includes line 1. We can regard that p_1, k and p_0 flow along C, K and $-L$(L means the other branch of C, i.e. $C-K$) plus line 0, respectively (see Fig. 5). Then a_{1j}, $(b_j)_{q=k}$ and a_{0j} correspond to the intersections of $C(j)$ and C, K and $-L$, respectively.

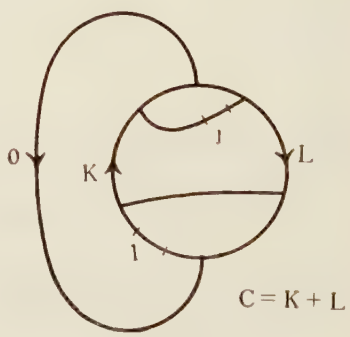

Fig. 5.

Thus (B·5) is valid for both cases

Appendix C

The calculation of $\rho(u)$ in § 5 is rather inconvenient on account of the presence of improper self-energy graphs. Hence we introduce a propagator defined only by proper self-energy, $\sum^*$:

$$\Delta_F^* \equiv \Delta_F \sum^* \Delta_F . \tag{C·1}$$

As is well known

$$\Delta_F' = \Delta_F / (1 - \Delta_F^{-1} \Delta_F^*). \tag{C·2}$$

The renormalized propagator Δ_{F1}^* corresponding to Δ_F^* is defined by

$$\Delta_{F1}' = Z^{-1} \Delta_F' = \Delta_F / (1 - \Delta_F^{-1} \Delta_{F1}^*). \tag{C·3}$$

Then from (C·2) and (C·3)

$$\Delta_{F1}^* = (1 - Z) \Delta_F + Z \Delta_F^* . \tag{C·4}$$

By virtue of the same argument as in § 5, the spectral representation of Δ_{F1}^* is possible:

$$\Delta_{F1}^*(k^2) = -2i \int \frac{\rho^*(u)}{k^2 + u} du. \tag{C·5}$$

Substituting the spectral representations in (C·3),

$$\int \frac{\rho(u)}{k^2 + u} du = \frac{1}{k^2 + m^2} \left\{ 1 - (k^2 + m^2) \int \frac{\rho^*(u)}{k^2 + u} du \right\}^{-1} . \tag{C·6}$$

In (C·6) multiplying by $k^2 + m^2$ and taking the limit $-k^2 \to \infty$, we obtain

$$\int \rho^*(u) du = 1 - Z, \tag{C·7}$$

provided the theory is consistent. From the imaginary part of (C·6)

$$\rho(u) = \delta(u - m^2) + \rho^*(u) \left\{ (1 + (u - m^2) \int \frac{\mathrm{P}}{v - u} \rho^*(v) dv)^2 + (\pi (u - m^2) \rho^*(u))^2 \right\}^{-1} . \tag{C·8}$$

Thus $\rho^*(u) \geqq 0$ if $\rho(u) \geqq 0$.

From (C·4) and (C·7) we obtain the explicit expansion of $\rho^*(u)$ corresponding to (5·10):

$$\left.\begin{aligned} \rho^*(u) &= \sum_{m=1}^{\infty} g^{2m} \rho_m^*(u), \\ \rho_m^*(u) &= \sigma_m^*(u) - \sum_{i=1}^{m-1} z_{m-i}^* \sigma_i^*(u), \\ z_m^* &= \int_0^{\infty} \rho_m^*(u) du, \end{aligned}\right\} \tag{C·9}$$

where $\sigma_m^*(u)$ are the sums of σ_G of order $2m$ over proper graphs only. If $\rho^*(u)$ has been calculated, $\rho(u)$ is given immediately by (C·8).

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Note added in proof: The inadequacies of Dyson's proof of the power-counting theorem are mainly caused by the unreasonable application of Cauchy's theorem to the multiple integral. According to his original proof the contributions from the "displaced poles" may diverge even for "primitive" integrals, contrary to his assertion. On these details the author will communicate privately to interested readers.
[To § 2] In (2·5) and (2·6) it should be understood that $p_i \equiv 0$ for dotted lines. In (2·24), $\mathfrak{A}_{kk}$ means the value when all x_i belonging to C are equal to zero.

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Quantum Field Theory of Unstable Particles

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Two methods of the renormalization are proposed for unstable particles. One is connected with the resonance in scattering and the other with the time development of the unstable particle state. The relation between the two formulations is discussed. Arbitrariness in the choice of the renormalization procedure comes from the uncertainty principle of the quantum mechanics.

§ 1. Introduction

Ever since the establishment of Tomanaga and Schwinger's covariant field theory in 1948, we have a consistent method for treating the quantized fields with renormalizable interactions. However, we have so far considered only stable particles such as electrons or nucleons, hence it would be worth while to discuss here the field theories concerned with unstable particles. Lack of knowledge of the renormalization theory of unstable particles will be fulfilled in the present paper.

In formulating the renormalization theory for unstable particles, we would refer to two theories, i. e. the renormalization theory for the stable particles and the resonance theory for the atomic nuclei. There are, however, some essential differences between stable and unstable particles. One of them is the uncertainty relation between the mass and the life-time of the unstable particles, which is due to the fundamental nature of quantum mechanics.

Taking account of such essential differences we shall construct a theory of unstable particles satisfying the following conditions :

i) Our theory must coincides with the current one when the width γ tends to zero.

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ii) When the coupling constant g is small, our theory has to give the same result as obtained by the perturbation method.

iii) After the renormalization procedure is performed, the theory contains no divergences.

vi) Needless to say, the theory must be simple and convenient and physically intuitive.

Under such wide restrictions, theories that we can construct consistently are not unique. We believe that the following two theories are most reasonable and simplest. It is our aim to construct general theories, but for an illustrative purpose, we shall first confine ourselves to the well-known Lee model[1] with a change in the mass relation, i.e., $m_V > m_N + \mu$. This permits the natural decay of the V particle.

One of the most reasonable methods treating the unstable particle V is based on the resonance scattering of θ and N through the formation of V. As in the resonance theory for the atomic nuclei, we can define the "observed" mass of V by the energy at which the phase shift δ becomes $\pi/2$, (the pole of the reaction matrix[2] K). The lifetime of V can be defined as the reciprocal of the width of the resonance. Above procedure can be performed compactly in terms of the modified propagator $S_V'(E)$ of the V particle. And the renormalization of the coupling constant can be performed analogously to the stable case. (§ 2.)

The second method is based on the analytic continuation of the S-matrix[3] for the $N+\theta$ scattering. $S_V'(E)$ has a pole at $E=m_S-i\gamma_S/2$ in the lower half plane of the complex variable E. This pole can be shown to coincide with that of the S-matrix, and the "observed" mass and the width of V can be defined by m_S and γ_S. The renormalization constant of the coupling constant can be defined as the residue of $S_V'(E)$ at this point. Contrary to the first method, this turns out to be a complex number. This fact may be the major defect of the present procedure. Using the above defined m_S and γ_S, it can be shown that the amplitude for the one particle V state obeys the radioactive decay law. (§ 3.)

The difference between above two procedures is due to the difference in the choice of the parameters, such as the mass and the coupling constant. Both theories give exactly the same value for the "renormalization invariants", e. g., scattering amplitudes and the phase shifts. They give actually different values to the mass and the life-time of the V particle. This circumstance originates in the difference of the definitions for those quantities, the measurements of which correspond to different experimental methods. In principle, the mass and the life-time given by one of the two theories can be expressed as the known functions of them given by another one. We can regard the two theories as equivalent in this sense.

Further, it will be shown that the difference between two definitions is of the order $(\gamma/m)^2$, so that it can be neglected on account of the uncertainty principle in the case of long life-time. In the actual case, the experimental observation is performed on either the resonance phenomena (e. g. $\pi^+ + p$ scattering) or on the time development of the unstable particle (e. g. π-μ decay). It is also a well-known fact[4] that in order to

investigate the energy dependence of the scattering cross section one must use a monochromatic and therefore very long train of waves while in order to investigate the time development of the decay process one must produce such unstable state by using non-monochromatic and therefore short train of waves. In the above sense, it would be concluded that the relation between two theories is rather complementary. The first method is convenient for describing the scattering, the second for the description of the time development of the unstable particle.

In § 4 we try to generalize these theories to the cases more usual than the Lee model. We use a formulation similar to but a little more general than that of Arnous and Zienau.[5] Then every-thing goes parallel with § 2 and § 3 using general Hamiltonian, except that we must neglect vacuum polarization. Treatment of the time development of V particle state is more natural here than in § 3 as we start from the formation of V by a collision. It justifies the procedure in § 3. The result thus obtained is the same as in § 2 and § 3 with the addition of an expression for the dissociation probability of the unstable particle.

In § 5 we compare and discuss the two theories. Appendices A and B are devoted to the mathematical manipulations which are necessary in the text.

§ 2. Renormalization procedure in terms of the K-matrix

In this section we consider first the scattering process between θ and N particles, defining the "observed" mass of the V particle by the resonance energy, and perform the appropriate renormalization of the coupling constant. The Hamiltonian of our system with which we are concerned is that of the Lee model[1];

$$\begin{aligned} H &= H_0 + H_1 \\ &= m_0 \psi_V^+ \psi_V + m_N \psi_N^+ \psi_N + \int \omega_k \alpha_k^+ \alpha_k \, d\boldsymbol{k} \\ &\quad + g\left(\psi_V^+ \psi_N \int d\boldsymbol{k} \frac{G(\omega_k)}{\sqrt{2\omega_k}} \alpha_k + \psi_N^+ \psi_V \int d\boldsymbol{k} \frac{G(\omega_k)}{\sqrt{2\omega_k}} \alpha_k^+ \right), \end{aligned} \tag{2·1}$$

where ψ_V, ψ_N and α_k are the annihilation operators of the V, N and θ (with momentum $\boldsymbol{k}$) particles, and m_0, m_N and μ are their "bare" masses respectively, and ω_k is $\sqrt{\boldsymbol{k}^2+\mu^2}$. To avoid the troublesome appearance of the "Ghost" state,[6] we introduce the cut-off factor $G(\omega)$ which suppresses the ultra violet divergences. Of course, after the renormalization procedure for the mass and the coupling constant is performed, there should be no divergence in the limit of no cut-off $G(\omega) \to (2\pi)^{-3/2}$. The only difference between Lee's case and ours is the mass relation between V, N and θ:

$$m_V > m_N + \mu \tag{2·2}$$

which permits the natural decay of the V particle. In (2·2) m_V is the "observed" mass of the V particle the definition of which contains some ambiguity as we shall see later.

Setting

$$|N+\theta, \boldsymbol{p}\rangle=\int \chi(\boldsymbol{k}, \boldsymbol{p})\alpha_k^+ d\boldsymbol{k}|N\rangle+C(\boldsymbol{p})|V\rangle, \tag{2·3}$$

we solve the Schrödinger equation

$$(H-m_N-\omega_p)|N+\theta, \boldsymbol{p}\rangle=0 \tag{2·4}$$

with the boundary condition that $\chi(\boldsymbol{k}, \boldsymbol{p})$ represents the incident plane wave with momentum $\boldsymbol{p}$ plus the outgoing spherical wave. In (2·3), we denote the state of a "bare" V particle and that of a "bare" N particle by $|V\rangle$ and $|N\rangle$. Substituting (2·1) and (2·4), one can readily obtain

$$\begin{aligned}\chi(\boldsymbol{k}, \boldsymbol{p})&=\delta(\boldsymbol{k}-\boldsymbol{p})-gC(\boldsymbol{p})\frac{G(\omega_k)}{\sqrt{2\omega_k}}\frac{1}{\omega_k-\omega_p-i\epsilon},\\ C(\boldsymbol{p})&=\frac{g}{\sqrt{2\omega_k}}G(\omega_p)S_V'(\omega_p+m_N),\end{aligned} \tag{2·5}$$

where $S_V'(E)$ represents the modified propagator of the V particle and is defined as usual by $[S_V'(E)]^{-1}=[S_V(E)]^{-1}-\Sigma(E)$. On account of the special form of H, there exists only one proper self-energy graph as given in Fig. 1 and $\Sigma(E)$ $S_V'(E)$ are given by

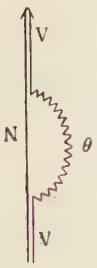

Fig. 1.

$$\Sigma(E)=-g^2\int\frac{G^2(\omega)}{2\omega}\cdot\frac{d\boldsymbol{k}}{\omega-E+m_N-i\epsilon}, \tag{2·6}$$

$$[S_V'(E)]^{-1}=E-m_0+i\epsilon+g^2\int\frac{G^2(\omega)}{2\omega}\frac{d\boldsymbol{k}}{\omega-E+m_N-i\epsilon}. \tag{2·7}$$

From (2·5) one obtains the scattering phase shift $\delta(\omega_p)$ by using the standard method:

$$p\cot\delta(\omega_p)=-\frac{1-2i\pi^2g^2pG^2(\omega_p)S_V'(\omega_p+m_N)}{2\pi^2g^2G^2(\omega_p)S_V'(\omega_p+m_N)}. \tag{2·8}$$

On the other hand one gets from (2·7) the following relation:

$$\begin{aligned}\mathfrak{Re}|S_V'(E)|^{-1}&=E-m_0+g^2\int\frac{G^2(\omega)}{2\omega}\mathfrak{p}\frac{d\boldsymbol{k}}{\omega-E+m_N},\\ \mathfrak{Im}[S_V'(E)]^{-1}&=\begin{cases}2\pi^2g^2\sqrt{(E-m_N)^2-\mu^2}\,G^2(E-m_N) & \text{for } E>m_N+\mu,\\ 0 & \text{for } E<m_N+\mu,\end{cases}\end{aligned} \tag{2·9}$$

where $\mathfrak{p}$ denotes the principal part of the integral. By the help of (2·9), (2·8) is rewritten as

$$\cot\delta(\omega_p)=-\frac{\mathfrak{Re}[S_V'(\omega_p+m_N)]^{-1}}{\mathfrak{Im}[S_V'(\omega_p+m_N)]^{-1}}. \tag{2·10}$$

According to Lippman and Schwinger[2] the reaction matrix K is given by

$$K(E)=-\frac{1}{\pi}\tan\delta(\omega_p)=\frac{\Im\mathfrak{m}[S_V'(E)]^{-1}}{\pi\Re\mathfrak{e}[S_V'(E)]^{-1}},$$
$$E=m_N+\omega_p. \tag{2·11}$$

The resonance of the N-θ scattering occurs at the energy satisfying $\delta(\omega_p)=\pi/2$ (the pole of the K-matrix), and one may define the "observed" mass of the V particle m_K by this energy. From (2·11) and (2·9) m_K statisfies

$$\Re\mathfrak{e}[S_V'(m_K)]^{-1}$$
$$=m_K-m_0+g^2\int\frac{G^2(\omega)}{2\omega}\mathfrak{p}\frac{d\boldsymbol{k}}{\omega-m_K+m_N}=0, \tag{2·12}$$

and the correction of the mass δm_K which must be renormalized is given by

$$\delta m_K=m_0-m_K=g^2\int\frac{G^2(\omega)}{2\omega}\mathfrak{p}\frac{d\boldsymbol{k}}{\omega-m_K+m_N}. \tag{2·13}$$

In the case that V be a stable particle, $m_V<m_N+\mu$ and m_V and δm_V are given by (2·1) and (2·13) with the omission of the $\mathfrak{p}$-symbol. Conversely, the root m_K of (2·12) corresponds to the mass value of V for the stable or the unstable case according as its value is smaller or larger than $m_N+\mu$.

The renormalization constant of the coupling constant for the unstable case can again be defined* by

$$Z_K^{-1}=\frac{\partial}{\partial E}\Re\mathfrak{e}[S_V'(E)]^{-1}\Big|_{E=m_K}$$
$$=1+g^2\lim_{\epsilon\to 0}\int\frac{G^2(\omega)}{2\omega}\mathfrak{p}\frac{d\boldsymbol{k}}{(\omega-m_K+m_N)(\omega-m_K+m_N-\epsilon)}, \tag{2·14}$$

in conformity to the stable case. In contrast to the stable case the above defined Z_K does not necessarily satisfy the relation $0\leq Z_K\leq 1$ and one cannot interpret Z_K as the probability of finding "bare" V particle in the "observed" V state. This fact seems to originate from the following circumstance; as will be proved in the Appendix B, for the case of unstable particle the scattering states (2·3) form the complete set in the subspace $\mathfrak{N}_V+\mathfrak{N}_\theta=\mathfrak{N}_V+\mathfrak{N}_N=1$ ($\mathfrak{N}$'s are the number of the particles.), and there is no room to treat the "observed" V state as the stationary state of H.

The renormalized coupling constant g_K is defined, as in the stable case, by

$$g_K^2=Z_Kg^2, \tag{2·15}$$

and the renormalized propagator for the V particle by

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* In (2·16) the order of the limit and the integration cannot be exchanged and the integrand is not positive definite.

$$S'_{VK}(E)=Z_K^{-1}S'_V(E). \tag{2·16}$$

Using (2·9), (2·13), (2·14) and (2·15) one gets

$$\begin{aligned}[S'_{VK}(E)]^{-1}=&(E-m_K)\Big(1+g_K^2(E-m_K)\\ &\times\lim_{\epsilon\to 0}\int\frac{G^2(\omega)}{2\omega}\,\mathfrak{p}\,\frac{d\boldsymbol{k}}{(\omega-E+m_N)(\omega-m_K+m_N)(\omega-m_K-\epsilon+m_N)}\Big)\\ &+i\pi g_K^2\int\frac{G^2(\omega)}{2\omega}\,\delta(\omega-E+m_N)\,d\boldsymbol{k}.\end{aligned} \tag{2·17}$$

One observes that $S'_{VK}(E)$ contains no divergence even in the case of no cut-off $G^2(\omega)\to(2\pi)^{-3}$.

Further we can define the width (the reciprocal of the life-time) of the V particle by

$$\begin{aligned}\gamma_K/2=&\Im\mathfrak{m}[S'_{VK}(m_K)]^{-1}\\ =&2\pi^2 g_K^2 G^2(m_K-m_N)\sqrt{(m_K-m_N)^2-\mu^2}.\end{aligned} \tag{2·18}$$

One sees that γ_K just coincides with the result of perturbation theory except the substitution of $g^2\to g_K^2$, $m_0\to m_K$.

Finally the scattering cross-section for $N+\theta$ is written as

$$\begin{aligned}\sigma(E)=&4\pi\sin^2\delta(\omega_p)/p^2\\ =&\frac{4\pi}{p^2}\,\frac{(\Im\mathfrak{m}[S'_V(E)]^{-1})^2}{(\Re\mathfrak{e}[S'_V(E)]^{-1})^2+(\Im\mathfrak{m}[S'_V(E)]^{-1})^2},\\ E=&m_N+\omega_p.\end{aligned} \tag{2·19}$$

Multiplying the denominator and the numerator of (2·19) by Z_K^2, one can express the $\sigma(E)$ by using $S'_{VK}(E)$ in the exactly similar form. This shows that the $\sigma(E)$ is "renormalization invariant". In the neighbourhood of $E\approx m_K$, one can approximate $\Re\mathfrak{e}[S'_{VK}(E)]^{-1}\approx E-m_K$ and $\Im\mathfrak{m}[S'_{VK}(E)]^{-1}\approx\gamma_K/2$ and $\sigma(E)$ reduces the well-known one-level formula for the atomic nuclei:

$$\sigma(E)\approx\frac{4\pi}{p^2}\,\frac{(\gamma_K/2)^2}{(E-m_K)^2+(\gamma_K/2)^2}. \tag{2·20}$$

Evidently the above discussed renormalization procedure satisfies the four conditions mentioned in the introduction and is convenient to treat the scattering problem of $N+\theta$.

§ 3. Renormalization procedure in terms of the S-matrix

In this section we find the S-matrix for the $N+\theta$ scattering and try to define the mass and the life-time of the V particle by the method of analytic continuation[3] and perform the renormalization procedure adapting to this definition.

The S-matrix for the $N+\theta$ scattering is defined by

$$S(E)\equiv e^{2i\delta(\omega_p)}=\frac{\cot\delta(\omega_p)+i}{\cot\delta(\omega_p)-i}\,,$$
$$E=m_N+\omega_p, \tag{3·1}$$

where the phase shift $\delta(\omega_p)$ is given by (2·10). Substitution of (2·10) in (3·1) yields

$$S(E)=[S_V'(E)]^{*-1}/[S_V'(E)]^{-1}, \tag{3·2}$$

therefore, to find the pole of the $S(E)$ in the lower half plane of the complex variable E is equivalent to find the zero point of $|S_V'(E)|^{-1}$ in the lower half-plane. For this purpose we study the properties of $\Sigma'(E)$ given by (2·6). Performing the angular integration we get

$$\Sigma'(E)=-2\pi g^2\int_\mu^\infty \frac{G^2(\omega)\sqrt{\omega^2-\mu^2}}{\omega-E+m_N-i\epsilon}d\omega. \tag{3·3}$$

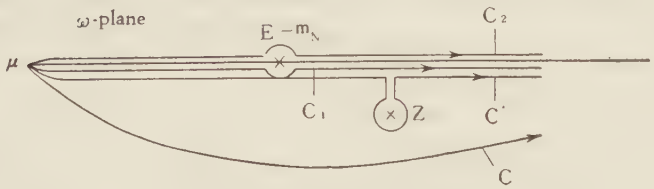


Fig. 2. Integration paths for $\Sigma'(E)$.

For the sake of $i\epsilon$ in (3·3), we must take the integration path C_1 in Fig. 2 for the real value of $E>m_N+\mu$. To perform the analytic continuation of $\Sigma'(E)$ for the general value of $Z=E-m_N(\Im\mathrm{m}\,E<0)$ from the real axis $E\geq m_N+\mu$ to the lower half-plane of complex E one must take the integration path C in Fig. 2. Deforming the path C into C' one can separate $\Sigma'(E)$ into the integral along real axis and the contribution from the residue at the point $\omega=Z$. Putting $E=E_1-iE_2(E_2>0)$, and performing the above procedure one gets*

$$\Sigma'(E_1-iE_2)=-2\pi g^2\left[\int_\mu^\infty \frac{G^2(\omega)\sqrt{\omega^2-\mu^2}}{\omega-E_1+m_N+iE_2}d\omega\right.$$
$$\left.+2\pi i G^2(E_1-iE_2-m_N)\sqrt{(E_1-iE_2-m_N)^2-\mu^2}\right]. \tag{3·4}$$

At this point one can define the "observed" mass m_S and the width γ_S of the V particle through the equation

$$[S_V'(m_S-i\gamma_S/2)]^{-1}=0. \tag{3·5}$$

* To perform this procedure rigorously it is sufficient to demand $G(\omega)$ that $\int_\mu^\infty G^2(\omega)\sqrt{\omega^2-\mu^2}\,d\omega<\infty$, and $G(\omega)$ has analytic continuation in the lower half-plane.

From (2·7), (3·4) and (3·5) one gets the following as the determining equation* for m_S and γ_S

$$\delta m_S = m_0 - m_S = g^2 \int \frac{G^2(\omega)\,(\omega - m_S + m_N)}{(\omega - m_S + m_N)^2 + \gamma_S^2/4}\,\frac{d\boldsymbol{k}}{2\omega} + (2\pi)^2 g^2 \eta,$$

$$\frac{\gamma_S}{2}\left\{1 + g^2 \int \frac{G^2(\omega)}{(\omega - m_S + m_N)^2 + \gamma_S^2/4}\,\frac{d\boldsymbol{k}}{2\omega}\right\} = (2\pi)^2 g^2 \xi, \qquad (3\cdot 6)^{**}$$

with

$$G^2(m_S - m_N - i\gamma_S/2)\sqrt{(m_S - m_N - i\gamma_S/2)^2 - \mu^2} \equiv \xi - i\eta. \qquad (3\cdot 7)$$

In actual case $m_S - m_N \gg \gamma_S$, and so

$$\xi \simeq \sqrt{(m_S - m_N)^2 - \mu^2}\, G^2(m_a - m_N) \quad \text{and} \quad \eta \simeq 0.$$

In this procedure it is natural to define the constant of the coupling constant renormalization by

$$Z_S^{-1} = \frac{\partial}{\partial E}[S_V'(E)]^{-1}\Big|_{E = m_S - i\gamma_S/2} = 1 + g^2 \int_C \frac{G^2(\omega)}{2\omega}\,\frac{d\boldsymbol{k}}{(\omega - m_S + m_N + i\gamma_S/2)^2}, \qquad (3\cdot 8)$$

instead of (2·4). The path C in (3·8) is given in Fig. 2. The renormalized coupling constant defined by

$$g_S^2 = Z_S g^2 \qquad (3\cdot 9)$$

naturally becomes complex number, the imaginary part of which is smaller than the real part by the factor $(\gamma_S/m_S)^2$. This is one major defect of this procedure. The renormalized propagator $S_{VS}'(E)$ defined by

$$[S_{VS}'(E)]^{-1} = Z_S [S_V'(E)]^{-1} = (E - m_S + i\gamma_S/2) \times \left(1 - g_S^2 \int_C \frac{G^2(\omega)}{2\omega}\,\frac{(E - m_S + i\gamma_S/2)\,d\boldsymbol{k}}{(E - m_N - \omega)\,(\omega + m_N - m_S + i\gamma_S/2)^2}\right) \qquad (3\cdot 10)$$

contains also no ultra-violet divergence in the limit of no cut off.

The above discussed procedure evidently satisfies the four conditions in § 1 as in the

* In the following we assume for simplicity that (3·6) has only one solution.

** At first sight one feels that the result of (3·6) does not reduce to the stable case since the residue part of (3·4) is multiplied by $2i\pi$ and the imaginary part of (2·7) only by $i\pi$. This is due to the overlook of the finite contribution from the second term of the left side of (3·7) in the limit of $\gamma_S \to 0$, because $\lim_{\epsilon \to 0} \epsilon/x^2 + \epsilon^2 = \pi\delta(x)$.

case of § 2, and the scattering cross-section $\sigma(E)$ for $N+\theta$ reduces to (2·20) with the interchange of suffices K to S in the neighbourhood of $E \approx m_S$, if one neglects the second term in the second bracket of the right side of (3·10). It would be worth-while to note that $S_V'(E)$ analytically continued to the upper half-plane of E from the real axis has neither zero nor singularity in this domain. So that

$$S_V'(t) \equiv \frac{i}{2\pi}\int S_V'(E)e^{-iEt}dE = 0 \quad \text{for} \quad t<0, \tag{3·11}$$

and the pole of $S(E)$ in the lower half-plane has one-to-one correspondence to zeros of $[S_V'(E)]^{-1}$, since $[S_V'(E)]^{*-1}$ has no singularity in this domain.

Finally in order to see what physical meanings have m_S and γ_S defined by (3·5), we treat the one-particle state unstationarily and follow its time development. Assume that a "bare" V particle is produced at the rate $f(t_0)$ at each instant, over the time interval T defined by

$$1/m_S \ll T \ll 1/\gamma_S. \tag{3·12}$$

At the time t our system can be repesented by

$$\Psi(t) = \int_0^T dt_0 e^{-iH(t-t_0)} f(t_0)|V\rangle, \quad t>0. \tag{3·13}$$

Substituting the special form of H in (3·13), we find that $\Psi(t)$ contains the component of V and that of N and θ. As will be shown in the Appendix A and § 4, $S_V'(t)$ defined by the first equation of (3·11) and (2·7) can be written as

$$\begin{aligned} S_V'(t_1-t_2) &= \langle V|e^{-iH(t_1-t_2)}|V\rangle \quad \text{for} \quad t_1>t_2, \\ &= 0 \qquad\qquad\qquad\quad \text{for} \quad t_1<t_2. \end{aligned} \tag{3·14}$$

Therefore one obtains

$$\psi_V(t) \equiv \langle V|\Psi(t)\rangle = \int_0^T dt_0 S_V'(t-t_0)f(t_0). \tag{3·15}$$

Further by the help of (A·11) in Appendix A it is possible to represent $S_V'(t)$ as

$$S_V'(t) = Z_S e^{-i(m_S-\gamma_S/2)t} + \frac{i}{2\pi}\int_{-\infty}^{m_N+\mu} Q(E)e^{-iEt}dE, \tag{3·16}$$

where m_S, γ_S and Z_S are defined by (3·5) and (3·8). On the other hand $f(t_0)$ can be decomposed as the Fourier integral:

$$f(t_0) = \int e^{-i\nu t_0} g(\nu)d\nu, \tag{3·17}$$

and one may choose $g(\nu)$ such that it has a peak at $\nu \approx m_S$ and its width is much smaller than m_S and $m_S-(m_N+\mu)$, for $f(t_0)$ has non-vanishing value only in the interval (3·12). Substituting (3·16) and (3·17) in (3·15) and performing the time integration on t_0, we can neglect the contribution from the second term of (3·16) in virtue of the

interference. For the first term of (3·16) we use the following relations:

$$\int_0^T dt_0 e^{-i(m_S - i\gamma_S/2)(t-t_0)} \int e^{-i\nu t_0} g(\nu) d\nu$$

$$= e^{-i(m_S - i\gamma_S/2)t} \int g(\nu) \frac{e^{i(m_S - i\gamma_S/2 - \nu)T} - 1}{i(m_S - i\gamma_S/2 - \nu)} d\nu$$

$$\approx \pi e^{-i(m_S - i\gamma_S/2)t} g(m_S), \quad (\because m_S \gg \gamma_S).$$

One obtains

$$\psi_V(t) \approx \pi Z_S e^{-i(m_S - i\gamma_S/2)t} g(m_S). \tag{3·18}$$

Equation (3·18) shows nothing but the radioactive decay law of the V particle and it is natural to interpret m_S and γ_S as the "observed" mass and width of the V particle.

From the above analysis it is concluded that the method of the analytic continuation of the S-matrix is adapted to the description of the time-development of the unstable particle. It is the most amusing point of this procedure that the defining equation (3·5) is very simple and the mass and the life-time of the unstable particle are determined on the equal footing.

§ 4. Generalization

In this section, we generalize the results of the foregoing two sections. The formalism is restricted, however, to the case where the vacuum polarization can be neglected.

(a) *Preliminary formulations*

We will first develop a time dependent formalism[5] which is suitable for the description of an unstable particle. An essential step in the formalism is the separation of a scattering process into two parts, according as it takes place through the formation of an unstable particle state or not. In the following, we denote this unstable particle as V, as in § 2 and § 3.

Suppose a certain state vector $\Psi(t)$ in the Schrödinger representation be $|I>$ at $t=0$, then

$$\Psi(t) = e^{-iHt}|I\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE \frac{e^{-iEt}}{E^+ - H}|I\rangle \quad \text{for} \quad t>0, \tag{4·1}$$

where

$$E^+ \equiv E + i\epsilon, \quad H = H_0 + H_1 \quad \text{(total Hamiltonian)}. \tag{4·2}$$

Let a projection operator to the state of one "bare" V particle be denoted by $\Lambda(|\Lambda H_0| = 0)$ and put

$$G \equiv \frac{1}{E^+ - H}, \quad G_1 \equiv G(1-\Lambda), \quad G_2 \equiv G\Lambda. \tag{4·3}$$

Then G_1 satisfies

$$(E^+-H_0)G_1=H_1G_1+(1-\Lambda). \tag{4·4}$$

Solving (4·4) with respect to ΛG_1 and $(1-\Lambda)G_1$, we get

$$\begin{aligned}
&\Lambda G_1=S_V'(E)\,U(E)\,(1-\Lambda)\,(E^+-H_0)^{-1},\\
&(1-\Lambda)G_1=\frac{1-\Lambda}{E^+-H_0}+\frac{1-\Lambda}{E^+-H_0}\,U(E)\,\frac{1-\Lambda}{E^+-H_0}\\
&\qquad+\frac{1-\Lambda}{E^+-H_0}\,U(E)S_V'(E)\,U(E)\,\frac{1-\Lambda}{E^+-H_0},
\end{aligned} \tag{4·5}$$

where

$$\begin{aligned}
&U(E)\equiv H_1[1-(1-\Lambda)\,(E^+-H_0)^{-1}H_1]^{-1}=[1-H_1(1-\Lambda)\,(E^+-H_0)^{-1}]^{-1}H_1,\\
&S_V'(E)\equiv\Lambda[E^+-H_0-\Sigma(E)]^{-1},\\
&\Sigma(E)\equiv\Lambda U(E)\Lambda=\Lambda H_1\Lambda\\
&\qquad+\Lambda H_1(1-\Lambda)[1-(1-\Lambda)\,(E-H_0)^{-1}H_1]^{-1}(1-\Lambda)\,(E^+-H_0)^{-1}H_1\Lambda.
\end{aligned} \tag{4·6}$$

In the same way we get for G_2

$$\begin{aligned}
&\Lambda G_2=S_V'(E),\\
&(1-\Lambda)G_2=(1-\Lambda)\,(E^+-H_0)^{-1}U(E)S_V'(E).
\end{aligned} \tag{4·7}$$

(4·7) is equivalent to the results of Heitler[5] and Arnous and Zienau. The first equation of (4·7) means, in view of the equation (4·1) and the definition (4·3),

$$S_V'(E)=-i\int_0^\infty dt\,e^{iEt}\langle V|e^{-iHt}|V\rangle.$$

In the case where the free vacuum coincides with the true vacuum, we have $|V\rangle=\psi_V|\mathrm{vac}\rangle$ and putting $\phi_V(t)=e^{iHt}\psi_V e^{-iHt}$ we get the well-known expression[7]

$$S_V'(E)=-i\int_{-\infty}^\infty dt_2 e^{iE(t_2-t_1)}\langle T(\phi_V(t_2)\phi_V^+(t_1))\rangle_{\mathrm{vac}}. \tag{4·8}$$

Equations (4·1), (4·5) and (4·7) constitute the desired time dependent formulation.

In addition, by utilizing the foregoing results, we can write the S-matrix with the use of operators defined in (4·6) in a form convenient for our purposes. According to Lippman and Schwinger,[2] the S-matrix is given by

$$\begin{aligned}
&S_{ab}=\delta_{ab}-2\pi i\delta(E_a-E_b)\,T_{ab}(E_a),\\
&T_{ab}(E)=(H_1[1-(E^+-H_0)^{-1}H_1]^{-1})_{ab},
\end{aligned} \tag{4·9}$$

where a and b denote final and initial states respectively and they are eigenstates of H_0. On the other hand, we have

$$G=(E^+-H_0)^{-1}+(E^+-H_0)^{-1}\,T(E)\,(E^+-H_0)^{-1}.$$

If $\Lambda|a\rangle=\Lambda|b\rangle=0$, that is, if we are dealing with a scattering of particles (and not the

development of one particle state), we get by using (4·5)

$$T_{ab}(E)=(U(E)S_V'(E)U(E)+U(E))_{ab}. \tag{4·10}$$

By the same procedure, we can write the K-matrix which is given by

$$K_{ab}=\left(H_1\left[1-\frac{\mathfrak{p}}{E-H_0}H_1\right]^{-1}\right)_{ab} \tag{4·11}$$

in the following form:

$$K_{ab}=(U_p(E)S_p'(E)U_p(E)+U_p(E))_{ab} \tag{4·12}$$

where $U_p(E)$ and $S_p'(E)$ are obtained by replacing $(E^+-H_0)^{-1}$ with $\mathfrak{p}/(E-H_0)$ in (4·6).

Finally we add a relation concerned with the imaginary part of $\Sigma(E)$.

$$\begin{aligned}2i\mathfrak{Im}\,\Sigma(E)&=\Lambda(U(E)-U^+(E))\Lambda\\&=\Lambda H_1[1-(1-\Lambda)(E^--H_0)^{-1}H_1]^{-1}((E^+-H_0)^{-1}-(E^--H_0)^{-1})\\&\quad\times(1-\Lambda)H_1[1-(1-\Lambda)(E^+-H_0)^{-1}H_1]^{-1}\Lambda\\&=-2\pi i\Lambda U^+(E)\delta(E-H_0)(1-\Lambda)U(E)\Lambda.\end{aligned} \tag{4·13}$$

In a special case where the N-θ-V interaction acts only for s wave of $N+\theta$ and the channels are open only to the two-particle state $N+\theta$, we get in the centre of mass system

$$2i\mathfrak{Im}\,\Sigma(E)=-2\pi i\frac{1}{v_N+v_\theta}\frac{4\pi k^2}{(2\pi)^3}|\langle\boldsymbol{k}|U(E)|V\rangle|^2\Lambda, \tag{4·13′}$$

where $\boldsymbol{k}$ denotes the momentum of θ, v_θ the velocity of θ.

(b) *Resonance in scattering.*

In terms of T defined by (4·9) the differential cross section for scattering is given by

$$d\sigma_{ab}=\frac{2\pi}{v_N+v_\theta}|T_{ab}|^2\rho(E_a)d\Omega. \tag{4·14}$$

Since the first term in (4·10) can be interpreted as the scattering amplitude through the formation of an unstable particle, we first consider the case where the second term in (4·10) vanishes as is the case in § 2. In such a case, when E equals to m_K which is the root of

$$\mathfrak{Re}[S_V'(E)]^{-1}=0 \tag{4·15}$$

and if the conditions stated above (4·13′) hold, we get by using (4·10), (4·13′) and (4·14)

$$T_{ab}=-\frac{(v_N+v_\theta)(2\pi)^3}{4\pi^2k^2i},\quad d\sigma=\frac{1}{k^2}d\Omega. \tag{4·16}$$

This shows that s wave of $N+\theta$ is just in resonance at this energy. Consequently, the K-matrix has a pole at this energy and m_K is an eigenvalue of

$$\left(1-\frac{\mathfrak{p}}{E-H_0}H_1\right)\Psi=0. \tag{4·17}$$

Further, putting as in (2·14)

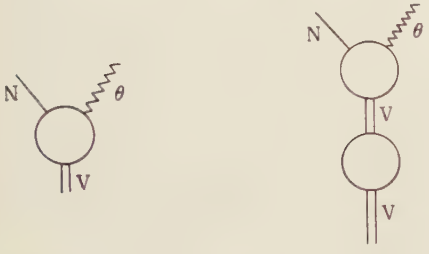
$$Z_K^{-1}=\frac{\partial}{\partial E}\mathfrak{Re}[S_V'(m_K)]^{-1}, \tag{4·18}$$

we have in the neighbourhood of $E=m_N$

$$k^2\pi^{-1}(v_N+v_\theta)^{-1}T\approx-\gamma_K(E-m_K+i\gamma_K/2)^{-1}, \tag{4·19}$$

$$\gamma_K=Z_K 2\pi|\langle V|U(E)|k\rangle|^2\rho(E). \tag{4·20}$$

(4·19) is the well-known one-level formula of resonance scattering and m_K and γ_K are usually called energy level and level width respectively. (4·20) means that the level width is obtained if we calculate all graphs of the type shown in Fig. 3a with the exception of those shown in Fig. 3b and multiply its result by Z_K. The last procedure is the so-called external line or wave function renormalization.



(a) (b)

Fig. 3

Next we consider the case where the second term in (4·10) does not vanish. In this case a pole of the K-matrix or an eigenvalue of (4·17) is no longer given by m_K but, in view of (4·12), by the root of

$$\begin{aligned}S_p'(E)^{-1}&=E-m_0-\langle V|H_1\left[1-\frac{\mathfrak{p}}{E-H_0}(1-\Lambda)H_1\right]^{-1}|V\rangle\\&=E-m_0-\langle V|H_1\Big(1+\frac{\mathfrak{p}}{E-H_0}(1-\Lambda)H_1\\&\quad+\frac{\mathfrak{p}}{E-H_0}(1-\Lambda)H_1\frac{\mathfrak{p}}{E-H_0}(1-\Lambda)H_1+\cdots\Big)|V\rangle=0.\end{aligned} \tag{4·21}$$

From the relation between the S- and K-matrices $S=(1-i\pi K)/(1+i\pi K)$, the scattering amplitude, the level width and the renormalization constant are given by the same expression as (4·17), (4·18) and (4·16) excepting that U and S_V' should be replaced by U_p and S_p'.

In Heitler's damping theory the energy levels are determined by (4·15), and not by (4·21) which gives the poles of the K-matrix. But (4·15) and (4·21) give the same results for the beginning few terms in their perturbation expansion.

(c) *Time dependent consideration* (1)

In this subsection we will investigate a process where two particles N and θ collide with each other producing an unstable particle V which again decays into N and θ, namely, we will consider $N+\theta$ scattering time-dependently to find expressions for mass and life-time.

Let the initial state be a wave packet with regard to the relative coordinate of N and θ which is an eigenstate of H_0 and be denoted by $|I\rangle$. Since the total momentum conserves, we consider it in the centre of mass system. We make the thickness of the wave packet much smaller than the life-time of the V particle[4] in the unit $c=\hbar=1$. Consequently the energy range of the incident wave is much broader than the width and we can assume it to contain the mass of the V particle.

We denote the amplitude of V by $\psi(t)$ and the amplitude that the relative coordinate of N and θ is x by $\phi(x, t)$. Then by using (4·1) and (4·5), we obtain

$$\begin{aligned}
&\psi(t)=-(2\pi i)^{-1}\int_{-\infty}^{\infty}dE e^{-iEt}\langle V|S_V'(E)U(E)(E^+-H_0)^{-1}|I\rangle,\\
&\phi(x, t)=\phi_0(x, t)+\phi_1(x, t)+\phi_2(x, t),\\
&\phi_0(x, t)=\langle x|e^{-iH_0t}|I\rangle,\\
&\phi_1(x, t)=-(2\pi i)^{-1}\int_{-\infty}^{\infty}dE e^{-iEt}\langle x|(E^+-H_0)^{-1}U(E)(E^+-H_0)^{-1}|I\rangle,\\
&\phi_2(x, t)=-(2\pi i)^{-1}\int_{-\infty}^{\infty}dE e^{-iEt}\\
&\qquad\times\langle x|(E^+-H_0)^{-1}U(E)S_1'(E)U(E)(E^+-H_0)^{-1}|I\rangle.
\end{aligned}\tag{4·22}$$

ϕ_0 is the incident wave itself, ϕ_1 is a scattering amplitude without forming a V-particle state. It can be shown, by the same argument as below (4·25), that the latter is scattered away from the centre as soon as N and θ collide, in so far as it has no resonance denominator in the energy range of the incident wave. Therefore it nearly vanishes far behind the front of scattered waves where ϕ_2 dominates. For this reason we discard ϕ_1 in the following. Making use of the formula

$$(E^+-H_0)^{-1}=-i\int_0^{\infty}ds e^{+i(E-H_0)s}$$

we have for ψ and ϕ_2

$$\begin{aligned}
&\psi(t)=-i\int_0^{\infty}ds\langle V|I(t-s)\int|x\rangle d^3x\phi_0(x, s),\\
&\phi_2(x, t)=-\int_0^{\infty}dr\int_0^{\infty}ds\langle x|e^{-irH_0}J(t-s-r)\int|x\rangle d^3x\phi_0(x, s),
\end{aligned}\tag{4·23}$$

with

$$\begin{aligned}
&I(t)=-(2\pi i)^{-1}\int_{-\infty}^{\infty}dE e^{-iEt}S_V'(E)U(E),\\
&J(t)=-(2\pi i)^{-1}\int_{-\infty}^{\infty}dE e^{-iEt}U(E)S_1'(E)U(E).
\end{aligned}$$

In general $S_V'(E)$ has branch points at $E=m+n\mu_0$ $(n=1, 2, \ldots)$, and owing to the $i\epsilon$ in (4·1), the path of integration with regard to E has its detours at branch points in

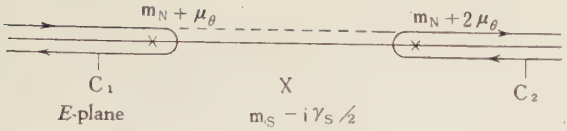


Fig. 4

upper half-plane. If the energy range of the wave packet lies between $m_N+\mu_\theta$ and $m_N+2\mu_\theta$ we get (4·24) below, by changing the integration path as shown in Fig. 4.

$$
\begin{aligned}
&I(t)=J(t)=0 && \text{for} \quad t<0,\\
&I(t)=e^{-i(m_S-i\gamma_S/2)t}\,AU(m_S-i\gamma_S/2)Z_S+I_1(t) && \text{for} \quad t>0, \qquad (4\cdot24)\\
&J(t)=e^{-i(m_S-i\gamma_S/2)t}\,U(m_S-i\gamma_S/2)AU(m_S-i\gamma_S/2)Z_S\\
&\qquad\qquad +J_1(t) && \text{for} \quad t>0,
\end{aligned}
$$

with

$$
[S_V'(m_S-i\gamma_S/2)]^{-1}=0, \quad Z_S^{-1}=\frac{\partial}{\partial E}[S_V'(E)]^{-1}\Big|_{E=m_S-i\gamma_S/2}. \qquad (4\cdot25)
$$

In (4·24), I_1 and J_1 are the integrations along C_1 and C_2. Since we assumed the energy of the incident wave to lie between $m_N+\mu_\theta$ and $m_N+2\mu_\theta$, $I_1(t)$ vanishes approximately owing to the s integration when t is large. $J_1(t)$ also vanlishes approximately except at the front of scattered waves owing to s and r integrations. Consequently we get

$$
\begin{aligned}
\psi(t)\approx\psi^{\mathrm{res}}\equiv&-i\int_0^\infty ds\,\theta(t-s)\exp[i(s-t)(m_S-i\gamma_S/2)]\\
&\times Z_S\langle V|U(m_S-i\gamma_S/2)\int|x\rangle d^3x\,\phi_0(x,s),\\
\phi_2(x,t)\approx\phi^{\mathrm{res}}\equiv&-i\int_0^\infty dr\,\langle x|e^{-irH_0}U(m_S-i\gamma_S/2)|V\rangle\psi(t-r),\\
&(t-t_0-R/(v_N+v_\theta)\gg 1/m_S\,.)
\end{aligned}
\qquad (4\cdot26)
$$

where $\theta(t)=1$ or 0 according as $t\geq 0$ or $t<0$. Since we assumed that the wave packet is thin, if $N+\theta$ interaction has a short range (both as compared with the life-time) we get

$$
\begin{aligned}
&\psi(t)\approx A\theta(t-t_0)\exp[-i(m_S-i\gamma_S/2)(t-t_0)],\\
&A=-i\int_0^\infty ds\,Z_S\langle V|U\int|x\rangle\phi_0(x,s)\,dx\exp[is(m_S-i\gamma_S/2)],
\end{aligned}
\qquad (4\cdot27)
$$

where t_0 is an instant of the $N+\theta$ collision. This approximation depends on the shortness of collision time of N and θ to form V as compared with the lifetime of V. Actually the exponential decay of ψ becomes obscure and the value of t_0 is undetermined to the extent of the collision time.

As for the $N+\theta$ amplitude, we assume, for the sake of simplicity, that decay products of V are only s wave of $N+\theta$. From (4·26) we have

$$
\phi^{\mathrm{res}}(x,t)=-i\int_0^\infty dr\int_0^\infty\frac{4\pi k^2dk}{(2\pi)^3}\,\frac{e^{ikR}-e^{-ikR}}{2ikR}\cdot e^{-iE_kr}\langle k|U(E)|V\rangle\psi(t-r).
$$

The term e^{-ikR} is of higher order in $1/R$ and we can neglect it in the region $R \gg 1/m$. Substituting (4·27) we get

$$\begin{aligned}
&\phi^{\mathrm{res}}(x,\ t) = A\int_{m_N+\mu_\theta}^{\infty} dE_k\, a(E_k) \exp i[kR - E_k(t-t_0)]\, \theta(t-t_0)/2\pi, \\
&a(E_k) = k(v_N+v_\theta)^{-1}_{E_k} \langle \boldsymbol{k}|U(E)|V\rangle/2\pi, \\
&E_k = \sqrt{k^2+m_N^2} + \sqrt{k^2+\mu_\theta^2}, \\
&(v_N+v_\theta)_{E_k} = k(k^2+m_N^2)^{-1/2} + k(k^2+\mu_\theta^2)^{-1/2}.
\end{aligned} \tag{4·28}$$

The term (-1) in the first equation is again of higher order in $1/R$ and we drop it. In the neighborhood of $E=m_S$, we make the expansion

$$k_{Ek} = k_{m_S} + (E_k - m_S)(v_N+v_\theta)^{-1}_{m_S}$$

and move integration path upward or downward as $R/(v_N+v_\theta) - (t-t_0)$ is positive or negative so as to drive it away far from the pole. Then the remaining integrals vanish approximately owing to the damping (near $E=m_S$) or oscillating (off $E=m_S$) factors. The contribution from the pole becomes

$$\begin{aligned}
\phi^{\mathrm{res}}(x,\ t) = {} & i a(m_S - i\gamma_S/2)\psi(t - R(v_N+v_\theta)^{-1}_{m_S}) \\
& \times \exp i[k_{m_S} R - i m_S R (v_N+v_\theta)^{-1}_{m_S}].
\end{aligned} \tag{4·29}$$

Consequently

$$\begin{aligned}
&|\psi^{\mathrm{res}}(t)|^2 \propto e^{-\gamma_S t}, \\
&|\phi^{\mathrm{res}}(R,\ t)|^2 = |a|^2 |\psi(t - R(v_N+v_\theta)^{-1}_{m_S})|^2.
\end{aligned} \tag{4·30}$$

(4·27) and (4·29) allow us to interpret m_S and γ_S defined in (4·25) as the mass and an inverse of the lifetime of V.

We add here the justification for the treatment in § 3. ψ and ϕ_2 in (4·22) can be written in the following form without any approximation

$$\begin{aligned}
\psi(t) = {} & -\int_{-\infty}^{\infty} dE \langle S_V'(E)\rangle_V \int_{-\infty}^{\infty} \exp[-iE(t-t_0)] f(t_0)\, dt_0/2\pi i, \\
\phi_2(x,\ t) = {} & \int_{-\infty}^{\infty} dE \langle x|(E^+ - H_0)^{-1} U(E) S_V'(E)|V\rangle \\
& \times \int_{-\infty}^{\infty} \exp[-iE(t-t_0)] f(t_0)\, dt_0/2\pi i, \\
f(t_0) = {} & \int_{-\infty}^{\infty} dE\, e^{-iEt} \langle V|U(E)(E^+ - H_0)^{-1}|I\rangle 2\pi.
\end{aligned} \tag{4·31}$$

This is identical with amplitudes which we will get if we take as initial state "bare" V states superposed chronologically with a weight of $f(t_0)$. This also means that "bare" V particles are produced at a rate of $f(t_0)$. Even if the V particle is produced through

other mode, it may be expected that we can write its amplitude in the form (4·31). In that case we can carry out the same calculation as above, if we assume suitable properties for $f(t_0)$. This justifies to make a calculation concerning V starting from "bare" V states superposed chronologically as in § 3.

(d) *Time dependent consideration* (2)

In this subsection we will find an expression for the dissociation probability for an unstable particle V by using (4·30). We consider a sphere of a radius R_0 in the relative coordinate space of N and θ. As stated below (4·22) and (4·25), after the front of scattered waves have passed the surface of the sphere, there remains only decay products and the "cloud" around V inside the sphere. Since (4·30) gives the total amplitude of $N+\theta$ at a sufficiently large distance from the centre as compared with $1/m$, we obtain the following results:

Outward probability flow at the surface:

$$J(t,\ R_0)=4\pi R_0^2(v_N+v_\theta)m_S|\phi^{\mathrm{res}}(R_0,\ t)|^2,$$
$$=4\pi R_0^2(v_N+v_\theta)m_S|a|^2|\psi(t-R_0(v_N+v_\theta)^{-1})|^2,$$

Total probability within the sphere (including the probability of V):

$$P_1=\int_t^\infty J(t,\ R_0)\,dt=J(t,\ R_0)/\gamma,$$

Total probability of outgoing waves within the sphere:

$$P_2=\int_0^{R_0}4\pi R^2|\phi^{\mathrm{res}}(R,\ t)|^2\,dR=[J(t,\ R_0)-J(t,\ 0)]/\gamma,$$

Total probability of "dressed" unstable particle:

$$P=P_1-P_2=J(t,\ 0)/\gamma=4\pi R^2(v_N+v_\theta)|a|^2|\psi(t)|^2/\gamma,$$

Dissociation probability = ("bare" V probability)/("dressed" V probability):

$$Z'\equiv[4\pi R^2(v_N+v_\theta)|a|^2/\gamma]^{-1}. \tag{4·33}$$

In the above calculation we extended the integration of P_2 to $R=0$. Actually (4·30) hold only if $R\gg 1/m_S$ and in the region $R\sim 1/m_S$ the distinction between decay products and the "cloud" around V are obscure. The contribution to the total integral from that region is, however, of the order γ/m and the uncertainty of this order seems to be unavoidable. So we extended the region of the integral to $R=0$. (4·22) may, therefore, be greater than 1 by this order.

Next we evaluate (4·33). By substituting $a(E_k)$ of (4·28) in (4·33) and making use of (4·13′) we get

$$Z'=\gamma_S[2(v_N+v_\theta)m_S|(v_N+v_\theta)^{-1}\Im\mathrm{m}\,\Sigma(E)\,|_{E=m_S-i\gamma_S/2}]^{-1}.$$

$\Im\mathrm{m}$ means that we first calculate the imaginary part when E is real and then consider its analytic continuation. By neglecting a quantity of the order $(\gamma/m)^2$ we get

$$Z'=\gamma_N/|2\,\mathfrak{Im}\,\Sigma'(E)|_{E=m_N-i\gamma_N/2}. \tag{4·34}$$

§ 5. Discussion

We have discussed the renormalization procedures for the unstable particle, in terms of the reaction matrix K in § 2, and in terms of the S-matrix in § 3, and generalized these procedures in § 4. All procedures satisfy the conditions in § 1.

In the first procedure, the mass and the width of the unstable particle are defined by the level and the width of the resonance scattering, respectively. In the second procedure they correspond to the frequency and the damping coefficient of the probability amplitude for the unstable particle.

We have found it most difficult to attribute a physical meaning to the renormalization constant of the coupling constant. The following discussions are devoted to make clear its physical meanings and the ambiguities in its definition. Defining the mass and the width of V by the first procedure, using (2·9) and (2·13), we can write (2·8) as

$$p\cot\delta(\omega_p)=-(E-m_K)(1+g^2F(E))/2\pi g^2G^2(\omega_p), \tag{5·1}$$

with

$$F(E)=\int\frac{G^2(\omega)}{2\omega}\mathfrak{P}\frac{d\boldsymbol{k}}{(\omega-E+m_N)(\omega-m_K+m_N)}.$$

Let λ be any real constant, and putting

$$\begin{aligned} Z^{-1}(\lambda)&=1+g^2F(\lambda),\\ g_\lambda^2&=Z(\lambda)g^2, \end{aligned} \tag{5·2}$$

one can rewrite (5·1) as

$$p\cot\delta(\omega)=-\frac{(E-m_K)(1+g_\lambda^2(F(E)-F(\lambda)))}{2\pi g_\lambda^2G^2(\omega_p)}. \tag{5·3}$$

The renormalization procedure has been performed by this process. The reasons why we have taken $\lambda=m_K$ and adopted the definition (2·14) are the following:

i) The second bracket of the left-hand side of (5·3) becomes 1 at $E=m_K$, and the one level formula (2·20) is deduced in the neighbourhood of $E=m_K$;

ii) From (4·20), Z_K corresponds to the renormalization for the external line.

Further, in the second procedure, Z_N has only the physical meaning that it corresponds to "the coupling constant renormalization for the production rate of the V particle," as can be seen from (4·27). For the practical purpose it is very inconvenient that Z_N becomes a complex number. This defect may be removed by using the ambiguities analogous to (5·2). At all rates it is impossible to discuss rigorously the probability of finding "bare" V particle in the "observed" V state, since there exists no stationary state including unstable particle. Only by neglecting the order of (γ/m) one can discuss the above probability as in (4·33).

Next, we shall discuss the differences of the various definitions about m, γ and Z. Writing (2·17) as

$$[S_V'(E)]^{-1}=Z_K^{-1}(E-m_K+\Delta(E)+i\gamma(E)/2), \tag{5·4}$$

and using (2·18), one sees that

$$\gamma(m_K)=\gamma_K, \quad \Delta(m_K)=0 \quad \text{and} \quad \partial\Delta(E)/\partial E \Big|_{E=m_K}=0. \tag{5·5}$$

$\Delta(E)$ and $\gamma(E)$ have no divergence. From (3·5) one gets

$$[S_V'(m_S-i\gamma_S/2)]^{-1}=Z_K^{-1}[m_S-m_K$$
$$+\Delta(m_S-i\gamma_S/2)+i(\gamma(m_S-i\gamma_S/2)-\gamma_S)/2]=0. \tag{5·6}$$

Taking $\partial\gamma/\partial E\approx O(\gamma/m)$ one finds from (5·5) and (5·6) that

$$m_S=m_K(1+O(\gamma/m)^2), \quad \gamma_S=\gamma_K(1+O(\gamma/m)^2). \tag{5·7}$$

From (3·8) and (5·5) one gets also

$$Z_S^{-1}=Z_K^{-1}(1+O(\gamma/m)^2). \tag{5·8}$$

Finally, as $2|\Im\mathfrak{m}\,\Sigma(m_S-i\gamma_S/2)|=Z_K^{-1}|\gamma(m_S-i\gamma_S/2)|$, one finds from (4·34)

$$Z'=Z_K(1+O(\gamma/m))=Z_S(1+O(\gamma/m)). \tag{5·9}$$

From (5·7), (5·8) and (5·9) one sees that the various definitions about m, γ and Z coincide with each other except for the quantities of the order γ/m.

From the preceding analysis we can draw the following conclusion: The two procedures in § 2 and § 3 attribute different physical meanings to the "observed" mass and the width of the unstable particle, and their differences are of the order $(\gamma/m)^2$. This fact owes its origin to the uncertainty principle. When γ is small, their differences cannot be detected by experiments. In principle, the mass and the lifetime given by one procedure can be expressed by those given by another. In this sense, one can regard two procedures as equivalent. These circumstances are quite similar to the case of the virtual level of deuteron, which has been summarized by Ma.[8] In that case it is also known that the definitions of the virtual level in terms of the K- and S-matrices are most reasonable.[9]

It is interesting, according to Lehmann,[10] to decompose $S_V'(E)$, the modified propagator of the unstable particle, in terms of the mass

$$S_V'(E)=\int_0^\infty \rho(\kappa)\,d\kappa\,(E-\kappa+i\epsilon)^{-1}. \tag{5·10}$$

From (A·17) in the Appendix we have

$$\rho(\kappa)=\theta(\kappa-m_N-\mu)\,|S_V'(\kappa)|^2\,\Im\mathfrak{m}[S_V'(\kappa)]^{-1}/\pi. \tag{5·11}$$

There is no discrete term in (5·11) as in the stable case (see (A·20)). In the neighbourhood of $\kappa\approx m_K\approx m_S$ we can approximate $[S_V'(\kappa)]^{-1}\approx Z^{-1}[(\kappa-m)+i\gamma/2]$, $\gamma/2=Z\Im\mathfrak{m}[S_V'(m)]^{-1}$. (The suffices S or K for m, γ and Z are ignored.) We get

$$\rho(\kappa) \approx Z \frac{\gamma/2}{(\kappa-m)^2+(\gamma/2)^2} \cdot \frac{1}{\pi}, \quad \kappa \approx m,$$

and see that $\rho(\kappa)$ represents the superposition of the various mass states, the centre of which is at m, the width of which is γ. In the limit $\gamma/2 \to 0$, $\rho(\kappa) \to Z\delta(\kappa-m)$; $\rho(\kappa)$ reduces to the stable case.

Appendix A Mathematical properties of $S(t)$

In this section we shall study the mathematical properties of $S_V'(t)$ which has been used in the text. To make clear the differences between unstable and stable cases, we shall designate them respectively as a) and b), and consider both cases at the same time.

As in § 4, we define $S_V'(t_1-t_2)$[7] by

$$S_V'(1-2) = \langle 0 | T(\phi_V(1)\phi_V^+(2)) | 0 \rangle, \tag{A·1}$$

where $\phi_V(1) = e^{iHt_1}\phi_V e^{-iHt_1}$ is the Heisenberg operator at time t_1, and $|0\rangle$ represents the true-vacuum. As in § 4 we have from (A·1)

$$\begin{aligned} S_V'(1-2) &= \langle V | e^{-iH(t_1-t_2)} | V \rangle \quad \text{for} \quad t_1-t_2 > 0, \\ &= 0 \qquad\qquad\qquad\quad \text{for} \quad t_1-t_2 < 0, \end{aligned} \tag{A·2}$$

which has been used in (3·14). The Fourier transform of (A·2) yields

$$\begin{aligned} S_V'(E) &= -i\int_{-\infty}^{\infty} e^{iEt} S_V'(t)\,dt = -i\langle V | \int_0^{\infty} e^{i(E-H)t} dt | V \rangle \\ &= \langle V | (E-H_0+i\epsilon-H_1)^{-1} | V \rangle. \end{aligned} \tag{A·3}$$

Exploiting the relation $(Q-H_1)^{-1} = Q^{-1}+Q^{-1}H_1Q^{-1}+\cdots$ with $Q = E-H_0+i\epsilon$ and substituting (2·1) in (A·3), we have the relation used in (2·7),

$$S_V'(E) = \left[E-m_0-g^2\int \frac{G^2(\omega)}{2\omega} \frac{d\boldsymbol{k}}{E-m_N-\omega+i\epsilon} \right]^{-1}. \tag{A·4}$$

Next, we shall obtain the integral representation of $S_V'(t)$ for the both cases, a) and b).

Case a)

In § 3, we have shown that the $S_V'(E)$ has a pole in the lower half plane of the complex variable E. In that case we have started from the real value of $E > m_N+\mu$, and continued analytically $S_V'(E)$ defined by (A·4). On the other hand, it we start from the real value of $E < m_N+\mu$, and continue $S_V'(E)$ to the lower half-plane, the second term of (3·4) does not follow. For $\Sigma'(E)$ for $E < m_N+\mu$ defined by (3·3) is represented by an integral from μ to $+\infty$ along the real axis in the ω-plane. The value of $S_V'(E)$ for $E < m_N+\mu$ continued in this way is the complex conjugate of (A·4). (The path of $\Sigma'(E)$ must be taken as C_2 in Fig. 2, § 3.) This fact shows that the point $E = m_N+\mu$ is the branch point of $S_V'(E)$.

Remembering these facts we shall obtain $S_V'(t)$ for $t < 0$. The Fourier inversion of (A·3) yields

$$S_V'(t)=\frac{i}{2\pi}\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon}S_V'(E)e^{-iEt}dE=\frac{i}{2\pi}\int_{C_1}S_V'(E)e^{-iEt}dE \quad \text{for} \quad t>0, \qquad (A\cdot 5)$$

where the integration path C_1 is given in Fig. 5.

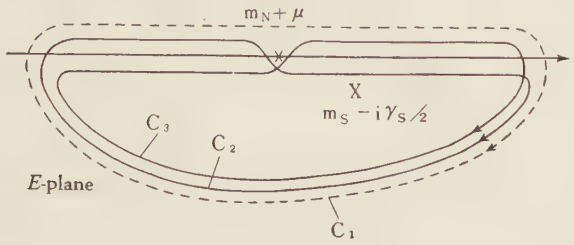


Fig. 5 Integration paths for $S_V'(E)$ for the Case a).

Along the path C_1, $S_V'(E)$ is continued from $E>m_N+\mu$ and has a pole at $E=m_S-i\gamma_S/2$. The value of integral (A·5), however, is not equal to the residue of $S_V'(E)$ at $E=m_S-i\gamma_S/2$ since the contour C_1 includes the branch point $m_N+\mu$. If we take the path C_3 which coincides with C_1 for $E>m_N+\mu$ we get

$$\frac{i}{2\pi}\int_{C_1}S_V'(E)e^{-iEt}dE=Z_S e^{-i(m_S-i\gamma_S/2)t} \quad \text{for} \quad t>0, \qquad (A\cdot 6)$$

where

$$Z_S^{-1}=\partial[S_V'(E)]^{-1}/\partial E\,|_{E=m_S-i\gamma_S/2} \qquad (A\cdot 7)$$

is given by (3·8). Next, if we take the path C_2 which coincides with C_1 for $E<m_N+\mu$, we obtain

$$\frac{i}{2\pi}\int_{C_1}S_V'(E)e^{-iEt}dE=0 \quad \text{for} \quad t>0. \qquad (A\cdot 8)$$

The proof of (A·8) is easy, since along the path C_2, $S_V'(E)$ is continued from $E<m_N+\mu$ and has no singularity in the lower half plane.

Finally we obtain two integral representations for $S_V'(t)$. The first one is obtained by decomposing the integral path C_1 into C_2 and the real axis for $E>m_N+\mu$. Using (A·8), and remembering that $S_V'(E)$ for $E>m_N+\mu$ becomes complex conjugate of (A·4) along the path C_2, we have

$$\begin{aligned}S_V'(t)&=\frac{i}{2\pi}\Big[\int_{m_N+\mu}^{\infty}[S_V'(E)-S_V'^{*}(E)]e^{-iEt}dE\\&=2\pi g^2\int_{m_N+\mu}^{\infty}|S_V'(E)|^2G(E-m_N)\sqrt{(E-m_N)^2-\mu^2}\,dE\,e^{-iEt}\\&=g^2\int d\boldsymbol{k}\frac{G^2(\omega)}{2\omega}|S_V'(E)|^2\exp[-i(m_N+\omega)t] \quad \text{for} \quad t>0. \qquad (A\cdot 9)\end{aligned}$$

On deriving (A·9) we have used (2·9). The physical meaning of (A·9) will be stated in Appendix B.

The second expression for $S_V'(t)$ is obtained by decomposing the path C_1 into C_2 and the real axis for $E<m_N+\mu$. We denote the difference of the values of $S_V'(E)$ on the two branches C_1 and C_2 as $Q(E)$. Using (A·6), we obtain

$$S_V'(t)=Z_S\exp\left[-i(m_S-i\gamma_S/2)\right]t+\frac{i}{2\pi}\int_{-\infty}^{m_N+\mu}e^{-iEt}dEQ(E), \qquad (A\cdot10)$$

which has already been used in § 3.

Case b)

In this case the pole of $S_V'(E)$ appears on the real axis at the point $E=m_V<m_N+\mu$, instead of the point $E=m_S-i\gamma_S/2$ in the Case a).

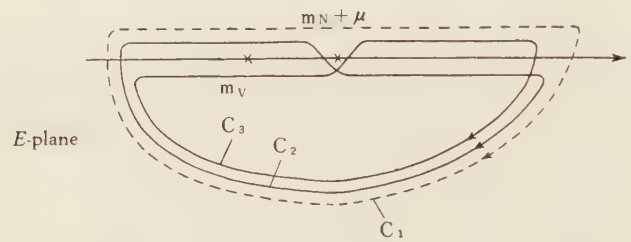


Fig. 6 Integration paths for $S_V'(E)$ for the Case b).

Instead of (A·6) and (A·8) we have

$$\frac{i}{2\pi}\int_{C_3}S_V'(E)e^{-iEt}dE=0 \qquad \text{for} \quad t>0, \qquad (A\cdot11)$$

$$\frac{i}{2\pi}\int_{C_2}S_V'(E)e^{-iEt}dE=Z_2e^{-im_Vt} \qquad \text{for} \quad t>0, \qquad (A\cdot12)$$

where

$$Z_2^{-1}=\partial[S_V'(E)]^{-1}/\partial E\,|_{E=m_V}=1+g^2\int\frac{G^2(\omega)}{2\omega}\frac{d\boldsymbol{k}}{(\omega-m_V+m_N)^2} \qquad (A\cdot13)$$

is the renormalization constant for the coupling constant given by Lee. $S_V'(t)$ is again given by (A·5) in this case. Corresponding to (A·9) and (A·10), we have

$$S_V'(t)=Z_2e^{im_Vt}+\frac{i}{2\pi}\int_{m_N+\mu}^{\infty}(S_V'(E)-S_V'(E)^*)e^{-iEt}dE$$

$$=Z_2e^{-im_Vt}+g^2\int d\boldsymbol{k}\frac{G^2(\omega)}{2\omega}|S_V'(m_N+\omega)|^2\exp\left[-i(m_N+\omega)t\right] \qquad (A\cdot14)$$

and

$$S_V'(t)=\frac{i}{2\pi}\int_{-\infty}^{m_N+\mu}e^{-iEt}dEQ(E). \qquad (A\cdot15)$$

It should be noted that another expression for Z_2 than (A·13) can be obtained by (A·15). As $\lim_{t\to+0} S_V'(t)=0$, we have from (A·14)

$$Z_2=1-g^2\int d\boldsymbol{k}\,\frac{G^2(\omega)}{2\omega}|S_V'(m_N+\omega)|^2. \tag{A·16}$$

Finally we calculate $\rho(\kappa)$[10] in (5·10).

Case a)

The Fourier inversion of (A·9) yields

$$S_V'(E)=2\pi g^2\int_{m_N+\mu}^{\infty} d\kappa\,\frac{|S_V'(\kappa)|^2G^2(\kappa-m_N)\sqrt{(\kappa-m_N)^2-\mu^2}}{E-\kappa+i\epsilon}$$

which proves that

$$\rho(\kappa)=2\pi g^2\theta(\kappa-m_N-\mu)\,|S_V'(\kappa)|^2G^2(\kappa-m_N)$$
$$\times\sqrt{(\kappa-m_N)^2-\mu^2}=|S_V'(\kappa)|^2\Im\mathfrak{m}[S_V'(\kappa)]^{-1}/\pi. \tag{A·17}$$

Case b)

Similarly, from (A·14) one gets

$$\rho(\kappa)=Z_2\delta(\kappa-m_V)+|S_V'(\kappa)|^2\Im\mathfrak{m}[S_V'(\kappa)]^{-1}. \tag{A·18}$$

Appendix B The complete set of state vectors for the system including unstable particles

In this section we shall prove that for Case a) the scattering states of $N+\theta$ form the complete set in the subspace $\mathfrak{N}_V+\mathfrak{N}_\theta=\mathfrak{N}_N+\mathfrak{N}_\theta=1$. For Case b) we must of course add the "observed" V particle state $|\boldsymbol{V}\rangle$ to the scattering states in order to get a complete set.

As the first step of this proof we can show that the scattering states $|N+\theta,\boldsymbol{p}\rangle$ defined by (2·3) and (2·5) are correctly ortho-normalized. Using the well-known formula

$$(\omega_k-\omega_p-i\epsilon)^{-1}(\omega_k-\omega_q+i\epsilon)^{-1}$$
$$=(\omega_q-\omega_p-i\epsilon)^{-1}((\omega_k-\omega_q+i\epsilon)^{-1}-(\omega_k-\omega_p-i\epsilon)^{-1}),$$

we get

$$\langle N+\theta,\boldsymbol{q}|N+\theta,\boldsymbol{p}\rangle=C(\boldsymbol{p})C^*(\boldsymbol{q})+\int\chi^*(\boldsymbol{k},\boldsymbol{q})\,d\boldsymbol{k}\chi(\boldsymbol{k},\boldsymbol{p})$$
$$=\delta(\boldsymbol{q}-\boldsymbol{p}) \tag{B·1}$$

where χ and C are defined by (2·5)

For Case a), using (A·9), (2·3) and (2·5) we have

$$S_V'(1-2)=\int C(\boldsymbol{p})C^*(\boldsymbol{p})\exp[-i(m_N+\omega_p)(t_1-t_2)]\,d\boldsymbol{p}$$

$$=\int\langle 0|\boldsymbol{\phi}_V(1)|N+\theta,\boldsymbol{p}\rangle d\boldsymbol{p}\langle N+\theta,\boldsymbol{p}|\boldsymbol{\phi}_V(2)|0\rangle \quad \text{for} \quad t_1-t_2>0. \tag{B·2}$$

Comparing (B·2) with (A·1), one observes indirectly that the scattering states $|N+\theta, \boldsymbol{p}\rangle$ form the complete set.

For Case b) the "observed" V particle state $|V\rangle$ is given by Lee[1] and

$$\begin{aligned} |V\rangle &= Z_2^{1/2}|V\rangle + \int \chi(\boldsymbol{k})\alpha_k^+ d\boldsymbol{k}|N\rangle, \\ \chi(\boldsymbol{k}) &= -g Z_2^{1/2} G(\omega_k)(2\omega_k)^{-1/2}(\omega_k - m_V + m_N)^{-1} \end{aligned} \tag{B·3}$$

with Z_2 given by (A·13). Using (A·14), (2·3), (2·5) and (B·3) we have also

$$\begin{aligned} S_V'(1-2) &= Z_2 \exp[-im_V(t_1-t_2)] + \int |C(\boldsymbol{p})|^2 \exp[-i(\omega_p+m_N)(t_1-t_2)]d\boldsymbol{p} \\ &= \langle 0|\boldsymbol{\psi}_V(1)|\boldsymbol{V}\rangle\langle \boldsymbol{V}|\boldsymbol{\psi}_V^+(2)|0\rangle \\ &\quad + \int \langle 0|\boldsymbol{\psi}_V(1)|N+\theta, \boldsymbol{p}\rangle d\boldsymbol{p}\langle N+\theta, \boldsymbol{p}|\boldsymbol{\psi}_V^+(2)|0\rangle \quad \text{for} \quad t_1-t_2>0. \end{aligned} \tag{B·4}$$

The comparison of (B·4) with (A·1) shows that the scattering states $|N+\theta, \boldsymbol{p}\rangle$ and "observed" V particle state $|V\rangle$ form the complete set.

The direct proof is also elementary but tedious, and we shall write the results only

$$\begin{aligned} &\int |N+\theta, \boldsymbol{p}\rangle d\boldsymbol{p}\langle N+\theta, \boldsymbol{p}| = \int d\boldsymbol{k}\,\alpha_k^+|N\rangle\langle N|\alpha_k + |V\rangle\langle V| \\ &\qquad\qquad \text{for Case a)}, \\ &\int |N+\theta, \boldsymbol{p}\rangle d\boldsymbol{p}\langle N+\theta, \boldsymbol{p}| + |V\rangle\langle V| = \int d\boldsymbol{k}\,\alpha_k^+|N\rangle\langle N|\alpha_k + |V\rangle\langle V| \\ &\qquad\qquad \text{for Case b)}. \end{aligned} \tag{B·5}$$

In the above proof, we have used the scattering states $|N+\theta, \boldsymbol{p}\rangle$ solved with the boundary condition that $\chi(\boldsymbol{k}, \boldsymbol{p})$ contains the incident plane wave and outgoing scattered wave. If we use the solution solved with the boundary condition that $\chi(\boldsymbol{k}, \boldsymbol{p})$ contains the incident plane wave and standing wave, we can reach the same conclusion. In this case the incident wave vanishes in the normalized standing wave solution at the resonance energy for Case a).

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The Magnetic Octupole Moments of Nuclei

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The magnetic octupole moment Ω of the nucleus is discussed on the basis of the shell and collective models. In the strong coupling case of the collective model, Ω becomes smaller than the shell model result because of the quenching of the moment along the symmetry axis of the deformed core. This fact is in agreement with experimental data (for $Ga^{71,73}$, In^{115}, and I^{127}) available at present.

Introduction

We have a good deal of information[1] about the magnetic dipole moment μ and the electric quadrupole moment Q of atomic nuclei. On the other hand, we know little about the next higher moments,[1)2)] i.e., the magnetic octupole and tht electric hexadecapole moments. Since experimental data of these higher moments have just begun to be published,[3] we think that it is a timely attempt to give a theoretical review on these moments referring to the standpoints of current nuclear physics. Thus in the present paper we shall discuss the magnetic octupole moments. The electric hexadecapole moments will be investigated in a forthcoming paper[4].

After stating the preliminary introduction (§ 1), we investigate the octupole moment in the cases of the shell model (§ 2) and the collective model (§ 3). The results are discussed in § 4. It is shown that if the strong coupling case is valid in the collective model, the deformed core plays an important role in suppressing the values of the octupole moments as compared with those for the shell model owing to a small projection factor P_{Ω}. Experimental data available at present (Ga, In, and I) are in more favour of the strong coupling case than of the weak coupling one (cf. Table 1).

§ 1. Preliminaries

The magnetic moment operator of order λ in unit of the nuclear magneton $e\hbar/2Mc$ is defined by[1]

$$M_{op}^{(\lambda)}=\sum_n[\nabla r_n^{\lambda}P_{\lambda}(\cos\theta_n)]\cdot\left[g_l^{(n)}\frac{2}{\lambda+1}\boldsymbol{l}^{(n)}+\frac{1}{2}g_s^{(n)}\boldsymbol{\sigma}^{(n)}\right], \tag{1}$$

where $\hbar\boldsymbol{l}^{(n)}$ and $\boldsymbol{\sigma}^{(n)}$ are the orbital angular momentum and spin operator for the n-th nucleon, $P_{\lambda}(x)$ is the Legendre function, and

$$\begin{cases} g_l=1, \quad g_s=5.587 & \text{for a proton,} \\ g_l=0, \quad g_s=-3.826 & \text{for a neutron.} \end{cases} \tag{2}$$

In our definition (1), we have neglected so-called interaction effects for brevity. If the wave function for a nucleus with spin I in the magnetic substate M is Ψ_I^M, then the magnetic moment $M^{(\lambda)}$ of order λ is given by

$$M^{(\lambda)}=(\Psi_I^I, M_{op}^{(\lambda)}\Psi_I^I).$$

This quantity vanishes for even λ and $\lambda>2I$. The magnetic dipole moment μ is $M^{(1)}$, and the magnetic octupole moment Ω is defined by Schwartz[2] as

$$\Omega=-M^{(3)} \tag{3}$$

to make the majority of moments positive. Evidently $\Omega\neq 0$ for $I\geq 3/2$. We shall list down typical expressions for Ω based on various nuclear models for comparison.

§ 2. Shell model

In the extreme single particle model, Ω is given by[2]

$$\Omega_{s.p.}=\frac{3}{2}\frac{(2I-1)(I+2)}{(2I+4)(2I+2)}\langle r^2\rangle\times\begin{cases}[(I-3/2)g_l+g_s] & \text{for } I=j=l+1/2 \\ [(I+5/2)g_l-g_s] & \text{for } I=j=l-1/2.\end{cases} \tag{4}$$

If we plot $\Omega_{s.p.}/\langle r^2\rangle$ versus I, we obtain the lines similar to Schmidt lines for the magnetic dipole moments. For a configuration $(j)^m$ of m (odd) equivalent nucleons in the ground state $(I=j)$, the expression for the octupole moment is again given by (4). In contrast to the magnetic dipole moment, all experimental values of Ω known so far are far from the shell model values (4) (see § 4). In this connection, it is interesting to see Ω in the collective model though there remains a possibility to explain these deviations by configurational mixing of some kinds.[5,6]

§ 3. Collective model

We want to examine the octupole moments in the collective model. The octupole moment operator is written as follows in the collective model.

$$\Omega_{op}=\Omega_{op}^{p}+\Omega_{op}^{c}. \tag{5}$$

The particle part, Ω_{op}^{p}, is given by eq. (1) with change of sign and the summation should be taken over the extra particles outside the core. If the nuclear surface is expressed by

$$R=R_0\left(1+\sum_{lm}\alpha_{lm}Y_{lm}(\theta,\varphi)\right),$$

then the core part becomes

$$\Omega_{op}^{c}=-\frac{i}{28}\frac{3AM}{4\pi\hbar}g_R R_0^4\sum_{m=-2}^{2}m(17-5m^2)\alpha_{2m}^{*}\dot{\alpha}_{2m},$$

where $g_R \simeq Z/A$ is the g-factor for the core and the higher order terms are neglected (see Appendix).* Then the Bohr-Mottelson's theory[7),8)] can be applied in straightforward manner to obtain $\mathcal{Q}=(\Psi_I^I, \mathcal{Q}_{cp}\Psi_I^I)$. *We shall adopt in the following the same notation as the reference 8.*

To see a main feature of the collective model, we shall take the simple case in which there is only one particle outside the core. We shall investigate two extreme cases, where the coupling, $H_{int}=-k(r)\sum_m \alpha_{2m}Y_{2m}$, between the extra particle and the nuclear surface is quite large or small.

(A) **Weak coupling case.**

Let us start with the weak coupling case. Using the perturbation procedure, the octupole moment correct up to the second order in H_{int} is easily calculated $(I=j)$;

$$\mathcal{Q}=\mathcal{Q}_p+\mathcal{Q}_c\,, \tag{6}$$

where

$$\mathcal{Q}_p=\mathcal{Q}_{s.p.}\Bigg[1+k^2\frac{\hbar\omega}{2C}\sum_{j'j''}\frac{(j|h_2|j'')\,(j|h_2|j')}{(\hbar\omega+\Delta_{jj''})\,(\hbar\omega+\Delta_{jj'})}$$
$$\times(-)^{j''+I-3}(2I+1)W(j''Ij'I;23)\frac{(j''\|\mathcal{Q}^p_{cp}\|j')}{(j\|\mathcal{Q}^p_{cp}\|j)}$$
$$-k^2\frac{\hbar\omega}{2C}\sum_{j'}\frac{(j|h_2|j')^2}{(\hbar\omega+\Delta_{jj'})^2}\Bigg]\,, \tag{7}$$

$$\mathcal{Q}_c=-\frac{1}{28}R_0^2 g_R\frac{k^2\hbar\omega}{C}\sum_{j'}\frac{(j|h_2|j')^2}{(\hbar\omega+\Delta_{jj'})^2}\times\sum_{m=-2}^{2}m(17-5m^2)\,(j'2I-mm|j'2II)^2. \tag{8}$$

Here, according to the Bohr-Mottelson's notation, the energy difference $\Delta_{jj'}$ of the particle levels with j and j', $(j|h_2|j')=(j\|Y_2\|j')/\sqrt{2j+1}$, and Racah and Clebsch-Gordan coefficients are used. $\mathcal{Q}_{s.p.}$ is given by eq. (4).

As a special case when $\Delta_{jj'}(j'\neq j)$ is large and j may be considered as a good quantum number, $\mathcal{Q}_p$ and $\mathcal{Q}_c$ are simplified as

$$\mathcal{Q}_p=\mathcal{Q}_{s.p.}\Bigg[1+\frac{5}{32\pi}\frac{k^2}{\hbar\omega C}\left\{\frac{4I^2(I+1)^2-75I(I+1)+234}{4I^2(I+1)^2}-\frac{(2I-1)\,(2I+3)}{4I(I+1)}\right\}\Bigg], \tag{9}$$

$$\mathcal{Q}_c=-\frac{45}{224\pi}g_R R_0^2\frac{k^2}{\hbar\omega C}\cdot\frac{(2I-1)\,(I-1)}{I(I+1)^2}\,. \tag{10}$$

(B) **The strong coupling case**

Next we take the strong coupling case where it is convenient to use the expression (A3) for $\mathcal{Q}^c_{cp}$. Then using the strong coupling wave function described in reference 8, we obtain the following result,

---

* $\mathcal{Q}^c{}_{cp}$ is in good approximation free from the higher order deformations than 2. This is in contrast to the case of hexadecapole moments discussed in reference 4.

$$\Omega = P_3(\Omega_{s.p.} + \Omega_0) \tag{11}$$

where P_3 and Ω_0 are given by,

$$P_3 = \frac{2I(2I-1)(2I-2)}{(2I+2)(2I+3)(2I+4)}, \tag{12}$$

$$\Omega_0 = -\frac{30}{49}\sqrt{\frac{2}{3}} g_R R_0^2, \tag{13}$$

and $\Omega_{s.p.}$ is given by eq. (4) if j is assumed to be a good quantum number, otherwise it is complicated. The value of the projection factor P_3 is quite small for small value of I.

Strictly speaking, eq. (11) is correct for higher spin I than 3/2. In the exceptional case, $I=3/2$, the degeneracy of two states [with $(\Omega=3/2, \gamma=\pi)$ and $(\Omega=1/2, \gamma=0)$, see Appendix III of reference 8] occurs and the wave function becomes complicated. Consequently the octupole moment for this case differs from the expression (11) and is given by

$$\Omega \simeq \frac{19}{35}\Omega_{s.p.}(I=3/2) + \frac{1}{35}\Omega_0 \qquad \text{(for } I=3/2) \tag{14}$$

where $\Omega_{s.p.}(I=3/2)$ and Ω_0 are given by eq. (4) (evaluated for $I=3/2$) and eq. (13).*)

§ 4. Comparison with experimental results and conclusion

Needless to say, much information about atomic nuclei is derived from the investigation of the magnetic dipole and the electric quadrupole moments. Similarly we can expect that the magnetic octupole moment is also useful for investigating nuclear properties. Unfortunately, not many data are available at the present making such analyses difficult to perform. Therefore, we content ourselves here with just comparing the experimental results with our theoretical ones derived in the preceding sections. Such comparison is done in Table 1. One can see that the observed values are very small compared with the predictions from the shell model presenting a striking contrast to the case of the magnetic dipole moment. This large discrepancy may be explained either by the configurational mixing method or by the collective model. If one takes the former, one has to introduce new configurations into the ground state which contribute to Ω but not to μ and Q. This would make the necessary computational work very complicated and has the disadvantage of introducing many configurations to be fitted to the nuclear data, i.e., I, μ, Q, Ω, etc.. On the other hand the collective model gives more clear-cut explanation of small observed values for Ω, as is seen in Table 1.

In the collective model, the second order deformation of the nuclear surface gives major contribution to Ω while higher order deformations are unimportant for our purpose

* On deriving eqs. (11)~(14), we have replaced the expectation value of $f(\beta, \gamma)$ with respect to the vibrational wave function by $f(\beta_1, \gamma_1)$, where β_1 and γ_1 are the equilibrium values of the shape parameters β and γ (see references 7 and 8). The values of Ω obtained from eqs. (11) and (14) will be a few percent larger than those obtained without employing this simplification.

Table 1

Comparison of the observed and the theoretical magnetic octupole moments

	I	shell assignment	observed	single particle model	collective model				
					weak coupling case			strong coupling case	
			$\mathcal{Q}/\langle r^2\rangle$	$\mathcal{Q}_{\text{s.p.}}/\langle r^2\rangle$	$\mathcal{Q}_p/\langle r^2\rangle$	$\mathcal{Q}_c/\langle r^2\rangle$	$\mathcal{Q}/\langle r^2\rangle$	$\mathcal{Q}_0/\langle r^2\rangle$	$\mathcal{Q}/\langle r^2\rangle$
Ga^{69}	3/2	$p_{3/2}$	0.502 ±0.019	1.68	1.19	−0.047	1.14	—	0.86*
Ga^{71}	3/2	$p_{3/2}$	0.679 ±0.019	1.68	1.20	−0.045	1.15	—	0.86*
In^{115}	9/2	$g_{9/2}$	1.148 ±0.037	4.69	3.03	−0.012	3.02	−0.39	1.26
I^{127}	5/2	$d_{5/2}$	0.533 ±0.094	2.84	0.70	−0.010	0.69	−0.35	0.29
Equation used				(4)	(9)	(10)	(6)	(12) (13)	(11) (14)*

$\mathcal{Q}$ is expressed in unit of nuclear magneton $e\hbar/2Mc$. We have used $\langle r^2\rangle=(3/5)R_0^2$, $R_0=1.44A^{1/3}\times10^{-13}$cm, $k=40$ Mev, and C, $\hbar\omega$ etc. have been calculated from eqs. (II 6), (II 6a) etc. of reference 8.

and neglected in § 3. When the strong coupling case is valid, the great reduction of the value of $\mathcal{Q}$ is caused by the projection factor P_3 which represents the transformation to the space fixed frame from the body fixed frame. Thus the present available data of $\mathcal{Q}$ are more favourable to the strong coupling results than the weak coupling ones. It is noted, however, that these values are obtained using the hydrodynamical estimations of various parameters, $\hbar\omega$ or C, etc., and thus they will change more or less by the use of more reliable estimations for them which have been discussed by Marumori et al.[8] Anyway one can conclude that as far as the observed nuclei are concerned their values of $\mathcal{Q}$ are fitted by the collective model. One should also remark the fact that the octupole moments in general give a useful test of the validity of the collective model. To analyze $\mathcal{Q}$ in more detail, we shall wait for further accumulation of experimental data.

Appendix

The core part of the octupole moment is given by

$$\mathcal{Q}^c_{op}=-\frac{1}{2}g_R\int\frac{1}{\hbar}\mathcal{M}(\boldsymbol{r})\nabla\{r^3P_3(\theta)\}d\tau \tag{A1}$$

where $g_R\simeq Z/A$ is the g-factor for the core and the angular momentum density $\mathcal{M}(\boldsymbol{r})$ is given by[7]

$$\mathcal{M}(\boldsymbol{r})=\frac{3AM}{4\pi R_0^3}[\boldsymbol{r}\times\nabla]\sum_{l\mu}\frac{r^l}{lR_0^{l-2}}\dot{\alpha}_{l\mu}Y_{l\mu}\,.$$

After a straightforward calculation, we find

$$\mathcal{Q}^c_{op}=-i\frac{1}{28}\cdot\frac{3AM}{4\pi\hbar}R_0^4g_R\sum_{m=-2}^{2}m(17-5m^2)\alpha^*_{2m}\dot{\alpha}_{2m}, \tag{A2}$$

where higher order terms are neglected. We also notice the relation

$$\dot{\alpha}_{lm}=\pi^*_{lm}/B_l\,, \qquad B_l=\frac{3}{4\pi}\frac{AMR_0^2}{l}\,.$$

In the strong coupling case, it is convenient to use the following representation (again using the notation of reference 7),

$$\Omega^c_{\nu\mu}=-\frac{15}{7}\sqrt{\frac{2}{7}}B_2R_0^2\eta_K\sum_{\lambda}\mathfrak{D}^{\lambda}_{0\lambda}\sum_{\nu\nu'\mu}\delta_{\lambda,\nu-\mu}a_\nu a_{\nu'}(32\nu-\mu\mu|322\nu)\sum_{\kappa}(M_\kappa)_{\mu\nu'}\frac{I_\kappa-j_\kappa}{\vartheta_\kappa}\,. \tag{A3}$$

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The Hexadecapole Moments of Atomic Nuclei

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The electric hexadecapole moments $Q^{(4)}$ of atomic nuclei are investigated on the basis of three nuclear models, i.e., shell, liquid drop, and collective models. The effect of the deformed core on this higher moment is discussed in detail. It is shown that experimental values of $Q^{(4)}$ shall be used to test the validity of the collective model (Bohr-Mottelson's theory), and the core part of $Q^{(4)}$ is useful to derive information about the core deformation of order 4.

§ 1. Introduction

Following to the previous paper[1] which has dealt with the magnetic octupole moment, we would like to investigate the electric hexadecapole moment in this paper. Although experiments as to the latter so far performed are scanty[2], we may expect in the near future the accumulation of experimental data with advanced techniques at the present.

Another motive of this work came from a theoretical aspect. As is well known, values of the quadrupole moments of many nuclei are much larger than those expected from the simple shell model calculation. This fact has led to the collective model of the nucleus in which the deformation of the core plays an important role. Analyses given so far show that the more the nuclear core deforms, the farther the quadrupole moment deviates from the shell model value. We consider that such a situation should be valid also for the case of the electric hexadecapole moment. In the latter case, however, some effects ignorable for the case of the quadrupole moments (e. g., higher order deformations) should be taken into account.

The electric moment operator of order λ is defined by[3]

$$Q_{op}^{(\lambda)}=\sum_{n=1}^{A}\left\{\epsilon_n+(-1)^{\lambda}\frac{Z}{(A-1)^{\lambda}}\right\}P_{\lambda}(\cos\theta_n)r_n^{\lambda}, \tag{1}$$

where $\epsilon_n=1$ for a proton, $\epsilon_n=0$ for a neutron, Ze denotes total charge, A mass number, and $P_{\lambda}(x)$ Legendre Polynomial. The second term in the braces represents the recoil effect which has to be omitted for $\lambda=0$. This term is negligible for a proton compared with the first term but not negligible for a neutron.

If we take the wave function as Ψ_I^M for a nucleus with spin I and its z-component M, the electric moment of order λ is given by*

* We sometimes call this quantity the spectroscopic moment to avoid confusion with the intrinsic moment discussed later.

$$Q^{(\lambda)}=(\Psi_I{}^I, Q_{op}^{(\lambda)}\Psi_I{}^I). \tag{2}$$

$Q^{(\lambda)}=0$ for odd λ and $\lambda > 2I$. $eQ^{(0)}=Ze$ is the total charge of the nucleus and $2Q^{(2)}=Q$ is the electric quadrupole moment.

In the following, we will present the electric hexadecapole moment $Q^{(4)}$ in the case of the shell model, Rainwater model, and the collective model (§ 2), and then, try to discuss on the effect of the core deformation to $Q^{(4)}$ (§ 3).

§ 2. The hexadecapole moments in the various nuclear models.

A. Shell model

The shell model result may be considered as a standard with which the results from other models are compared. Therefore we start from the case of shell model.

In the independent particle model, the hexadecapole moment of $(j)^m$ configuration (m=odd) is given by[4]

$$Q_p{}^{(4)}=\left(1-\frac{2m-2}{2j-1}\right)Q_{sp}^{(4)}, \qquad (I=j), \tag{3}$$

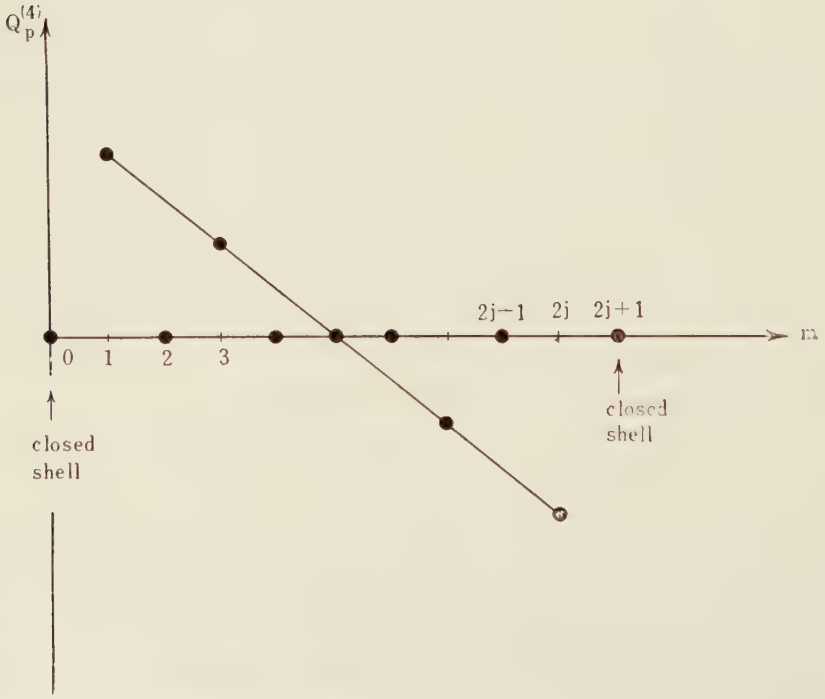

Fig. 1.

Expected trend of $Q^{(4)}$ in the independent particle model as a subshell is being filled, see eq. (3).

where $Q_{sp}^{(4)}$ is the hexadecapole moment in the extreme single particle model, and it is given by

$$Q_{sp}^{(4)}=\frac{3}{8}\frac{(2I-1)(2I-3)}{(2I+2)(2I+4)}\langle r^4\rangle\left[\epsilon+\frac{Z}{(A-1)^4}\right],\quad (I=j=l\pm 1/2). \tag{4}$$

Obviously $Q^{(4)}=0$ for even-even nuclei in this model. As is shown from eqs. (3) and (4), $Q_{sp}^{(4)}$ is positive for a less than half-filled shell and negative for a more than half-filled shell (see Fig. 1). This characteristic is opposite to the one for the quadrupole moments.

The independent particle model which neglects interaction between nucleons has failed to interpret the values of the quadrupole moments of nuclei. To improve this circumstance, the idea of the configurational mixing has been introduced.[5] For the electric hexadecapole moments, we can introduce the configuration mixing in an analogous manner. However, we think that such an analysis should be performed after sufficient experimental data has been accumulated.

B. Rainwater model

For many nuclei, the values of the quadrupole moments are quite larger than those predicted by the shell model. This fact has led to the deformed core model.[6] Accordingly, we consider that for those nuclei with large quadrupole moments the hexadecapole moments should also be affected by the deformed core.

First, to examine the hexadecapole moment of the deformed core from a simple geometrical consideration, we take a uniformly charged liquid drop of an ellipsoidal shape. The hexadecapole moment of this ellipsoid is always positive irrespective of the lengths of the three axes. If we restrict the deformation to the spheroidal shape, we can find the simple relation between the intrinsic hexadecapole moment $Q_c^{(4)}$ and the intrinsic quadrupole moment Q_c

$$Q_c^{(4)}=\frac{15}{28}\frac{1}{Z}(Q_c)^2. \tag{5}$$

This suggests that for the nuclei with large Q_c (i. e., with large core deformations), the values of $Q_c^{(4)}$ should also be large. This relation (5), however, is valid for the intrinsic moments. The observed moments are obtained by multiplying to them projection factors which transform the quantities from the body fixed frame to the space fixed one. It must be noted here that the projection factor P_4 for the hexadecapole moment is generally smaller than the one P_2 for the quadrupole moment.

The result of the above-mentioned ellipsoidal model is valid for any amount of deformation as far as the deformed shape is ellipsoidal. However, it may be too special to restrict the deformation to the ellipsoid. Therefore, we take more general shape of the nuclear surface of the core*

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* For the case of the spheroid, there is a relation $a_4\simeq 27/35\,(a_2)^2$ if we neglect the higher order terms.

$$R=R_0[1+a_0+a_2P_2(\cos\theta)+a_4P_4(\cos\theta)+\cdots] \tag{6}$$

and examine the contribution from each mode of the surface deformation. Here a_0, a_2, a_4,$\cdots$are assumed to be small quantities, and a_0 is introduced so as to make the total volume constant ($=4\pi R_0^3/3$). Neglecting the higher order terms than a_4, we can find the intrinsic moment of the core,

$$Q_c^{(4)}\fallingdotseq ZR_0^4\left\{\frac{1}{3}a_4+\frac{18}{35}(a_2)^2\right\}. \tag{7}$$

Information about the deformation of the order 2 has been obtained from the investigation of the quadrupole moment. Therefore the contribution from the second term in eq. (7) are supposed to have close relation to the quadrupole moment. In fact, if we denote the contribution of a_2 to $Q_c^{(4)}$ as $Q_c'^{(4)}$, we can find a relation similar to eq. (5)

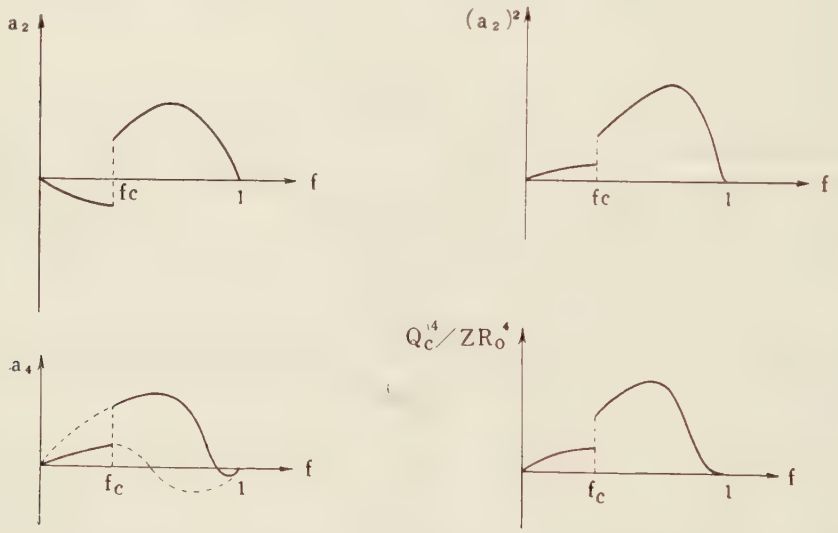

Fig. 2.

Expected trend of deformation parameters a_2, a_4 and of $Q_c^{(4)}/ZR_0^4=1/3a_4+18/35(a_2)^2$, as a subshell is being filled. f=fraction of the subshell filled. For $0<f<f_c$ (or $f_c<f<1$) the substates are filled from j downward; $\pm j$, $\pm(j-1)$,$\cdots$(or from 1/2 upward; $\pm1/2$, $\pm3/2$, $\cdots$). For details see Moszkowski and Townes, ref. 7). With their notation, a_2 and, a_4 are approximately given by

$$a_2\simeq\frac{2\epsilon_1\langle P_2\rangle+2\epsilon_2'\langle P_2\rangle^2}{DR_0^3\Re_{nl}^2(R_0)},$$

$$a_4\simeq\eta_1(a_2)^2+\eta_2\langle P_4\rangle,$$

where

$$\eta_1=\frac{0.1143+0.3429x}{2(1-0.3704x)},$$

$$\eta_2=\frac{DR_0^3\Re_{nl}^2(R_0)}{2(1-0.3704x)4\pi R_0^2\mathfrak{D}},$$

$$Q_c'^{(4)}=\frac{5}{14}\frac{1}{Z}(Q_c)^2, \qquad \left|\begin{array}{l} Q_c'^{(4)}=\dfrac{18}{35}ZR_0^4(a_2)^2, \\ Q_c=\dfrac{6}{5}ZR_0^2a_2. \end{array}\right. \tag{8}$$

Information about the deformation of the order 4 which contributes to the hexadecapole moment has not been furnished by the investigation of the quadrupole moment. Here we can obtain it by analyzing the hexadecapole moment.

To proceed further we must know the values of the parameters, a_2 and a_4, of the deformation. For this purpose we use the Rainwater model. (In practical calculations we have followed the way described in the appendix to the paper by Moszkowski and Townes.[7]) Then, we see the general trend of a_2, a_4, and $Q_c^{(4)}$ as a subshell is being filled (these are shown schematically in Fig. 2). a_4 is of the order of $(a_2)^2$. $Q_c^{(4)}$ is usually positive and may have negative value just before a shell is filled. Furthermore, the value of $Q_c^{(4)}$ becomes quite large about a half-filled shell, and there it is larger than the one predicted from the extreme single particle model (e. g., for rare earth elements).* This shows that the effect of the core deformation is very important as expected before.

C. Collective model

Now we consider a nucleus consisting of a core and extra nucleons and introduce the surface coupling between them. Bohr-Mottelson's collective model[5),9)] furnishes effective means for this purpose.** The nuclear surface is described by***

$$R(\theta,\varphi)=R_0\left[1+\sum_{l=2,4}\sum_m\alpha_{lm}Y_{lm}(\theta,\varphi)\right] \tag{9}$$

and the surface coupling between the deformed core and extra nucleons is given by

$$\begin{aligned} &H_{int}=H^{(2)}+H^{(4)}, \\ &H^{(l)}=-\sum_p k_l(r_p)\sum_m\alpha_{lm}Y_{lm}(\theta_p,\varphi_p). \end{aligned} \tag{10}$$

Here the summation over p means the summation over the extra nucleons outside the core. The hexadecapole moment operator in this case takes the form,

$$Q_{op}^{(4)}=Q_{p,op}^{(4)}+Q_{c,op}^{(4)}. \tag{11}$$

The particle part $Q_{p,op}^{(4)}$ has the same expression as eq. (1) but the summation must be taken over the nucleons outside the core. The core part is

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* If we take $j=7/2$, $Z=70$, $A=170$, $4\pi R_0^2\mathfrak{D}=14\times A^{2/3}$ Mev, $x=0.60$, and $DR_0^3\Re_{nl}^2\simeq 40$ Mev (see reference 7), then the maximum values of a_2, a_4, and $Q_c^{(4)}/ZR_0^4$ are ~ 0.27, ~ 0.086, and ~ 0.067, respectively.

** For the sake of convenience we use the same notation as Bohr-Mottelson's[9)] but with small alteration. We have two types of deformation corresponding to $l=2$ and 4. Therefore, we distinguishbetween corresponding quantities $\hbar\omega_l$, C_l, B_l, etc., by a suffix $l(l=2$ and $4)$.

***In the summation $\sum_l$ in eq. (9), terms of $l=3$ and $l\geq 5$ can de neglected.

$$Q^{(4)}_{c,op}=\frac{ZR_0^4}{\sqrt{4\pi}}\left[\alpha_{40}+\frac{9}{\sqrt{4\pi}}\sum_m \alpha_{2m}\alpha^*_{2m}(420m|2m)\,(4200|20)\right], \tag{12}$$

where higher order terms are neglected.

The argument given in the preceding section shows that the deformation of order 4 is small or the second order quantity as compared with the deformation of order 2 which is the first order quantity. Furthermore the energy $\hbar\omega_1$ of a surfon associated with the deformation of order 4 is much larger than the surfon energy $\hbar\omega_2$ of the deformation of order 2. These considerations make it plausible to treat the surfon phenomena corresponding to the deformation of order 4 by the perturbation theory*. Considering the lack of the experimental data, we shall restrict our following discussions within a simple case in which a single nucleon exists outside a closed shell. More general case will be discussed briefly in § 3.

C-1. *The weak coupling case*

First we deal with the weak coupling case assuming both $H^{(2)}$ and $H^{(4)}$ are small. Since $H^{(2)}$ and $H^{(4)}$ may be regarded as small quantities of the first order and the second order respectively, we calculate up to the second order by the perturbation theory using the Racah's technique. Then,

$$Q^{(4)}=Q_p^{(4)}+Q_c'^{(4)}+Q_c''^{(4)}. \tag{13}$$

The particle part is

$$\begin{aligned} Q_p^{(4)}&=Q_{sp}^{(4)}(1+\delta),\\ \delta&=k_2^2\frac{\hbar\omega_2}{2C_2}\sum_{j'j''}\frac{(j|h_2|j'')\,(j|h_2|j')}{(\hbar\omega_2+\Delta_{jj''})\,(\hbar\omega_2+\Delta_{jj'})}\cdot\frac{(j''||Q^{(4)}_{p,op}||j')}{(j||Q^{(4)}_{p,op}||j)}\\ &\quad\times(-1)^{j'+1}(2I+1)W(j''Ij'I;24)\\ &\quad-k_2^2\frac{\hbar\omega_2}{2C_2}\sum_{j'}\frac{(j|h_2|j')^2}{(\hbar\omega_2+\Delta_{jj'})^2};\quad (I=j) \end{aligned} \tag{14}$$

where δ is a correction term arising from the interaction of the extra nucleon and the surface, and $Q^{(4)}_{sp}$ is given by eq. (4). $(j|h_l|j')\equiv(j||Y_l||j')/\sqrt{2j+1}$, and $\Delta_{jj'}$ is the energy difference between the single particle states whose total angular momenta are j and j'. The core part $Q_c'^{(4)}$ and $Q_c''^{(4)}$ are the contributions from the surface deformation of orders 2 and 4 respectively, and are given by

* We have assumed that the effect of the deformation of order 4 can be treated as perturbation. This assumption, however, may not always hold and a more rigorous treatment of deformations of higher order than 2 becomes necessary. It brings forward a new problem which we shall leave for the forthcoming paper. However, we can immediately expect that, in such a case, the moment of inertia of the deformed core, the energy levels of nuclei, etc., will deviate from those predicted by Bohr-Mottelson theory.[9] Also the deformation of order 3 may be useful to explain the lowest excited level with the opposite parity to that of the ground state.

$$Q_c'^{(4)}=\frac{9}{8\pi}\sqrt{\frac{2}{7}}\left(\frac{\hbar\omega_2}{C_2}\right)^2 Z R_0^4 \sum_{j'}\frac{(j|h_2|j')^2}{\hbar\omega_2+\Delta_{jj'}}\left\{\frac{1}{\hbar\omega_2+\Delta_{jj'}}+\frac{1}{\hbar\omega_2}\right\}$$

$$\times(-1)^{j'-I}\sqrt{5(2I+1)}(I4\,I0|I0)W(I2\,I2\,;j'4), \tag{15}$$

and

$$Q_c''^{(4)}=\frac{9}{32\pi}Z R_0^4\frac{k_4}{C_4}\frac{(2I-1)(2I-3)}{(2I+2)(2I+4)}. \tag{16}$$

If all $\Delta_{jj'}$'s $(j'\neq j)$ are large enough and j is regarded as a good quantum number, eqs. (14) and (15) can be more simplified.

$$\delta=1+\frac{5}{128\pi}\frac{k_2^2}{\hbar\omega_2 C_2}\cdot\left\{\frac{4I^2(I+1)^2-123I(I+1)+630}{I^2(I+1)^2}\right.$$

$$\left.-\frac{(2I-1)(2I+3)}{I(I+1)}\right\} \tag{14'}$$

$$=1+\frac{75}{4\pi}\cdot\frac{k_2^2}{\hbar\omega_2 C_2}\cdot\frac{21-2I(2I+2)}{(2I)^2(2I+2)^2},$$

$$Q_c'^{(4)}=\frac{45}{224\pi^2}\left(\frac{k_2}{C_2}\right)^2 Z R_0^4\frac{(I-1)(2I-1)(2I-3)}{4I(I+1)^2}. \tag{15'}$$

C-2. The strong coupling case

Next we proceed to examine the case where the strong coupling treatment for $H^{(2)}$ and the weak coupling treatment for $H^{(4)}$ are appropriate. The unperturbed Hamiltonian H_0 and the wave function $\Psi_I^{(0)M}$ were given by Bohr (eqs. (94) and (118) in ref. 8) and are not stated here. There are two types of perturbing terms in the Hamiltonian, the one being $H^{(4)}$ and the other U (cf. eq. (94) of ref. 8). The latter, U, contains only the non-diagonal elements with respect to the strong coupling states. In this section we would rather like to know relative importance of two contributions from the deformation of orders 2 and 4 (denoted by $Q_c'^{(4)}$ and $Q_c''^{(4)}$, see below) to the intrinsic hexadecapole moment. Since the major part of the former, $Q_c'^{(4)}$, is given by the zeroth order wave function $\Psi_I^{(0)M}$ itself, the effect of the perturbing term U may be ignored. More precisely, one can say that the better the strong coupling approximation holds the smaller the effect of U. Moreover we introduce additional assumption that the total angular momentum j of outer particle is a good quantum number (eq. (19), however, is free from this assumption). Resulting expression then will be read as

$$Q^{(4)}=P_4[\{Q_{sp}^{(4)}+Q_c'^{(4)}\}(1+\delta)+Q_c''^{(4)}] \tag{17}$$

where $Q_{sp}^{(4)}$ is given by eq. (4), and

$$Q_c'^{(4)}=\frac{45}{224\pi^2}\left(\frac{k_2}{C_2}\right)^2 Z R_0^4\left(\frac{2I-1}{2I+2}\right)^2, \tag{18}$$

$$Q_c'^{(4)} = \frac{9}{32\pi} \frac{k_4}{C_4} Z R_0^4 \frac{(2I-1)(2I-3)}{(2I+2)(2I+4)}, \tag{19}$$

$$P_4 = \frac{2I(2I-1)(2I-2)(2I-3)}{(2I+2)(2I+3)(2I+4)(2I+5)}. \tag{20}$$

In these calculations the expectation value of the function $f(\beta, \gamma)$ of the shape parameters β and γ has been replaced by $f(\beta_1, \gamma_1)$, where β_1 and γ_1 are the equilibrium values of β and γ (cf. ref. 8 and 9). A very crude estimate for δ is given by

$$\delta = (-)^{2I}(2I+1) W(IIII; 44) (I|h_4|I)^2 (I4\,I0|II)^2 \cdot \frac{k_4^2}{2\hbar\omega_4 C_4}. \tag{21}$$

Fortunately this expression gives so small value for δ that it will not be necessary to improve it further ($|\delta| \lesssim 0.01$).

The *intermediate coupling* case can be worked out but it is not discussed here.

As we have seen above, we can state the following irrespective of the weak or the strong coupling when the total angular momentum j of the extra nucleon is a good quantum number. Between the core part of the hexadecapole moment due to the deformation of order 2, $Q_c'^{(4)}$, and the core part of the quadrupole moment, Q_c, there holds the same relation as eq. (8) for large values of nuclear spin I. (For the strong coupling case, the relationship holds between the intrinsic moments $Q_c'^{(4)}$ and Q_c.) The value of $Q_c'^{(4)}$ is always positive.

We notice that, for the case of a closed shell plus one hole, we may just change the signs of $Q_{sp}^{(4)}$ and k_l in the expressions obtained above.

§ 3. Discussion

In the preceding paragraph, we have examined the hexadecapole moment in various nuclear models. Let us now discuss the result obtained. In particular, we shall emphasize the important effect of the deformed core on the hexadecapole moment.

A deformability of the core and accordingly the value of the quadrupole moment strongly depend upon the strength of the coupling between the deformed core and the extra nucleons. Quite similar situation should occur for the hexadecapole moment and, in fact, it is confirmed by the argument presented in § 2 C. The core moment due to the surface deformation of order 2, $Q_c'^{(4)}$, is directly connected to the quadrupole moment. The deformation of order 4 has nothing to do with the quadrupole moment but contributes to the hexadecapole moment. Thus, the experimental data of the hexadecapole moment will provide more detailed information about the core deformation than those of the quadrupole moment do. According to Rainwater model and collective model, $Q_c'^{(4)}$ is always positive but $Q_c''^{(4)}$ is not always so. For example, $Q_c''^{(4)} > 0$ for the configuration consisting of a closed shell and one extra nucleon while $Q_c''^{(4)} < 0$ for the one consisting of a closed shell and one hole. Therefore $Q^{(4)}$ becomes quite small or even negative

just before a shell is filled (see Fig. 2).

For the sake of reference to our present discussion, let us exhibit some numerical examples of the hexadecapole moments. Choosing some typical nuclei such as tabulated in Table I, we have calculated their hexadecapole moments using the expressions obtained above and shown in Table II. It should be noted here that results for the collective

Table I Quadrupole moments in unit of 10^{-24} cm^2*

nucleus	spin I	shell assignment used to calculate $Q^{(4)}$ in Table II	observed quadrupole moment Q	quadrupole moment in the shell model Q_p	Q/Q_p
$_{13}Al^{27}$	5/2	$(d_{5/2})^{-1}$	0.15	0.04	4
$_{51}Sb^{121}$	5/2	$d_{5/2}$	−0.5	−0.13	4
$_{51}Sb^{123}$	7/2	$g_{7/2}$	−0.7	−0.15	5
$_{68}Er^{167}$	7/2	$(f_{7/2})^{-1}$	10.2		
$_{71}Lu^{175}$	7/2	$(g_{7/2})^{-1}$**	5.7±0.3	0.18	32±1
$_{73}Ta^{181}$	7/2	$(g_{7/2})^{-1}$**	4.3±0.4	0.18	24±2
$_{83}Bi^{209}$	9/2	$h_{9/2}$	−0.4	−0.2	2

* Taken from the review article by Blin-Stoyle[3].

** According to Mayer and Jensen[11], the configurations of these nuclei are as follows:

Lu^{175}: $(g_{7/2})^{-1}(d_{5/2})^6(h_{11/2})^8$,

Ta^{181}: $(g_{7/2})^{-1}(d_{5/2})^6\ (h_{11/2})^{10}$.

model in Table II were obtained from the expressions valid for the case where only one particle (or a hole) exists outside a closed shell. Also the assumptions that (a) the total angular momentum j of the extra particle is a good quantum number, and (b) the values of $\hbar\omega_l$ and C_l are taken from the hydrodynamical estimations* are adopted. Therefore, these results should not be taken too seriously.**

When there are many particles outside the closed shell, one may use the following estimate for the hexadecapole moment avoiding such an orthodox manner based on the collective model as was done in § 2 **C**. Here, $Q^{(4)}$ will be expressed as

$$Q^{(4)} = Q_p'^{(4)} + P^{(4)} Q_c^{(4)}, \tag{22}$$

where $Q_p'^{(4)}$ is the contribution from the extra particles outside the closed shell (which is regarded as a core) and the core moment $P^{(4)} Q_c^{(4)}$ is given by

$$Q_c^{(4)} \simeq \text{eq. (7)},$$

$$P^{(4)} = \begin{cases} 1 \text{ for the weak coupling case,} \\ P_4 \text{ for the strong coupling case.} \end{cases}$$

* The hydrodynamical estimation of C_2 or $\hbar\omega_2$ has recently been criticized by Marumori et al.,[10] who have obtained theoretically the strong dependence of C_2 on the shell structure in agreement with our experience. Similar situation is expected to occur for C_4 or $\hbar\omega_4$. Moreover we do not have good informamation about the strength of coupling $k_l(r)$.

** Since the configurations[11] of Lu^{175}, Ta^{181}, and Er^{167} are not "closed shell ± one nucleon", their values of $Q^{(4)}$ in Table II are not correct, see Table III.

Table II

Examples of hexadecapole moments of nuclei, calculated from various nuclear models.

Nucleus	shell model $Q_p^{(4)}/\langle r^4\rangle$	weak coupling case of the collective model				strong coupling case of colleceive model			
		particle moment $Q_p^{(4)}/\langle r^4\rangle$	core moment due to the deformation of		net result $Q^{(4)}/\langle r^4\rangle$	core moment due to the deformation of		projection factor P_4	net result $Q^{(4)}/\langle r^4\rangle$
			order 2 $Q_c'^{(4)}/\langle r^4\rangle$	order 4 $Q_c''^{(4)}/\langle r^4\rangle$		order 2 $Q_c'^{(4)}/\langle r^4\rangle$	order 4 $Q_c''^{(4)}/\langle r^4\rangle$		
equation used	(3)	(3), (14′)	(15′)	(16)	(13)	(18)	(19)	(20)	(17)
Al^{27}	−0.048	−0.022	0.064	−0.073	−0.03	0.21	−0.07	1/42	0.02
Sb^{121}	0.048	0.010	0.078	0.107	0.20	0.26	0.11	1/42	0.010
Sb^{123}	0.091	0.026	0.161	0.202	0.39	0.34	0.20	7/99	0.045
Er^{167}	0.000	0.000	0.227	−0.229	−0.002	0.48	−0.23	7/99	0.017
Lu^{175}	−0.091	−0.0042	0.244	−0.236	0.004	0.51	−0.24	7/99	0.013
Ta^{181}	−0.091	0.0064	0.255	−0.238	0.02	0.54	−0.24	7/99	0.015
Bi^{209}	0.126	−0.0025	0.493	0.351	0.84	0.85	0.35	18/143	0.18

We have adopted the following values of parameters $\langle r^4\rangle = (3/7)R_0^4$, $R_0 = 1.44A^{1/3}\times 10^{-13}$ cm, $k_2 = k_4 = 40$ Mev, and C_2, C_4, $\hbar\omega_2$, etc. were calculated from eqs. (II. 6), (II. 6a), and (II. 6b) of ref. 9.

Heavier nuclei far from a closed shell have large values of core moment $Q_c^{(4)}$ and thus the shell dependence of the spectroscopic hexadecapole moment $Q^{(4)}$ may be approximately represented by that of $P^{(4)}Q_c^{(4)}$. Appreciable difference between $Q^{(4)}$ and $P^{(4)}Q_c^{(4)}$ would occur about a closed shell. Moreover if the strong coupling case is valid, $Q^{(4)}$ becomes smaller than the intrinsic moment, $Q_c^{(4)}$, because of $P^{(4)} \simeq P_1 (< 1)$. Especially for small values of the nuclear spin I, P_4 is so small that $Q^{(4)}$ might become comparable to or even smaller than $Q_{sp}^{(4)}$ given by eq. (4). For nuclei with large spin and far from a closed shell (these nuclei generally have large quadrupole moments), $Q^{(4)}$ will still be larger than $Q_{sp}^{(4)}$. These circumstances may be used in testing experimentally the validity of the collective model and, particularly, its strong coupling case.

For nuclei with large quadrupole moments, the strong coupling case of the collective model is expected to hold in a good approximation. Then eqs. (5) and (22) suggest a "semi-empirical" mean to estimate the value for $Q^{(4)}$. Here the quadrupole moment will be written as

$$Q \simeq P_2 (Q_p + Q_c)$$

$$P_2 = \frac{I(2I-1)}{(I+1)(2I+3)} \quad \text{in the strong coupling case}$$

so that one has*

$$Q^{(4)} \simeq P_4 \left[Q_p^{(4)} + \frac{1}{Z} \frac{15}{28} \left(\frac{1}{P_2} Q - Q_p \right)^2 \right]. \tag{23}$$

The results listed in Table III are based on this semi-empirical formula with observed quadrupole moments and eq. (3) for $Q_p^{(4)}$. If one wishes to present the core moment $Q_c''^{(4)}$ due to the deformation of order 4 explicitly, one may take

$$Q^{(4)} \simeq P_4 \left[Q_p^{(4)} + \frac{1}{Z} \frac{5}{14} \left(\frac{1}{P_2} Q - Q_p \right)^2 + Q_c''^{(4)} \right] \tag{24}$$

and can find $Q_c''^{(4)}$ from observed values of the quadrupole moment and the hexadecapole moment.

Table III

Values of the hexadecapole moments calculated by semi-empirical method eq. (23)

(observed values of the quadrupole moments used are taken from Table I.)

nucleus	$Q^{(4)}/\langle r^4 \rangle$
Er^{167}	1.5
Lu^{175}	0.4
Ta^{181}	0.2

We want to make some remarks on experimental possibilities to get the hexadecapole moment other than hfs technique. As was emphasized by Wheeler,[12] the μ^--mesonic atoms are the excellent tool of measuring the charge distribution of the nucleus. Therefore the hexadecapole moments can in principle be obtained from μ^--mesonic atoms though

* Strictly speaking, the particle moment $P_4 Q_p^{(4)}$ is not correct. For large value of Q, however, contribution from $Q_p^{(4)}$ is so small that it does not practically affect the result in Table III.

in practice there remain many technical difficulties to be overcome, e. g., a new accurate method of measuring the energy of γ-ray must be developed. Next, the matrix elements of γ-transitions are known to be proportional to the intrinsic electro-magnetic moments. Thus the E4 transition probabilities between the states belonging to the same rotational series will be quite enhanced as compared with Weisskopf's estimate[13] if the nucleus has a large hexadecapole moment. The cross-over transition $(4^+ \to 0^+)$ in heavy even-even nuclei, if observable, may be an example to be checked. On the other hand, the E4 transitions between different rotational levels should occur with very much slower rate than Weisskopf's rate on account of the poor overlap of core wave functions. We may be able to pick up many other phenomena concerning the hexadecapole moments but leave them out of the present scope.

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$$Q^{(\lambda)} = \left(1 - \frac{2m-2}{2j-1}\right)(-1)^{l-1/2-\lambda-j}\sqrt{(2l+1)(2j+1)}\,(l\lambda 00|l0)$$

$$\times W(ljlj, \tfrac{1}{2}\lambda)\langle r^\lambda\rangle\left[\epsilon + \frac{Z}{(A-1)^\lambda}\right]; \quad I=j=l\pm 1/2.$$

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On the Wave Propagation in the Non-Linear Fields

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The wave propagation in the two non-linear fields is studied, one of which is the Born-type field and another the relativistic hydrodynamics. The discussion is restricted to one dimensional propagation; the formation of shock waves and the energy distribution are investigated with special attention on the basis of the method of characteristics.

§ 1. Introduction

The several non-linear field theories have been proposed by many authors in order to overcome the difficulties inherent in the field theories of elementary particles, such as the saturation of nuclear forces, the infinite self-energies, the mass spectrum of elementary particles, and the meson multiple production in high energy nuclear events. In the non-linear theories, the familiar law of superposition ceases to be valid, and dynamics of these fields in which the non-linear character of the basic equations plays a decisive role, is not yet in the mathematical perfection. This is one of the reasons why the attempts on the non-linear theories have been discouraged inspite of their physical realities, which might be expected by physicists. It seems to be desirable first to investigate the solutions of non-linear wave equations before we go over into the physical discussion on them and the problem of quantization.

However, discussions on *general* non-linear wave equations, even under the special physical conditions, are very difficult. Hence it seems to be most adequate in the present stage to discuss only some typical equations which have been considered in the field theories and to recapitulate the non-linear wave propagation in these equations.

In this paper, discussions will be done for the two special equations of the scalar variable ϕ: one is the so-called Born type[1] (§ 3), the other is the equation of the relativistic hydrodynamics proposed by Landau and Khalatnikov[2,3] (§ 4). We moreover restrict discussions to propagation in a one-dimensional space of violent disturbances initially localized in a small spatial region and invading the quiet state (the vacuum). Corresponding to this physical phenomenon, we may consider the following initial condition for ϕ: $\phi=\partial\phi/\partial x=0$ at $t=0$ for all x and $\partial\phi/\partial t$ is not equal to zero only in the small region.

For example, if the Born type equation is assumed to represent a complicated non-linear motion of some string and ϕ is interpreted as the displacement from the equilibrium, the above specified initial value problem turns out to be that of the motion of the string

struck suddenly at the center of it. In the case of the Landau-Khalatnikov's equation, using the relations between the scalar potential and the hydrodynamical quantities settled down by them[3], we can restate the above condition as follows: the fluid is initially at rest and localized in a small domain and proceeds to expand adiabatically into the vacuum.

It is worthwhile to explain the corresponding physical meanings of the above condition in the meson field theory. Heisenberg[1] considered the Born-type equation to describe the motion of meson cloud suddenly excited by the nucleon-nucleon collision at extremely high energy and invading the vacuum from the nucleon source. The relativistic hydrodynamical equation was originally introduced into the field theory by Landau[2] in order to account for the meson multiple production and its detailed physical correspondence has already been given in his paper.[2]

The purpose of this paper is to obtain the solutions of these two equations satisfying the above specified initial conditions. Especially, the following two subjects will be investigated with the special attention:

(i) Do shock waves occur?

(ii) Energy distribution at each instance.

In linear wave motion any initial discontinuity is preserved and propagated along the characteristics (light cone). However, non-linear wave motion behaves in a different manner. Suppose there is an initial discontinuity between two regions of the field quantity. Then there are the following alternative possibilities: either the initial discontinuity is resolved immediately and the disturbance, while propagated, becomes continuous, or the initial discontinuity is propagated through one or two shock fronts. In fact, shock fronts are the most conspicuous phenomena occurring in non-linear wave propagation; even without being caused by initial discontinuities they may appear and be propagated; the underlying mathematical fact is that, unlike linear partial differential equations, non-linear equations often do not admit solutions which can be continuously extended whenever the differential equations themselves remain regular. In the succeeding investigations, these possibilities will be discussed for the above two equations.

The energy distribution is concerned with the structure of self-interaction which may give rise to the concentration or dissipation of wave energy during propagation and here is another striking difference between linear and non-linear waves. The following calculations will give for both equations the analytical expressions to the variation of energy distribution in the course of propagation. This result may serve as the criterion for the basic equation proposed if the comparison with experimental data is possible.*

§ 2. Linear equation as preliminaries

The propagation characters of the linear hyperbolic equations with constant coefficients have been known in detail. For the better understanding of the succeeding discussions

*) The unit $\hbar=c=\mu=1$ is used, in which μ is the pion compton wave length.

we shall review, in this section, their main features in connection with the physical condition under consideration. Consider

$$\Box\phi=0, \tag{2·1}$$

with the initial condition:

$$\frac{\partial\phi}{\partial x}(=u)=0, \qquad \frac{\partial\phi}{\partial t}(=-v)=-v_0(x), \qquad \text{at} \quad t=0, \tag{2·2}$$

in which

$$v_0(x)\begin{cases} =\alpha\,(=\text{const.}) & \text{for } |x|\leqq a, \\ =0 & \text{for } |x|>a. \end{cases}$$

The solution of the above initial value problem may easily be obtained by the Fourier transformation. However, this method can not be applied to non-linear equations. In place of this approach, we shall use throughout this paper the method of characteristics which may be well applicable to both cases. The latter method is especially effective for the equations with two independent variables and not containing the potential ϕ explicitly.

The equation (2·1) can easily be transformed into the following system of differential equations:

$$\begin{aligned} &u_x+v_t=0, \\ &u_t+v_x=0\,; \end{aligned} \tag{2·1a}$$

or

$$\begin{aligned} &(u_x-u_t)-(v_x-v_t)=0, \\ &(u_x+u_t)+(v_x+v_t)=0. \end{aligned}$$

By introducing a set of new variables

$$x-t=\sqrt{2}\ \eta, \quad x+t=\sqrt{2}\ \xi, \tag{2·3}$$

the above equations become

$$u_\eta-v_\eta=0, \tag{2·1a)''}$$

$$u_\xi+v_\xi=0. \tag{2·1b)''}$$

The two sets of lines in the $(x,\ t)$ plane specified by $\xi=const.$ and $\eta=const.$ are called C^- and C^+ characteristics respectively. The whole $(x,\ t)$ space can be divided into three kinds of regions α, β and γ by the characteristics issuing from some special points on the initial line $t=0$ as illustrated in Fig. (1·1).

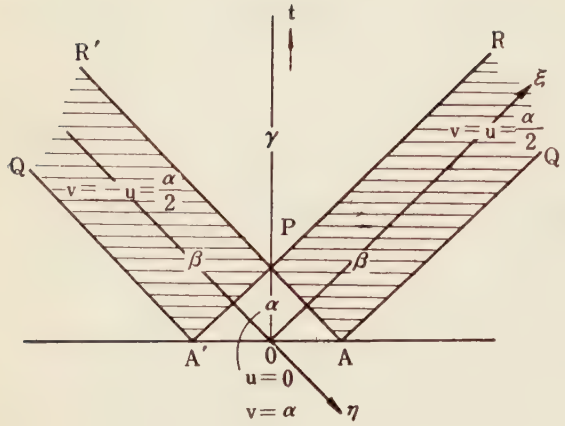

Fig. 1·1

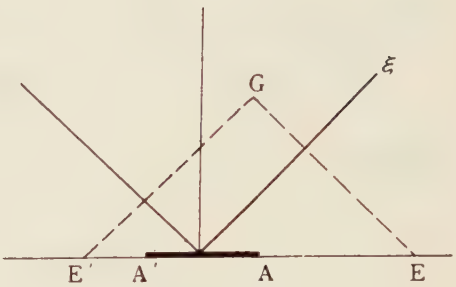

Fig. 1·2

Let us now consider the meaning of the equations (2·1a, b)″ in more detail. These equations lead to

$$u-v=\text{const.} \qquad \text{along } C^-$$

and

$$u+v=\text{const.} \qquad \text{along } C^+,$$

when ξ and η are respectively kept constant.

Then, we have the following relations for the values of u and v at any point G in γ (cf. Fig. 1·2)

$$u-v=\text{const.} \qquad \text{along } \overline{GE}\ (C^-), \tag{2·4a}$$

$$u+v=\text{const.} \qquad \text{along } \overline{GE'}\ (C^+). \tag{2·4b}$$

The initial condition yields

$$u=v=0 \qquad \text{at } E \text{ and } E',$$

and consequently we have

$$u=v=0 \qquad \text{in } \gamma \text{ region.}$$

Let us proceed to the analysis in "β-zone." This zone consists of two parts $\beta^{\pm}$ specified by $x \gtrless 0$ respectively. However the present initial condition and the differential equation are invariant under the inversion of x and therefore v is symmetric and u is antisymmetric, *i.e.*, $v(x, t)=v(-x, t)$, $u(x, t)=-u(-x, t)$. Hence it may be sufficient to consider only β^+ zone. In this zone, the initial condition gives

$$u-v=0,$$

$$u+v=\alpha, \qquad \text{at any point,}$$

namely

$$v=u=\alpha/2, \qquad \text{throughout } \beta^+ \text{ zone.}$$

Similarly we have

$$v=-u=\alpha/2 \qquad \text{throughout } \beta^- \text{ zone.}$$

In a quite similar way, we have

$$u+v=\alpha \qquad \text{along } C^+,$$

$$u-v=-\alpha \qquad \text{along } C^-,$$

or, $u=0$, $v=\alpha$ in α-region.

The energy density $H=(1/2)(u^2+v^2)$ has the following value in each region:

$$H=\begin{cases} \alpha^2/2 \cdots\cdots\cdots\cdots \alpha\text{-region}, \\ \alpha^2/4 \cdots\cdots\cdots\cdots \beta\text{-zone}, \\ 0 \quad \cdots\cdots\cdots\cdots \gamma\text{-region}. \end{cases} \tag{2·5}$$

The above result shows that the wave energy is always concentrated near the wave front, after the wave is swept away from α-region. The initial discontinuities at A and A' are preserved and propagated with the light velocity.*

The essential point of the method developed here consists in the transformation of the original equation into the system of the ordinary differential equations with respect to the characteristic directions.

The transformations of this kind may be applicable to general differential equations. Since the characteristic lines are, in general, determined by the coefficients of the highest derivatives in the original equations, in the linear theories they are given without regard to the initial condition. However, in non-linear theories they may depend on the field quantities, accordingly on the initial conditions.

It will be shown in the succeeding discussions how the transformation into the characteristic direction can be performed in the non-linear differential equations and that the method of characteristics will be well applicable also to these non-linear theories.

§ 3. A Born type non-linear equation[5)1)]

Let us consider a non-linear field ϕ defined by the Lagrangian:

$$L=l^2\{1+l^{-2}(\phi_x{}^2-\phi_t{}^2)\}^{1/2} \tag{3·1}$$

where l is a constant with the dimension of $(\text{length})^{-2}$.

Then, we have the following wave equation for ϕ:

---

* The method of solution exhibited here cannot be justified in the rigorous mathematical sense, because the relations (24·a; b) are valid only if u and v are continuous along $C^{\pm}$. However this difficulty can easily be overcome if we start from the smoooth initial distribution for $v(x)$ and tend to the limit given by (2·2) after the all calculations were performed. This procedure, of course, gives the same result obtained here.

$$(1-l^{-2}\phi_t^2)\phi_{xx}+2l^{-2}\phi_x\phi_t\phi_{xt}-(1+l^{-2}\phi_x^2)\phi_{tt}=0, \tag{3·2}$$

or

$$(v^2-l^2)u_x+(u_t-v_x)uv-(u^2+l^2)v_t=0, \tag{3·3a}$$

$$u_t+v_x=0, \tag{3·3b}$$

in which u and v are defined by

$$u=\phi_x, \quad v=-\phi_t.$$

As was done in § 2, we shall transform (3·3a, b) into the two ordinary differential equations with respect to the characteristic directions. Multiplying them by indeterminate functions λ_1 and λ_2 respectively and adding, we have

$$\{\lambda_1(v^2-l^2)\partial/\partial x+(\lambda_1 uv+\lambda_2)\partial/\partial t\}u$$
$$+\{(-\lambda_1 uv+\lambda_2)\partial/\partial x-\lambda_1(u^2+l^2)\partial/\partial t\}v=0.$$

Hence the above equation may be transformed into

$$\lambda_1(v^2-l^2)u_\sigma+(-\lambda_1 uv+\lambda_2)v_\sigma=0, \tag{3·5a}$$

or

$$(\lambda_1 uv+\lambda_2)u_\sigma-\lambda_1(u^2+l^2)v_\sigma=0, \tag{3·5b}$$

if the characteristic direction at any point is defined by the following relations;

$$\lambda_1(v^2-l^2)/(\lambda_1 uv+\lambda_2)=x_\sigma/t_\sigma \tag{3·4a}$$

and

$$(-\lambda_1 uv+\lambda_2)/\{-\lambda_1(u^2+l^2)\}=x_\sigma/t_\sigma, \tag{3·4b}$$

where σ means the length of the characteristics, and $x_\sigma=dx/d\sigma$, $t_\sigma=dt/d\sigma$.

The direction x_σ/t_σ in (3·4a, b), however, is restricted by the following equation;

$$(-l^2+v^2)t_\sigma^2-2uvt_\sigma x_\sigma+(l^2+u^2)x_\sigma^2=0, \tag{3·6}$$

which is derived by eliminating λ_1 and λ_2 from (3·4a; b). Corresponding to two real roots $t_\sigma/x_\sigma=\lambda_+$ and λ_- of (3·6), we have two characteristic curves with the length ξ and η respectively. Namely, these curves C^+ and C^- are defined by

$$\begin{aligned} t_\xi-\lambda_+ x_\xi&=0 \ : \qquad C^+, \\ t_\eta-\lambda_- x_\eta&=0 \ : \qquad C^-, \end{aligned} \tag{3·7a}$$

where

$$\lambda_\pm=(uv\pm l^2 s)/(v^2-l^2),$$
$$s=Ll^{-2}. \tag{3·8}$$

Now the original equations become a couple of the ordinary differential equations with respect to these characteristic directions σ: (ξ, η). In fact, eliminating λ_1 and λ_2 in (3·4a), (3·5a) and (3·4b), (3·5b), we obtain the equations:

$$u_\xi - \lambda_- v_\xi = 0, \qquad (3\cdot 9a)$$

$$u_\eta - \lambda_+ v_\eta = 0. \qquad (3\cdot 9b)$$

The solutions of the above equations give the relations between u and v and these correspond to the families of curves in the (u, v) plane. These curves are denoted by $I'^{\pm}$ characteristics corresponding to the $C^{\pm}$ characteristics in the (x, t) plane respectively.

Finally, we have the following system of equations equivalent to the original equation ;

$$\begin{aligned} I'^{+}: &\quad v_\xi - \lambda_- v_\xi = 0, \\ C^{+}: &\quad t_\xi - \lambda_- x_\xi = 0 ; \\ I'^{-}: &\quad v_\eta - \lambda_+ v_\eta = 0, \\ C^{-}: &\quad t_\eta - \lambda_- x_\eta = 0. \end{aligned}$$

It follows also from the above equations that

$$\begin{aligned} t_\xi v_\eta - x_\xi u_\eta = 0, \\ t_\eta v_\xi - x_\eta u_\xi = 0. \end{aligned} \qquad (3\cdot 10)$$

This relation indicates that the angle between C^{+} and x-axis and the angle between I'^{-} and u-axis are complementary to each other and the similar relation holds for I'^{+} and C^{-}, and the knowledge of I' characteristics may give the behaviours of C characteristics in the $(x$-$t)$ plane and *vice versa*. However, it should be kept in mind that this relation of representation has its meaning only if the transformation Jacobian is not zero and this condition can not necessarily be satisfied for the physically realizable state (*i.e.*, the constant state and the simple wave region which will be explained in the succeeding discussion).

In the following treatment, it may be more convenient to refer to the equation

$$(\mathcal{E}^2 - u^2) u_\sigma^2 + 2uvu_\sigma v_\sigma - (\mathcal{E}^2 + v^2) v_\sigma^2 = 0,$$

$$\mathcal{E}^2 = l^2 + u^2 - v^2,$$

or

$$\mathcal{E}^2 (du^2 - dv^2) = (udu - vdv)^2 ; \qquad \text{on } I'\text{-characteristics,} \qquad (3\cdot 11)$$

which can be derived from $(3\cdot 5a)$ and $(3\cdot 5b)$. The integration of $(3\cdot 11)$ is easy and it gives the general solution :

$$v = mu \pm pl, \qquad (3\cdot 12)$$

in which $p = \sqrt{1 - m^2}$, and m is an arbitrary real constant whose absolute value is smaller than unity. The above relation shows that $I'^{\pm}$ are straight lines. The somewhat detailed calculations show that

$$\left.\begin{aligned} v = mu + pl \quad &\text{corresponds to} \\ I'^{-}, \text{ if } pu - lm > 0 \quad &(\lambda^{+} = 1/m), \\ I'^{+}, \text{ if } pu - lm < 0 \quad &(\lambda^{-} = 1/m), \end{aligned}\right\} \qquad (3\cdot 13a)$$

and $v=mu-pl$ corresponds to

$$\left.\begin{aligned} &\Gamma^{+}, \text{ if } pu+lm>0 \quad (\lambda_{-}=1/m), \\ &\Gamma^{-}, \text{ if } pu+lm<0 \quad (\lambda_{+}=1/m). \end{aligned}\right\} \qquad (3\cdot 13\text{ b})$$

The above relations indicate that λ^{+} is constant along C^{-}, therefore all the C^{+} curves are parallel to each other at their intersecting points with an arbitrary C^{-}curve. The same relation also holds for C^{-} curves. This leads to the remarkable character of the solutions that the curves C^{-} can never cross each other and the same is true for C^{+}, namely C^{+} and C^{-} families are those composed of parallel curves.*

Let us now consider the general method of solution for an arbitrary initial value problem. When the initial conditions: $v=f(x)$, $u=g(x)$ at $t=0$, are given, we always have the two values of m, say $m^{\pm}(x_0)$, from the equations

$$f(x_0)=m(x_0)g(x_0)\pm p(x_0)l,$$

$$p(x_0)=\sqrt{1-m(x_0)^2}$$

for each point x_0 on the surface $t=0$. These values of m determine the $\Gamma^{\pm}$ families in the $(u\text{-}v)$ plane and accordingly the $C^{\pm}$ families in the $(x\text{-}t)$ plane can be drawn on the basis of the relation $(3\cdot 10)$. On the other hand, the relations $(3\cdot 9\text{a, b})$ or $(3\cdot 13\text{a, b})$ are valid along $C^{\pm}$, hence the values of u and v, at the cross point of the two curves C^{+} and C^{-}, can easily be obtained from $(3\cdot 13\text{a, b})$.

Fig. 3·1
The C^{+} characteristics of the equation (3·2).

It is especially interesting to consider the initial condition in which $f(x)$ and $g(x)$ are not arbitrary functions, but satisfy a relation

$$f(x)=\mu g(x)+pl, \qquad p=\sqrt{1-\mu^2},$$

with a parameter μ; in other words we are considering the initial distribution, in which one of the $m(x_0)$'s, say $m^{-}(x_0)$, is reduced to a given constant μ. If, moreover, $pg-l\mu>0$ is assumed, it turns out that the relation for Γ^{-} is valid on the initial line $t=0$. Accordingly, the relation

$$v(x,\ t)=\mu u(x,\ t)+pl,$$

holds over all (x, t) space, because the relation for Γ^{-} is preserved along C^{-}. On the other hand, we have

$$v(x,\ t)=m^{+}(x_0)u(x,\ t)-p^{+}(x_0)l$$

* This important character may largely exclude the possibility of the occurence of shock waves (cf. (4)).

along C^+. From these, u and v take some constant values along C^+ and C^+ becomes a straight line. As was already stated, the lines C^+ are parallel to each other, therefore the initial shape of distribution is shifted along C^+* without any deformation while

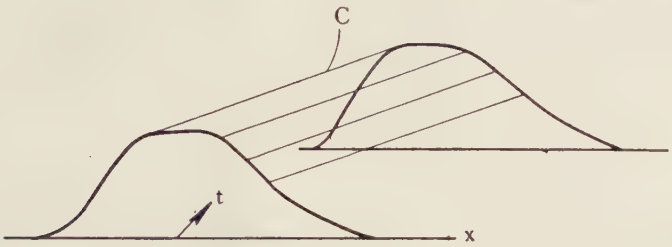

Fig 3·2 The propagation of the permanent simple wave.

propagated, that is, the solution is the permanent wave propagating with the phase velocity μ. (cf. Fig. 3·2, in which the lines denoted as C^+ are not the true C^+ but their projections on $(x\text{-}t)$ plane are true ones.) In the above example, contrary to the general one to one correspondence between Γ^- and C^--characteristics, all the C^--curves in $(x\text{-}t)$ plane correspond to a single Γ^- by virtue of our special initial relation between u and v. In general, such a domain of the (x, t) space, say D, as those corresponding to one Γ^+ or Γ^- line in the $(u\text{-}v)$ plane is called a simple wave region. As is obvious from the above discussion, the simple wave has the remarkable property that the characteristic line C^+ or C^- contained in its region is surely straight. However, in general, it is not true that those C^+ or C^- are parallel and the simple wave preserves its initial shape during propagation.**

In most cases of non-linear equations the C characteristics in the simple wave region are the families composed of the straight lines with different slopes, and in those cases shock waves may occur.[6] We also mention the following important theorem concerning the simple wave.

Theorem :

The region adjacent to a region of constant state is that of simple wave.····· (A)***

On the basis of the preceding discussions, the solution satisfying the following initial conditions will be given without any approximation :

at $t=0$, $\phi=u=0$,

$$v \begin{cases} =0 & \text{for } |x|>a+b, \\ =v_0(x) \quad (>0) & \text{for } a\leqq|x|\leqq a+b, \\ =\alpha \quad (>0) & \text{for } |x|\leqq a, \end{cases}$$

* In the present example this can be seen more easily as follows : Using the relation for Γ^-, we have $v_x=\mu u_x$, then it results immediately from (3·3b) that $u_t+\mu u_x=0$, or $u=f(x-\mu t)$.

** In the hydrodynamics the peculiar example of this kind has also been acknowledged as the Kármán-Tsiën's gas, which is fictitious and may not physically be realizable.

*** cf. R. Courant and K. O. Friedlichs *loc. cit.*; especially chapt. II. § 29.

in which α is a constant and $v(x)$ is a smooth function as a whole and symmetric with respect to x. The limit $b\to 0$ will be taken after all calculations are performed. In the first place, the C^- issuing out of $(a+b, 0)$ and C^+ out of $(-a-b, 0)$ are the straight lines whose tangents are $+1$ or -1 respectively, because in the vacuum $(u=v=0)$, $\lambda_\pm=\pm 1$. Refering to the procedure employed in § 2, we divide the (x, t) space into the three regions α, β, γ, by the $C^\pm$ characteristic curves mentioned above (cf. Fig. 3·3).**

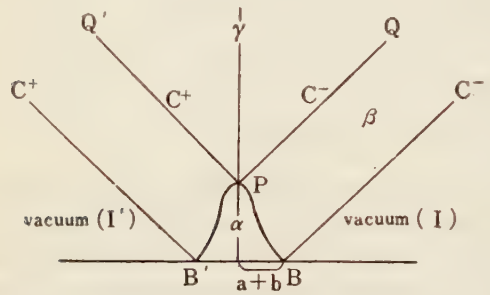

Fig. 3·3 The chracteristics of the equation (3·2)

(i) *γ-zone*

Any point in this region is connected with the vacuum (I) and (I)′ respectively by C^+ and C^- running across the β-zones.

Since

$$u=v=0 \quad \text{in (I)}$$

$$u=-v=0 \quad \text{in (I)}',$$

we see that

$$u=v=0$$

at every point in γ, and that $\overline{PQ}$ and $\overline{PQ}'$ in Fig. 3·3 are the straight lines of gradient $\pi/4$.

(ii) *α-region*

We subdivide this region into the four regions α_I α_{II} α_{III} and α_{IV} illustrated in Fig. 3.4.

(a) *α_I-region*

In this region, both $I'^\pm$ characteristics should be given by $v=mu+pl$ so as to be continuously connected with the initial conditions $v=\alpha>0$, $u=0$, at $t=0$.

Using the initial conditions, we have

$$m_I=\pm\sqrt{1-(\alpha/l)^2}\ (=\text{const.}),$$

and

$$v=m_I u+p_I l$$

representing

$$I'^+ \quad \text{for } m_I>0,$$

and

$$I'^- \quad \text{for } m_I<0.$$

Then, we obtain the relations

*) At the present stage of discussion, we can only say that among the C^+ characteristic lines drawn in Fig. 3·3, BC and B′C′ are straight lines and as for other lines we can say nothing whether they are straight or not unless more detailed calculations are performed.

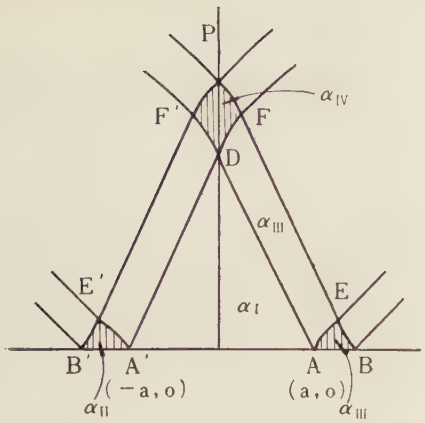

Fig. 3·4 The alalysis of α region

$$v=-|m_I|u+P_I l \qquad \text{along } C^-,$$
$$v=|m_I|u+P_I l \qquad \text{along } C^+,$$

which give $v=P_I l$, $u=0$ in the α_I-region and the straight characteristics AD, $A'D$ with $\lambda_\pm=\mp|m_\mp|$.

(b) *α_{II}-region* (cf. Fig. 3·5)

By the similar argument, we have

$$v=m_{II}(x_0)\,u+P_{II}(x_0)\,l,$$

with

$$m_{II}(x_0)=\pm\sqrt{1-(v_0(x_0)/l)^2};$$

or

$$v=-|m_{II}(x_0')|u+P_{II}(x_0')l \qquad \text{along } C^-,$$
$$v=|m_{II}(x_0'')|u+P_{II}(x_0'')l \qquad \text{along } C^+.$$

From the above equations, u, for example, becomes

$$u=l\cdot\{P_{II}(x_0')-P_{II}(x_0'')\}/[|m_{II}(x_0')|+|m_{II}(x_0'')|],$$

which shows the curved $C^\pm$ characteristics.

(c) *α_{III}-region*

This region is adjacent to the constant state α_I; therefore, according to the theorem (A), it is the region of simple wave. We have, in this case, the relations $v=-|m_I|u+P_I l$ along C^- prolonged from the α_I-region and $v=|m_{II}(x_0)|u+P_{II}(x_0)l$ along C^+ prolonged from the α_{II}-region.

The values of u and v at a cross point between C^- and C^+ can be given as follows;

$$v=l\{P_I|m_{II}(x_0)|+P_{II}(x_0)|m_I|\}/[|m_{II}(x_0)|+|m_I|],$$
$$u=l\{P_I-P_{II}(x_0)\}/[|m_I|+|m_{II}(x_0)|].$$

The values of u and v are, of course, not constant; however they are constant along C^-. Hence, C^+'s are the straight lines parallel to AP and C^-'s are the curves parallel to AE.

(d) *α_{IV}-region*

The values of u and v in this region can be settled down at each cross point between $\Gamma^+(C^+)$ issuing out of $\alpha_{II}(x>0)$ and $\Gamma^-(C^-)$ out of $\alpha_{II}(x<0)$.

We have

$$v=|m_{II}(x_0')|u+P_{II}(x_0')l \qquad \text{along } C^+,$$
$$v=-|m_{II}(x_0'')|u+P_{II}(x_0'')l \qquad \text{along } C^-,$$

with the solution

$$v=l\{P_{II}(x_0')\,|m_{II}(x_0'')\,|+P_{II}(x_0'')\,|m_{II}(x_0')\,|\}/[\,|m_{II}(x_0'')\,|+|m_{II}(x_0')\,|\,],$$

$$u=l\{P_{II}(x_0'')-P_{II}(x_0')\}/[\,|m_{II}(x_0'')\,|+|m_{II}(x_0')\,|\,].$$

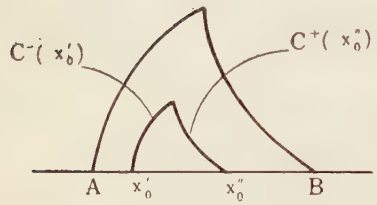

Fig. 3·5
The $C^{\pm}$ characteristics in α_{II} region.

C^+'s and C^-'s are the families of the curves parallel to AE and $A'E'$ respectively.

(iii) *β-zone*

We consider the solutions in the region $\beta^+(x>0)$ only; the solutions in another region, $\beta^-(x<0)$, can easily be obtained from the symmetry. In this case, we also have the three regions β_I, β_{II} and β_{III} as are shown in Fig. 3·6.

(a) *β_I-zone*

We have $v=u$ along C^+ prolonged from the vacuum (I) and also

$$v=-|m_{II}(x_0)\,|u+P_{II}(x_0)\,l$$

along C^- issuing out of α_{II}. The solution is

$$v=u=lP_{II}(x_0)/[1+|m_{II}(x_0)\,|\,].$$

These values are constant along C^-'s and give $\lambda^-=1$ along them. C^-'s are of course parallel straight lines of gradient $\pi/4$, while C^+'s are parallel to the curve $\overline{BE}$. This region is, of course, that of simple wave.

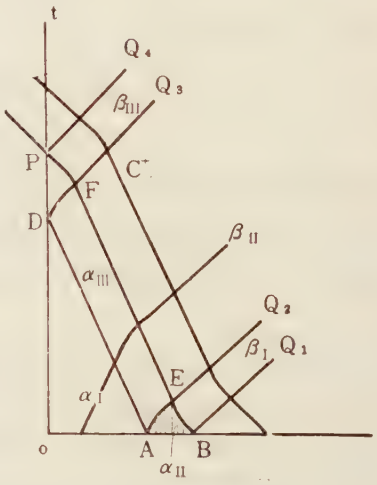

Fig. 3·6 The definition of β zone, β_I zone is $\sqsubset Q_1BEQ_2$ β_{II}, $\sqsubset Q_2ETQ_3$ and β_{III}, $\sqsubset Q_3FPQ_4$

(b) *β_{II}-zone*

By using the similar argument, we have in this case

$$v=u \qquad \text{along } C^+,$$

$$v=-|m_I|u+P_I\,l \qquad \text{along } C^-,$$

and

$$v=u=lP_I/(1+|m_I|) \qquad (=\text{constant}).$$

C^-'s are straight lines whose slopes are equal to unity and C^+'s are also straight and parallel to EF.

(c) *β_{III}-zone*

In this case, we see that

$$v=u \qquad \text{along } C^+,$$

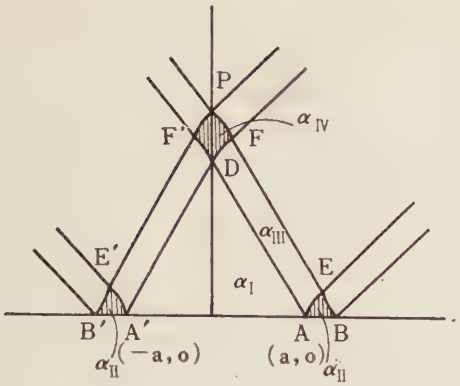

Fig. 3·7
The characteristics of the equation (3·2).

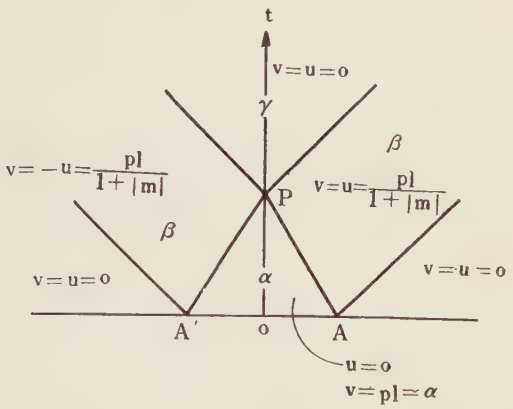

Fig. 3·8 The C characteristics and the spatial distribution of u and v in the limiting case: $b\to 0$.

$$v=-|m_{II}(x_0)|u+P_{II}(x_0)l \quad \text{along} \quad C^-,$$

and

$$v=u=lP_{II}(x_0)/[1+|m_{II}(x_0)|].$$

C^-'s are straight, C^+'s are parallel to PF and this region is of simple wave. Let us now consider the limit $b\to 0$. In this limit, as is obvious in the Fig. 3·7, the regions α_{II} and α_{IV} and the regions of simple wave tend to vanish and there remain only the regions of constant state, and on the boundaries between these α, β and γ, the values of u and v are discontinuous (cf. Fig. 3·8). It is quite interesting to compare this result with that of the linear equation (1·1) illustrated in Fig. 1. At a glance we see that the solution of the non-linear equation under consideration is quite similar to that of the linear equation. The differences are as follows:

(1) $\Delta APA'$ is not equilateral, hence $C^{\pm}$ are discontinuous at P.

(2) the breadth of the β-zone and the values of u and v in it are different from those of the linear case.

As for the energy distribution H, we have

$$H=\{l^2+u^2\}\cdot[1+l^{-2}(u^2-v^2)]^{-1/2}-l^2$$

$$\begin{cases} =\{[1-(\alpha/l)^2]^{-1/2}-1\}l^2 & \text{in the } \alpha\text{-region,} \\ =\alpha^2/[1+\{1-(\alpha/l)^2\}^{1/}]^2 & \text{in the } \beta\text{-zone,} \\ =0 & \text{in the } \gamma\text{-region.} \end{cases}$$

Let us now proceed, on the basis of the results obtained here, to discuss the two subjects cited in § 1: the occurrence of shock waves and the energy distribution.

As for the former, this non-linear equation yields the similar result as that of linear equation and the peculiar phenomena characteristic to most non-linear equations, the formation of shock waves, does not appear in this case, so far as the present initial conditions are concerned. Hence it may be said that this model belongs to the special class

of non-linear equations (for example, such as Kármán-Tsièn's gas in the hydrodynamics). However the so-called Born-type equations are rich in variety of the form of Lagrangian and it may be possible that the modification of Lagrangian destructs the above special character entirely.

We also investigate the difference between the non-linear and the linear equation existing in the energy distribution. The spatial breadth of β zone ($=a+OP$) increases as α increases, (α cannot exceed l), and a considerable amount of energy remains near the origin until the critical time t_c : $t_c=a/\sqrt{1-(\alpha/l)^2}$ which is larger by the factor $(1-(\alpha/l)^2)^{-1/2}$ than that of the linear equation. Therefore, if the value of α is so large as to be comparable to l, these effects may be expected to be observed. However, the following numerical calculation shows that these are, in general, too small to be recognized.

Let us assume that $l=6$, the initial energy $\bar{H}=360$, and $a=0.07$. The value of l can be interpreted as the inverse of the nucleon compton wave length. And the above values of a and H stand for the corresponding quantities in the event where a nucleon of energy 10^{11}ev collides with a nucleon at rest giving rise to a sudden multiple production of mesons which carry away a half of the total energy. For these values, we have

$$\alpha\approx 5.970,$$

hence

$$OP\approx 0.61\approx 0.85\times 10^{-13}\text{cm}.$$

§ 4. Relativistic hydrodynamics[2)3)7)]

Consider the equation

$$\frac{\partial}{\partial x_i}\left\{\frac{\partial\phi}{\partial x_i}\left(\frac{\partial\phi}{\partial x_k}\right)^2\right\}=0, \tag{4·1}$$

which may be derived from the Lagrangian

$$L=\tfrac{1}{2}(\phi_i\cdot\phi_i)^2. \tag{4·2}$$

This equation was proposed by Khalatnikov[3)] as describing a fluid with an extremely high temperature. In the present paper we shall investigate the mathematical behaviour of the above equation with the same initial condition as that of § 3 without regard to its physical background.* Putting

$$\phi_x=u,\quad \phi_t=-v,$$

we have

*) The result obtained in this paper is fairly good, even in the region of low temperature, compared with what is derived by using another equation which is confirmed to be valid in the region of low temperature.

$$\begin{cases} (3u^2-v^2)u_x+2uv(u_t-v_x)+(u^2-3v^2)v_t=0, & (4\cdot 3a) \\ u_t+v_x=0. & (4\cdot 3b) \end{cases}$$

The above equation is analogous to (3·3a, b), hence, applying the same method as was used in § 3, we can solve this equation.* For the C characteristic equation, we have

$$(3u^2-v^2)t_\sigma^2-4uvt_\sigma x_\sigma-(u^2-3v^2)x_\sigma^2=0, \qquad (4\cdot 4)$$

or

$$dt/dx=\lambda_\pm=\{2uv\pm\sqrt{3}(u^2-v^2)\}/(3u^2-v^2). \qquad (4\cdot 5)$$

We have also the relation

$$u^2-v^2=-T^{2\,8)} \qquad (4\cdot 6)$$

in which T is the temperature; then in the quiet gas of temperature T_0, λ reads

$$\lambda_\pm=\mp\sqrt{3}. \qquad (4\cdot 7)$$

This may, of course, be interpreted as the sound velocity in the rest medium. It should be noted that in the vacuum $T_0=0$ and $\lambda_\pm$ is indefinite, therefore the vacuum is the singular region of the original equation.** On the other hand, the Γ characteristics may be given by

$$\Gamma^+: \quad u_\xi-\lambda_- v_\xi=0 \qquad \text{along } C^+, \qquad (4\cdot 8a)$$

$$\Gamma^-: \quad u_\eta-\lambda_+ v_\eta=0 \qquad \text{along } C^-. \qquad (4\cdot 8b)$$

Hence, also in this case, we have the following relations of representation

$$t_\xi v_\eta-x_\xi u_\eta=0, \qquad t_\eta v_\xi-x_\eta u_\xi=0. \qquad (4\cdot 9)$$

Corresponding to (3·11), the equation for u and v turns out to be

$$2(udu-vdv)^2+(u^2-v^2)(du^2-dv^2)=0. \qquad (4\cdot 10)$$

This equation may easily be solved by putting $u=T\sinh\theta$ and $v=T\cosh\theta$.

The result is

$$Y^{(1-\sqrt{3})}-CX^{(1+\sqrt{3})}=0, \qquad (4\cdot 11a)$$

and

$$Y^{(1+\sqrt{3})}-C'X^{(1-\sqrt{3})}=0, \qquad (4\cdot 11b)$$

in which

*) As for the general scheme to transform the equations of these types into the characteristic equations we can, for example, refer to R. Courant and K. O. Friedlichs, *loc. cit.* The succeeding equations may directly be obtained, if the formulae cited there are used.

**) In the region in which the relation $u=v$ holds, we have $\lambda_+=\lambda_-$ and the equation is parabolic.

$$X=v-u, \tag{4·12a}$$

$$Y=v+u, \tag{4·12b}$$

and C's are the integration constants. The correspondence of (4·11a, b) to the $I'^{\pm}$ characteristics can be given by calculating dv/du on the basis of the above solutions. For this purpose, it is convenient to write $\lambda_{\pm}$ as

$$\lambda_{\pm}=-[X^2-Y^2\pm 2\sqrt{3}\ XY]/[X^2+Y^2-4XY], \tag{4·5}'$$

which will often be used in the following arguments. In this way we have

$$I'^{+}:\ Y^{(1+\sqrt{3})}-CX^{(1-\sqrt{3})}=0, \quad \text{along } C^{+}:\ dt-\lambda_{+}\,dx=0; \tag{4·13.a}$$

$$I'^{-}:\ Y^{(1-\sqrt{3})}-CX^{(1+\sqrt{3})}=0, \quad \text{along } C^{-}:\ dt-\lambda_{-}\,dx=0. \tag{4·13.b}$$

Let us investigate, in some illustrative examples, the peculiar characters of the simple wave. This may, on the one hand, be the answer to the subject (i) in § 1 and, on the other, be of use for the succeeding discussions.

We consider firstly the solution satisfying the following conditions: at $t=0$,

$$u=0 \qquad \text{for all } x,$$

$$v=\alpha_0 \ (=\text{const., not zero}) \qquad \text{for } x>a,$$

$$v=\alpha(x) \qquad \text{for } x\leqq a,$$

in which $v(x, 0)$ is smooth as a whole (cf. Fig. 4·1). As the region (I) is a constant state of temperature T_0, the region (II) is that of simple wave. According to (4·7), we have in (I),

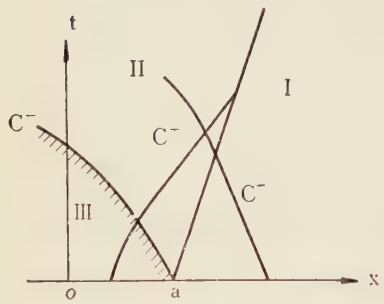

Fig. 4·1
The characteristics of the wave invading the quiet state. I is the constant (quiet) state, II the simple wave region.

$$C^{+}:\ t=\sqrt{3}\,x+\text{const.}$$

$$C^{-}:\ t=-\sqrt{3}\,x+\text{const.},$$

and in (II),

$$Y^{(1+\sqrt{3})}-C(\eta)X^{(1-\sqrt{3})}=0 \quad \text{along } C^{+}, \tag{4·14a}$$

$$Y^{(1-\sqrt{3})}-C_0X^{(1+\sqrt{3})}=0 \quad \text{along } C^{-}. \tag{4·14b}$$

Inserting the initial conditions, we have

$$C(\eta)=\alpha(\eta)^{2\sqrt{3}}, \quad C_0=\alpha_0{}^{-2\sqrt{3}};$$

and

$$X=\alpha^{(1-\sqrt{3})/2}\alpha_0{}^{(1+\sqrt{3})/2}, \tag{4·15a}$$

$$Y=\alpha^{(1+\sqrt{3})/2}\alpha_0{}^{(1-\sqrt{3})/2}. \tag{4·15b}$$

Accordingly λ_{+} becomes

$$\lambda_+ = \{V+(2-\sqrt{3})\}/\{V-(2-\sqrt{3})\}, \tag{4·16}$$

with

$$V=(\alpha/\alpha_0)^{\sqrt{3}} \neq (2+\sqrt{3}). \tag{4·17}$$

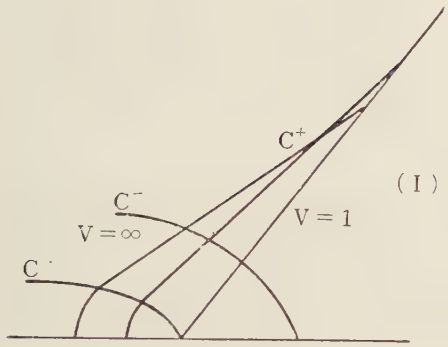

Fig. 4·2
The characteristics of a compression wave showing the occurence of a shock wave.

There exist the following two cases concerning the behaviour of λ_+, according as α is larger or smaller than α_0.

(1) $\alpha > \alpha_0$ (i.e. $V>1$)

λ_+ decreases monotonously from $\sqrt{3}$ as V increases monotonously and λ_+ tends to unity in the limit of $V=\infty$. (cf. Fig. 4·2)
This means that the velocity profile gradually steepens and it gives rise to discontinuity surfaces, that is, shock waves.[6]*

(2) $\alpha < \alpha_0$ (i.e. $V<1$)

λ_+ increases as V decreases monotonously

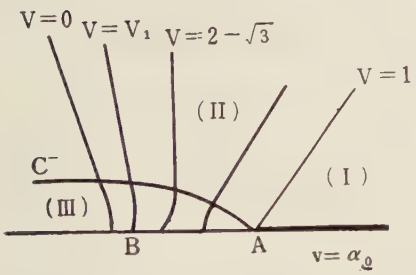

Fig. 4·3
The characteristics corresponding to $V \leqq 1$.

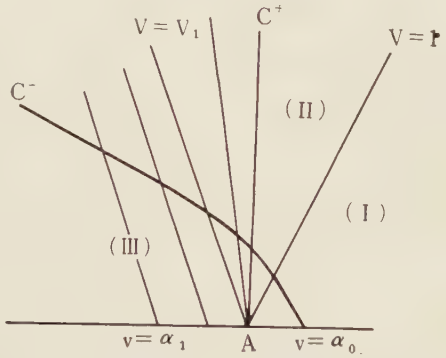

Fig. 4·4 The characteristics of a rarefaction wave resulting from the discontinuous initial condition: $v=\alpha_0$ (const.), for $x \geqq x_A$; $v=\alpha_1$ (const.) for $x \leqq x_A$.

from unity, and tends to $+\infty$ as V approaches $2-\sqrt{3}$, then jumps discontinuously to $-\infty$ as V exceeds $2-\sqrt{3}$ and increases monotonicly as V decreases. (cf. Fig. 4·3). Contrary to the case (1), we have, in this case, C^+'s spreading out of (III) as t increases.

We are especially interested in the discontinuity limit of $B \to A$ in which $v(x)$ decreases suddenly from α_0 to α_1. It may easily be understood that in this case, the C^+ characteristic lines become a bundle of lines issuing out of A as are illustrated in Fig.

*) In fact, this corresponds to the compression wave in the non-relativistic hydrodynamics.

4.4 and hence the region (II) is smoothly connected with the constant regions (I) and (III). This indicates the remarkable fact that the initial discontinuity at A is immediately resolved. The simple wave, which was discussed in the above example (2), corresponds to the rarefaction wave in the non-relativistic case which can be produced by the motion of receding piston. We illustrated in Fig. 4·5 the characteristics of a rarefaction wave in which AB is the trajectory of the piston with a constant terminal velocity. We shall encounter this case, when we shall deal with the initial value problem under the physical conditions explained in § 1. The analytical character of such centered rarefaction waves will be given later.

Now we are ready to answer the question (i) in § 1. As can be seen in the above example, the equation (4·1) has such conspicuous properties characteristic to non-linear equations that the initial discontinuity is resolved immediately and the disturbance, while propagated, becomes continuous while the discontinuities, shock fronts, may appear and be propagated even without being caused by initial discontinuities. It should, however, be noted that we have also the same characters of solutions in the non-relativistic hydrodynamics.[6]

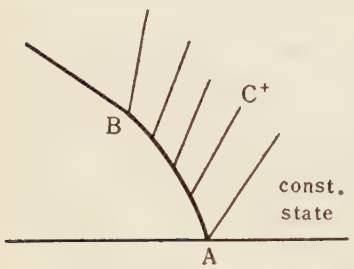

Fig. 4·5 The characteristics resulting from a receding piston.

Let us now consider the case of a flow invading the vacuum. The initial condition is supposed to be

at $t=0$,

$$v\begin{cases}=\alpha\ (=\text{const.}) \cdots\cdots\cdots & |x|\leqq a,\\ =v_0(x) \cdots\cdots\cdots & a\leqq|x|\leqq a+b,\\ =0 \cdots\cdots\cdots & |x|>a+b,\end{cases}\qquad u=0 \quad \text{for all } x, \tag{4·18}$$

that is, the initial velocity of fluid is zero and the enthalpy (or temperature) is given only in the region : $|x|\leqq a+b$. It was already stated that the vacuum is the singular region contrary to the case illustrated in Fig. 3.3. Hence, there exists the essential difference between the following two cases, the one that of invading the vacuum and the other the quiet gas.

(i) *α-region* (cf. Fig. 4·6)

The definition of α-region and β-zone are somewhat different from those given in § 3, and the α region is defined by the characteristic lines C^- and C^+ issuing out of two points A, $(0, a)$, and A', $(0, -a)$, respectively (cf. Fig. 4·6).
In this region, we have

$$Y^{(1+\sqrt{3})}-\alpha^2\sqrt{3}\,X^{(1-\sqrt{3})}=0 \qquad \text{along } C^+,$$

$$Y^{(1-\sqrt{3})} - \alpha^{-2\sqrt{3}} X^{(1+\sqrt{3})} = 0 \qquad \text{along } C^-;$$

hence, we obtain the solutions

$$X=\alpha, \qquad Y=\alpha, \tag{4·19}$$

or

$$v=\alpha, \qquad u=0,$$

and

$$\lambda^{\pm} = \pm 3.$$

(ii) *β_{II}-zone*

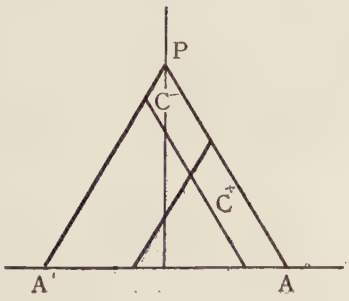

Fig. 4·6 Tha characteristics in α region.

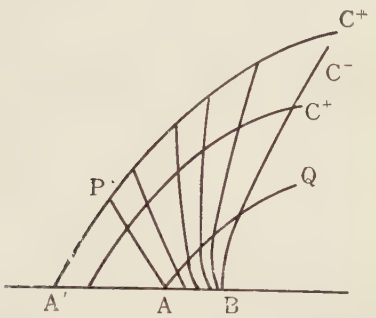

Fig. 4·7 The characteristics in β^+ zone.

β_{II}^+-zone is defined, in this case, as the region being adjacent to α-region and bounded by the characteristics C^+ and C^- issuing out of P and A respectively. (The definition of β_{II}^- may be obvious from this). (cf. Fig. 4·7)
This region is that of simple wave and we have

$$Y^{(1+\sqrt{3})} - \alpha^{2\sqrt{3}} X^{(1-\sqrt{3})} = 0$$

along C^+, and consequently over all β_{II}^+; and

$$Y^{(1-\sqrt{3})} - C(\xi) X^{1+\sqrt{3}} = 0$$

along C^- issuing out of AB. In this equation $C(\xi)$ can be given as $C(\xi) = v_0(\xi)^{-\sqrt{3}}$ from (4·8).

These two equations give

$$Y = \alpha^{(1+\sqrt{3})/2} \, v_0^{(1-\sqrt{3})/2},$$

$$X = \alpha^{(1-\sqrt{3})/2} \, v_0^{(1+\sqrt{3})/2};$$

or

$$v = (1/2)(\alpha v_0)^{\frac{1}{2}} \{(\alpha/v_0)^{\sqrt{3}/2} + (v_0/\alpha)^{\sqrt{3}/2}\},$$

$$u = (1/2)(\alpha v_0)^{\frac{1}{2}} \{(\alpha/v_0)^{\sqrt{3}/2} - (v_0/\alpha)^{\sqrt{3}/2}\}.$$

The values of u and v are constant along the straight C^-'s with gradient

$$\lambda_-=\{V+(2+\sqrt{3})\}/\{V-(2+\sqrt{3})\},\quad (V\neq 2-\sqrt{3})$$

in which $V=(\alpha/r_0)^{V_0}$ (cf. Fig. 4·7′). In this region ($x>0$), where $V\geqq 1$, the slope of C^- is equal to $-\sqrt{3}$ for $V=1$ (corresponding to AP) and becomes steeper as V increases up to $2+\sqrt{3}$, is parallel to t-axis for $V=2+\sqrt{3}$ and then tends to the line of gradient $\pi/4$ as V increases up to ∞. On the other hand, the value of λ^+ tends to unity as $V\to\infty$, hence the C^- starting from B tends to the C^+ issuing out of A in the limit of $(x,\ t)\to\infty$. Hence we have the infinitely extended region ABQ which is called β_I-zone, corresponding to the discussions in § 3.

However, we are interested only in the limiting case $b\to 0$, or $B\to A$. Noting the fact that the β_{II}-zone is just the region of rarefaction wave produced by the piston receding to the *positive* x direction whose trajectory is given by the boundary characteristic line AQ_{lim} (Fig. 4·7), we can easily perform this limiting process, the result of which is shown in Fig. 4.8 (the terminal velocity of piston, in this case, is the light velocity). It can be seen from the above example that the wave invading the vacuum is not the compression wave but the rarefaction wave, contrary to that invading the quiet gas. This indicates that the energy distribution near the wave fronts is entirely different from that of linear equation.

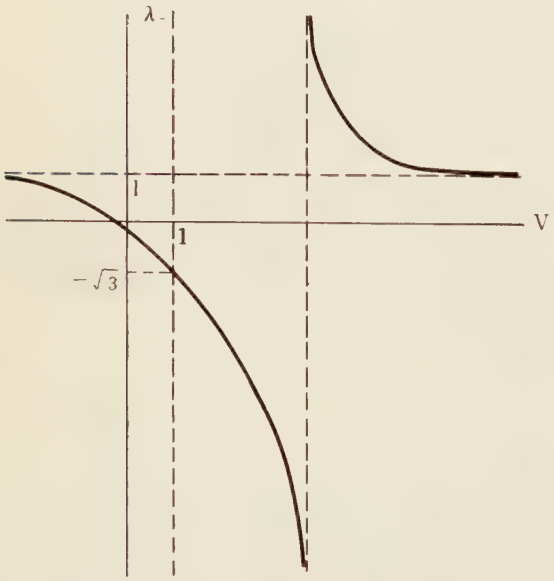

Fig. 4·7′ The change of λ^- due to the variation of V, $V=1$ corresponds to the curve AP in Fig. 4·7.

We investigate, now, the analytical behaviour in the β-zone introduced above. In the β-zone (Fig. 4·7′), we have:

$$\lambda_-=\text{const.}\quad \text{along } C^-,$$

or

$$t=\lambda^-\cdot(x-a),\qquad (4\cdot 20)$$

and

$$Y^{(1+\sqrt{3})}-\alpha^{2\sqrt{3}}X^{(1-\sqrt{3})}=0$$

over all β-zone.

Using the above relations and the expression $(4\cdot 5)'$ for λ^-, we can express X and Y in terms of λ^- as follows*:

$$X=\alpha[(\lambda-1)/\{(2+\sqrt{3})(\lambda+1)\}]^{(1+\sqrt{3})/2\sqrt{3}},\qquad (4\cdot 21a)$$

$$Y=\alpha[(\lambda-1)/\{(2+\sqrt{3})(\lambda+1)\}]^{(1-\sqrt{3})/2\sqrt{3}},\qquad (4\cdot 21b)$$

*) For the sake of convenience, we write hereafter λ in place of λ_-.

or

$$X=\alpha[(t-x+a)/\{(2+\sqrt{3})(t+x-a)\}]^{(1+\sqrt{3})/2\sqrt{3}}, \qquad (4\cdot21\text{a})'$$

$$Y=\alpha[(t-x+a)/\{(2+\sqrt{3})(t+x-a)\}]^{(1-\sqrt{3})/2\sqrt{3}}. \qquad (4\cdot21\text{b})'$$

This gives u and v in terms of x and t, and the solutions in β-zone can be settled down. On the other hand, the C^-'s are naturally not straight lines and they are given by the differential equation

$$dt/dx=-(X^2-Y^2+2\sqrt{3}XY)/(X^2+Y^2-4XY).$$

Inserting (4·21 a, b) into this equation we have

$$dt/dx=(2t+\sqrt{3}x-\sqrt{3}a)/(\sqrt{3}t+2x-2a),$$

or

$$(t+x-a)^{2-\sqrt{3}}/(t-x+a)^{2+\sqrt{3}}=\text{const.} \qquad (4\cdot22)$$

Introducing the new coordinates

$$t+x'=\sqrt{2}\,\eta, \qquad t-x'=\sqrt{2}\,\xi, \qquad x'=x-a,$$

we have finally

$$\eta=\text{const}\cdot\xi^{\sqrt{3}+1/\sqrt{3}}.$$

This equation gives the graph of C^+ in the (x, t) space : (Fig. 4·8). The integration constants can be given by the values at the cross points of C^+'s and AP. The boundary lines between β and γ regions are thus settled down : they are illustrated in Fig. 4.10.

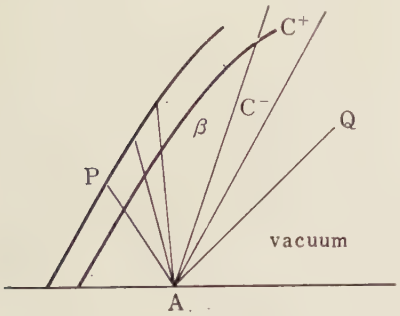

Fig. 4·8 The characteristics in β^+ zone in the limit : $b\to0$.

(iii) *γ-zone*

In this region, we can transform the original

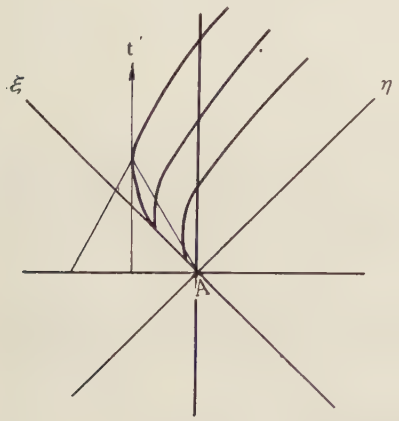

Fig. 4·9 The behaviours of C^+ curves iu β^+zone.

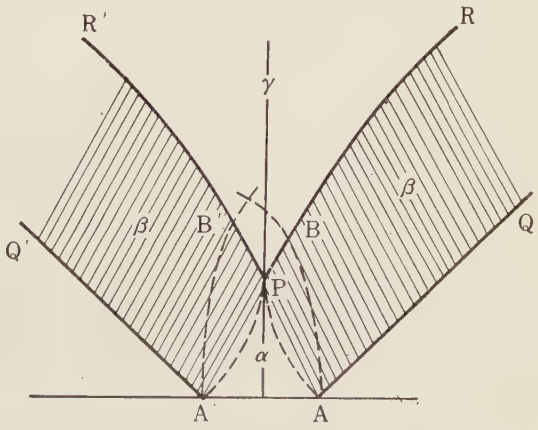

Fig. 4·10

equation into the linear equation by means of the Legendre transformation, which could not be used in the region of constant state or simple wave. It was shown by Khalatnikov[3] that by this transformation the original equation is reduced to the linear one with constant coefficients, which is easy to handle aside from the complexities in boundary conditions. By this procedure, he has obtained the solution corresponding to the flow invading vacuum. Employing the same method, Amai, Fukuda et al.[7] have also obtained the solutions under more general conditions, which include our initial conditions as the special case. They also investigated the solution in this γ region on the method of characteristics, but in some graphical approximation. In the present paper, we shall try to solve analytically by means of the same method.

In this region, we have

$$Y^{(1-\sqrt{3})} - C(\lambda) X^{(1+\sqrt{3})} = 0$$

along a C^- which is the prolongation of that in the β-zone, say AB (cf. Fig. 4·10). Inserting the relations (4·21a, b), which are valid in the β-zone into the above equation, we have

$$C(\lambda) = (2+\sqrt{3})^2 \, \alpha^{-2\sqrt{3}} (\lambda+1)^2 (\lambda-1)^{-2}. \tag{4·23a}$$

According to the similar calculation, we also have

$$Y^{(1+\sqrt{3})} - C(\lambda') X^{(1-\sqrt{3})} = 0 \quad \text{along } C^+,$$

in which $\lambda' = t/(x+a)$,

$$C(\lambda') = (2+\sqrt{3})^{-2} \, \alpha^{2\sqrt{3}} (\lambda'+1)^2 (\lambda'-1)^{-2}, \tag{4·23b}$$

and the symmetry $v(x) = v(|x|)$ $u(x) = -u(|x|)$ for $x < 0$, has been used. Hence we obtain the solutions

$$\begin{aligned} Y &= \alpha (2+\sqrt{3})^{-1/\sqrt{3}} [(\lambda+1)/(\lambda-1)]^{(3-\sqrt{3})/6} [(\lambda'+1)/(\lambda'-1)]^{(3+\sqrt{3})/6}, \\ X &= \alpha (2+\sqrt{3})^{-1/\sqrt{3}} [(\lambda-1)/(\lambda+1)]^{(3+\sqrt{3})/6} [(\lambda'-1)]/[(\lambda'+1)]^{(3-\sqrt{3})/6}. \end{aligned} \tag{4·24b}$$

The temperature T is given by

$$T^2 = \alpha^2 (2+\sqrt{3})^{-2/\sqrt{3}} \, |(\lambda-1)/(\lambda+1)|^{1/\sqrt{3}} \, |(\lambda'+1)/(\lambda'-1)|^{1/\sqrt{3}}. \tag{4·25}$$

Since in the above expressions, the field quantities are given as the functions of λ's, the shape of $C^\pm$ in the (x, t) space should be studied in order to obtain the spatial distribution of these values. In the following discussion we shall investigate the shape of C^- in this region; the knowledge about C^+ may be obtained from that of C^- because of the symmetry. Noting the representation relation (4·9) between I'^+ and C^-, we see that in the (Y, X) space*, the directions of I'^+'s and a C^- are symmetric with respect to Y-axis. If we draw a C^- in the (Y, X) space, where t and x axis is put to coin-

*) Coordinate space (Y, X) is given by the $\pi/4$ rotation of (u, v) space.

cide with v and u axis respectively, we have the following equation ;

$$dX/dY=-dX_1/dY_1$$

at a cross point between a Γ^+ and the C^-, in which dX/dY is the slope of the Γ^+ and X_1 and Y_1 is : $X_1=t-x$ and $Y_1=t+x$. Along Γ^+, we have

$$Y^{1+\sqrt{3}}-C(\lambda')X^{1-\sqrt{3}}=0\,,$$

i.e.,

$$dX/dY=-(\sqrt{3}+1)X/(\sqrt{3}-1)Y,$$

$$dX_1/dY_1=(\sqrt{3}+1)X_1/(\sqrt{3}-1)Y_1.$$

Hence we obtain the following equation for C^-;

$$Y_1=CX_1^{(\sqrt{3}-1)/(\sqrt{3}+1)} \quad (\text{or } X_1\approx Y_1^{3.732}); \tag{4·26}$$

or

$$(t+x)^{\sqrt{3}+1}-D(t-x)^{\sqrt{3}-1}=0, \tag{4·26}'$$

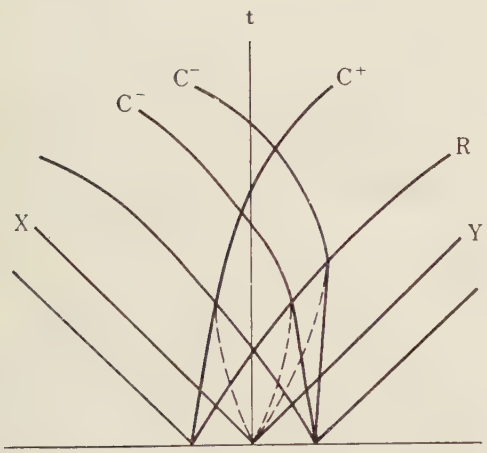

Fig. 4·11 The characteristics in γ region.

in which D is an integration constant and can uniquely be determined as the function of λ'. Following the similar way we can obtain the equation for C^+ in which the integration constant is expressed in terms of λ. Then, representing λ and λ' in terms of x and t, and inserting these expressions into (42·4a, b), we have the solutions X and Y (or u and v) as functions of x and t. However, as the final result obtained in this way is too complicated to evaluate, we rather prefer to use the following simple but strict graphical method with the aid of the analytical expression (4·26) for the curves $C^{\pm}$.

From (4·26), we have Fig. (4·11) for the graph of C in which each curve is determined by each λ and λ'. The values of X and Y are given by (4·24.a, b) in terms of λ and λ', hence the value at the cross points of a C^+ and a C^-, illustrated in Fig. 4.11, is to be uniquely determined.

(iv) *Energy distribution*

Let us now investigate the energy distribution. In the following discussion, we shall calculate the amount of energy contained in β-zone and γ-region after a sufficiently long time. From (4·2), the energy density H can be given by

$$H=(v^2-u^2)\left((3/4)v^2+\tfrac{1}{4}u^2\right). \tag{4·27}$$

At any instant, the energy contained in the whole γ region can be given as the total energy minus the energy contained in the whole β-region. Hence we try to calculate the energy contained in the β-zone.

From (4·22), the boundary curve $\overline{PR}$ can be given by

$$(t+x')^{2-\sqrt{3}}=(2-\sqrt{3})^2(2a^2)^{-\sqrt{3}}(t-x')^{2+\sqrt{3}}, \tag{4·22}'$$

in which $x'=x-a$, or

$$[1+(1/\lambda)]^{2-\sqrt{3}}=\rho[(1-(1/\lambda)]^{2+\sqrt{3}}\cdot t^{2\sqrt{3}} \tag{4·22}''$$

in terms of $\lambda=t/x'$, in which ρ is defined by

$$\rho=(2-\sqrt{3})^2(2a^2)^{-\sqrt{3}}. \tag{4·28}$$

The above equations give the value of x' on the $\overline{PR}$ at a given instant, $t=t_0$, or the corresponding value of $\lambda=\lambda_0$; λ_0 tends to 1 as $t_0\to\infty$. Therefore we may assume the following expression for the above λ_0:

$$\lambda_0\cong 1+kt^{\nu}\quad(\nu<0)\qquad\text{for } t\gg 1. \tag{4·29}$$

Inserting this into (4·22)″, we have

$$\nu\simeq -2\sqrt{3}/(2+\sqrt{3})\approx -0.93,$$

$$k\cong\{2(2+\sqrt{3})a^{\sqrt{3}}\}^{2(2-\sqrt{3})}\approx(7\cdot 46a^{\sqrt{3}})^{0.54}. \tag{4·31}$$

On the other hand, inserting (4·21a, b) into (4·27) and integrating over the β at an instant $t=t_0$, we have

$$\bar{H}^{+}_{\beta}=(a^4/4)\int_{x_0}^{t_0}\left\{Z^{(1+2/\sqrt{3})}+Z^{(-1+2/\sqrt{3})}+Z^{2/\sqrt{3}}\right\}dx', \tag{4·27}'$$

in which Z is given by

$$Z=(2+\sqrt{3})^{-1}(\lambda-1)(\lambda+1)^{-1}.$$

The total energy is equal to $\bar{H}$ at $t=0$,

i.e.
$$\bar{H}=2\times(a^4/4)3a. \tag{4·27}''$$

Denoting

$$\int_{x_0}^{t_0}Z^{\tau}\,dx'=I_{\tau},$$

and transforming x' into λ', we have

$$I_{\tau}=2t(2+\sqrt{3})^{-\tau}\int_0^{w_0}w^{\tau}(1+w)^{-2}\,dw,$$

in which

$$w=(\lambda-1)(\lambda+1)^{-1};$$

and $0 \leqq w \leqq 1$, $(1 \leqq \lambda \leqq \infty)$ for $t_0 \gg 1$.

Using the above property of w, the integral of (4·32) may be estimated as follows:

$$(4(\gamma+1))^{-1} w_0^{\gamma+1} \leqq \int_0^{w_0} w^\gamma (1+w)^{-2} dw \leqq (\gamma+1)^{-1} w_0^{\gamma+1}. \tag{4·33}$$

From (4·29), (4·30), (4·21), we have

$$w_0^{\gamma+1} \approx (k/2)^{\gamma+1} \, t_0^{-2\sqrt{3}(\gamma+1)/(2+\sqrt{3})}. \tag{4·34}$$

Inserting (4·33), (4·34) into (4·27)′, and neglecting the terms decreasing in the order of t_0^{-1} or more, we have

$$\frac{\alpha^4}{4} 8.66 a (t_0/a)^{-0.072} \gtrsim \bar{H}_\beta^+ \gtrsim \frac{\alpha^4}{4} 2 \cdot 14 a (t_0/a)^{-0.072}. \tag{4·35}$$

This indicates that the amount of energy contained in the β-zone decreases slowly with time, and is mainly determined by the factor $(t_0/a)^{-0.072}$. We have

$$(t/a)^{-0.072} \approx 1/2.1 \quad \text{for } t \approx 2.2 \times 10^3,$$

$$\approx 1/11 \quad \text{for } t \approx 2.2 \times 10^{13};$$

in the ordinary scale*, the above values of t correspond to 10^{-20} sec. and 10^{-10} sec. respectively. In other words, the amount of energy contained in the γ-zone is one-half at $t \approx 10^{-20}$ sec and gradually increases with time and at $t \approx 10^{-10}$ sec. the major part of energy is contained in γ-region, or the energies concentrated near the wave fronts dissipate gradually with time.

§ 5. Concluding remarks

We have proved that under the specified initial condition, the Born type equation treated in this paper has not shown any feature characteristic to non-linear theory. However, it might be doubtful that this conclusion holds even for other equations of Born type. Similarly the nature of the smoothing away of the energy distribution shown in the extremely relativistic hydrodynamics might be changed due to the slight modification of the original equation. However, for the study of these kinds of problems, it is desirable to do somewhat more general discussion.

The author would like to express his sincere thanks to Professor Y. Tanikawa, Professor S. Tomotika and Professor Z. Koba for their advices and to Mr. N. Mugibayasi for his discussions.

*) The value of a is taken to be $a=0.07$ corresponding to the nuclear dimension (cf. § 3).

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Theory of the Susceptibility and Specific Heat of He^3

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A gas model taking account of interatomic interactions is proposed for liquid He^3. Assuming that the interaction potential can be decomposed into Fourier components, $4\pi g(k)e^{ik\cdot r}$ (r: interatomic distance), we describe the motion of the system using the method of Bohm and Pines. If we regard the expansion coefficients c_0, c_2,$\cdots$ of $g(k)$ in powers of k as parameters and presume suitable values of these parameters with reference to the value of sound velocity proposed by Mikura, the experimental data of nuclear susceptibility χ and specific heat C_V, it can be shown that the number of collective modes is of the order of one percent of the total degrees of freedom, so that the influence of the collective motion on χ and C_V is negligible in the vicinity of 0^0K. The deviations of χ and C_V from those which follow from the simple free particles theory arise mainly from the exchange interactions. Considering only these exchange energies but neglecting the influence of the collective motion, we can explain the temperature dependence of χ in the whole temperature range provided that we take account of the k-dependence of the exchange energy per particle, while the temperature dependence of C_V is explained only up to 0.3^0K.

§ 1. Introduction

The specific heat of liquid He^3 has been measured by de Vries and Daunt,[1] Roberts and Sydoriak,[2] and Osborne, Abraham and Weinstock.[3] The experimental results are presented by the curves in Fig. 1. The remarkably high specific heat at low temperatures would seem to require a Fermi-Dirac type gas model for liquid He^3. We plot in Fig. 1 the specific heat of ideal Fermi-Dirac gas for which the degeneracy temperature is $T_0=5^0$K, which is determined from the actual mass of a free He^3 atom and the density of liquid He^3. On the other hand, the observed nuclear paramagnetic susceptibility χ of He^3 by Fairbank et al[4]. was shown to be in close agreement with that predicted for an ideal Fermi-Dirac gas with a degeneracy temperature of about 0.45^0K. We show this in Fig. 2. This corresponds to an ideal Fermi-Dirac gas of an effective mass of particles about 10 times as large as the mass of a free He^3 atom (cf. the plot in Fig. 1 of the specific heat of ideal Fermi-Dirac gas with a degeneracy temperature $T=0.45^0$K).

The aim of this paper is to see in what way this discrepancy for the simple Fermi-Dirac gas model is amended by taking account of the interactions between the particles in an approximate way, i.e. by describing these interactions according to the method developed by Bohm and Pines[5] for electrons. To start with free particle plane waves and to avoid the difficult problem of dealing with the hard core potential, we take a system of particles which interact with much milder potential $V^*(r)$ than the real one $V(r)$.

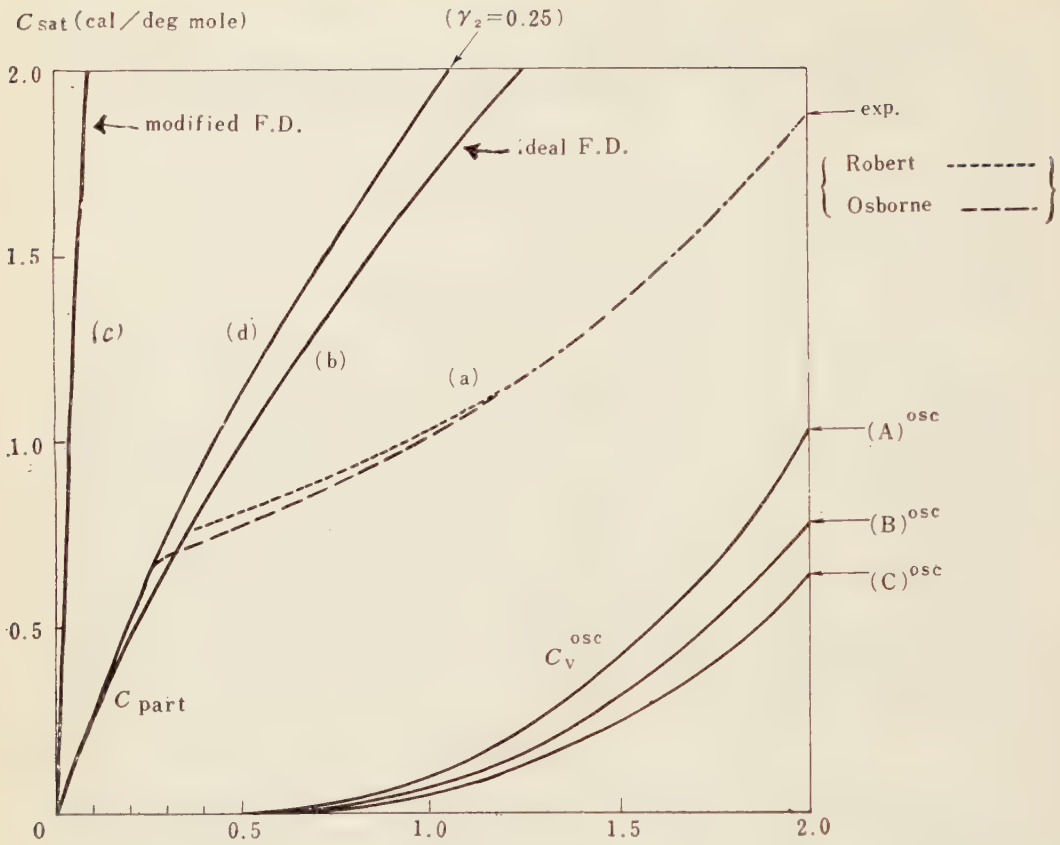

Fig. 1 Specific heat of He^3. (a), smoothed experimental results (dotted curve for Roberts, Dashed curve for Osborne); (b) and (c), C_V of ideal F.D. gas with degeneracy temperature of 5^0K and 0.45^0K, respectively; (d), calculated C_V curve for $\gamma_2=0.25$ in our statistical theory; (A), (B) and (C), calculated phonon part of the specific heat corresponding to the cases (A), (B) and (C), respectively.

We describe first in § 2 the collective motion of the system using the method of Bohm and Pines. It will be shown that the number of collective modes is of the order of one percent of the total degrees of freedom. In § 3, we then calculate the nuclear paramagnetic susceptibility χ^0 at 0 K and specific heat C_V in the vicinity of 0°K, using the collective description. From this calculation we conclude that the influence of the collective motion on χ and C_V is negligible and that the deviations of χ and C_V from those which follow from the simple free particles theory arise mainly from the exchange interaction. We neglect, in § 4, the influence of the collective motion over a considerable range of low temperatures and calculate the temperature dependence of χ and C_V by Fermi-Dirac statistics considering only the exchange energy. The results improves those

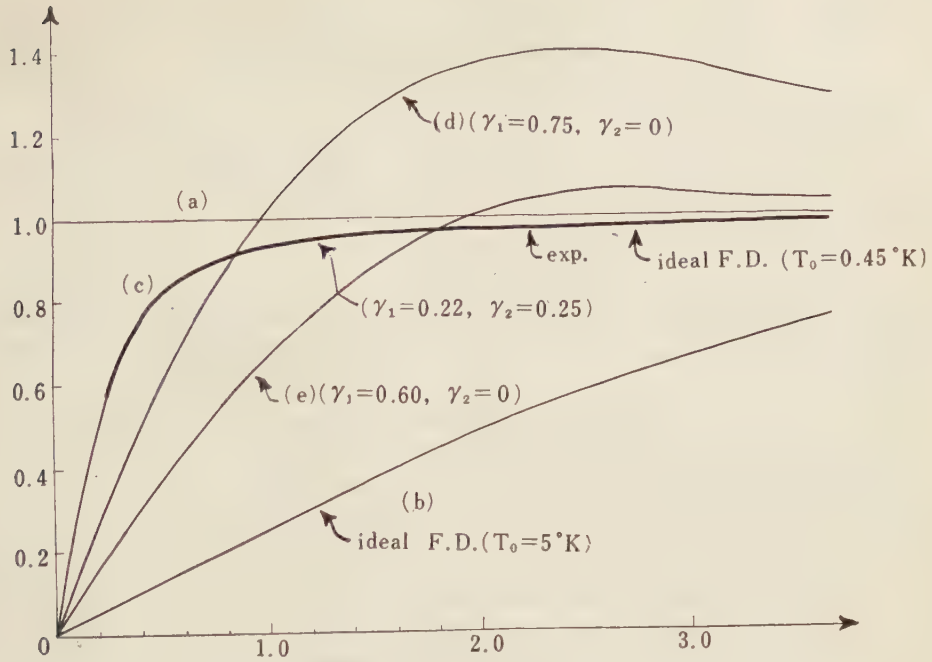

Fig. 2 Plot of $\chi T/C$ vs. T. (χ=nuclear magnetic susceptibility of He^3, T=absolute temperature, C=Curie constant); Experimental result of Fairbank et al. is presented by heavy line. (a) represents the Curie law, (b) is the curve for ideal F.D. gas with degeneracy temperature of 5°K, (c) is the curve for ideal F.D. gas with degeneracy temperature of 0.45°K and this coincides with the case (γ_1=0.22, γ_2=0.25) in our statistical theory. (d) and (e) correspond to the cases (γ_1=0.75, γ_2=0) and (γ_1=0.60, γ_2=0), respectively, in our theory.

which follow an ideal gas model, but the bending of the C_V curve in the vicinity of 0.3°K remains unexplained. Perhaps the coupling between particles and phonons would increase abruptly when the temperature exceeds this point. In § 5, some discussions are given on this point.

§ 2. Collective description

We assume that the interatomic potential $V^*(r)$ can be decomposed into Fourier components and denote its Fourier coefficients by $4\pi g(k)$. Then the Hamiltonian of our system can be written as

$$\sum_i \frac{\boldsymbol{P}_i^2}{2m} + 2\pi \sum_{k,i,j} g(k) e^{ik\cdot(r_i - r_j)} 2 - \pi n \sum_k g(k), \qquad (2\cdot1)$$

where n is the total number of particles in a box of unit volume. Denoting by q_k extra variables and by p_k their conjugate momenta, the Hamiltonian

$$H=\sum_i \frac{\boldsymbol{p}_i^2}{2m}+2\pi\sum_{k,i,j} g(k)e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}-\frac{1}{2}\sum_{k<k_c} p_k p_{-k}-i\sum_{i,k<k_c}(4\pi g(k))^{1/2}p_k e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}-2\pi n\sum_k g(k) \tag{2·2}$$

is equivalent to (2·1) if subsidiary conditions

$$p_{-k}\psi=0 \quad (k<k_c) \tag{4·3}$$

are imposed, where

$$p_k=-p_{-k}^*, \quad q_k=-q_{-k}^*. \tag{2·4}$$

To take out the collective field we apply after Bohm and Pines the unitary transformation $\psi=S\Phi$ with

$$S=\exp\left[\frac{1}{\hbar}\sum_{i,k<k_c}(4\pi g(k))^{1/2}q_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}\right]. \tag{2·5}$$

The transformed Hamiltonian and subsidiary conditions become

$$H=H_{\text{part}}+H_I+H_{\text{osc}}+H_{\text{s.r.}}, \tag{2·6}$$

$$H_{\text{part}}=\sum_i \frac{\boldsymbol{p}_i^2}{2m}-2\pi n\sum_{k<k_c} g(k), \tag{2·6a}$$

$$H_I=\frac{1}{m}\sum_{i\,k<k_c}(4\pi g(k)k^2)^{1/2}\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{p}_i-\frac{\hbar\boldsymbol{k}}{2}\right)q_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}, \tag{2·6b}$$

$$H_{\text{osc}}=-\frac{1}{2}\sum_{k<k_c}(p_k p_{-k}+\omega_k^2 q_k q_{-k}), \tag{2·6c}$$

$$H_{\text{s.r.}}=2\pi\sum_{\substack{k>k_c\\ i\neq j}} g(k)e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}, \tag{2·6d}$$

$$\Omega_k\Phi\equiv\{p_{-k}-i(4\pi g(k))^{1/2}\sum_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\}\Phi=0 \quad (k<k_c), \tag{2·7}$$

where

$$\boldsymbol{\varepsilon}_k=\boldsymbol{k}/k, \quad \omega_k=\left\{\frac{4\pi n}{m}k^2 g(k)\right\}^{1/2}. \tag{2·8}$$

In obtaining H_{osc} we have neglected terms like $q_k q_l e^{i(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{r}_i}(\boldsymbol{k}+\boldsymbol{l}\neq 0)$ by random phase approximation. Here k_c is the maximum wave vector beyond which the organized oscillation is not possible. Then the number of collective degrees of freedom is given by

$$n'=k_c^3/6\pi^2. \tag{2·9}$$

Let us introduce the creation and destruction operators for the collective field, a_k and a_k^*, which are defined by

$$q_k=(\hbar/2\tilde{\omega}_k)^{1/2}(a_k-a_{-k}^*), \qquad p_k=i(\hbar\tilde{\omega}_k/2)^{1/2}(a_k^*+a_{-k}), \tag{2·10}$$

$$[a_k, a_{k'}^*]=\delta_{kk'}, \qquad [a_k, a_{k'}]=[a_k^*, a_{k'}^*]=0. \tag{2·11}$$

In terms of these variables, we then write our Hamiltonian and subsidiary conditions as

$$H=H_{\text{part}}+H_I+H_{\text{coll}}+H_{\text{s.r.}}, \tag{2·12}$$

$$H_{\text{part}}=\sum_i \frac{\boldsymbol{P}_i^2}{2m}-2\pi n\sum_{k<k_c} g(k), \tag{2·12a}$$

$$H_I=\frac{1}{m}\sum_{i,k<k_c}\left(\frac{2\pi\hbar k^2 g(k)}{\tilde{\omega}_k}\right)^{1/2}\left\{\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{p}_i-\frac{\hbar\boldsymbol{k}}{2}\right)a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}+e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}a_k^*\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{p}_i-\frac{\hbar\boldsymbol{k}}{2}\right)\right\}, \tag{2·12b}$$

$$H_{\text{coll}}=\sum_{k<k_c}\frac{\hbar\tilde{\omega}_k}{2}(a_k^*a_k+a_ka_k^*)+\sum_{k<k_c}\frac{\hbar}{4\tilde{\omega}_k}(\omega_k^2-\tilde{\omega}_k^2)(a_ka_k^*+a_k^*a_k-a_ka_{-k}-a_{-k}^*a_k^*), \tag{2·12c}$$

$$H_{\text{s.r.}}=2\pi\sum_{\substack{k>k_c\\ i\neq j}} g(k)e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}, \tag{2·12d}$$

$$\Omega_k\Phi=\left\{a_k^*+a_{-k}-\left(\frac{8\pi g(k)}{\hbar\tilde{\omega}_k}\right)^{1/2}\sum_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}\right\}\Phi=0\quad(k<k_c). \tag{2·13}$$

In order to eliminate the particle-field interaction H_I, we perform a canonical transformation on our operators $(\boldsymbol{r}_i, \boldsymbol{p}_i, a_k, a_k^*)$ to get a new set of operators $(\boldsymbol{R}_i, \boldsymbol{P}_i, A_k, A_k^*)$. The relation between these two sets may be written as

$$\boldsymbol{r}_i=e^{-iS}\boldsymbol{R}_i e^{iS}\quad\text{etc.} \tag{2·14}$$

The generating function S is given by

$$S=-\frac{i}{m}\sum_{i,k<k_c}\left(\frac{2\pi\hbar k^2 g(k)}{\tilde{\omega}_k}\right)^{1/2}\left\{\frac{\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{P}_i-\frac{\hbar\boldsymbol{k}}{2}\right)A_k}{\tilde{\omega}_k-\frac{\boldsymbol{k}\cdot\boldsymbol{P}_i}{m}-\frac{\hbar k^2}{2m}}e^{i\boldsymbol{k}\cdot\boldsymbol{R}_i}-e^{-i\boldsymbol{k}\cdot\boldsymbol{R}_i}\frac{A_k^*\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{P}_i-\frac{\hbar\boldsymbol{k}}{2}\right)}{\tilde{\omega}_k-\frac{\boldsymbol{k}\cdot\boldsymbol{P}_i}{m}+\frac{\hbar k^2}{2m}}\right\}. \tag{2·15}$$

Putting the sum of the lowest order commutator arising from H_I and the zero-order term from $H_{\text{coll}}-\sum_{k<k_c}\frac{\hbar\tilde{\omega}_k}{2}(a_k^*a_k+a_ka_k^*)$ to be zero, we get the following dispersion relation:

$$1=\frac{4\pi k^2 g(k)}{m}\sum_i\frac{1}{(\tilde{\omega}_k-\boldsymbol{k}\cdot\boldsymbol{P}_i/m)^2-\hbar^2k^4/4m^2}. \tag{2·16}$$

Thus the lowest order terms of our canonically transformed Hamiltonian may be expressed as follows:

$$H_{\text{new}}^{(0)}=H_{\text{part}}+H_{\text{coll}}+H_{\text{r.p.}}+H_{\text{s.r.}}, \tag{2·17}$$

$$H_{\text{part}}=\sum_i\frac{\boldsymbol{P}_i^2}{2m}-\frac{1}{m^2}\sum_{i,k<k_c}2\pi k^2 g(k)\frac{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{P}_i)^2}{\left(\tilde{\omega}_k-\frac{\hbar k^2}{2m}\right)^2-\left(\frac{\boldsymbol{k}\cdot\boldsymbol{P}_i}{m}\right)^2}-2\pi n\sum_{k<k_c} g(k), \tag{2·17a}$$

$$H_{\text{coll}}=\sum_{k<k_c}\frac{\hbar\tilde{\omega}_k}{2}(A_k^*A_k+A_kA_k^*), \tag{2·17b}$$

$$H_{\text{r.p.}}=-\frac{\pi}{m^2}\sum_{\substack{k<k_c\\ i,j,i\neq j}} k^2 g(k)\left\{\frac{\left[\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{P}_i-\frac{\hbar\boldsymbol{k}}{2}\right)\right]\left[\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{P}_j+\frac{\hbar\boldsymbol{k}}{2}\right)\right]}{\tilde{\omega}_k[\tilde{\omega}_k-\boldsymbol{k}\cdot\boldsymbol{P}_j/m-\hbar k^2/2m]}\exp[i\boldsymbol{k}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)]\right.$$

$$\left.+\exp[-i\boldsymbol{k}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)]\frac{\left[\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{P}_i-\frac{\hbar\boldsymbol{k}}{2}\right)\right]\left[\boldsymbol{\varepsilon}_k\cdot\left(\boldsymbol{P}_j+\frac{\hbar\boldsymbol{k}}{2}\right)\right]}{\tilde{\omega}_k[\tilde{\omega}_k-\boldsymbol{k}\cdot\boldsymbol{P}_j/m-\hbar k^2/2m]}\right\}, \qquad (2\cdot17\text{c})$$

$$H_{\text{s.r.}}=2\pi\sum_{\substack{i,j,k>k_c\\ i\neq j}} g(k)e^{i\boldsymbol{k}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)}, \qquad (2\cdot17\text{d})$$

and our new subsidiary conditions in the lowest order reduce to

$$(\Omega)_{\text{new}}\Phi\equiv\sum_i\frac{\tilde{\omega}_k^2}{\tilde{\omega}_k^2-[\boldsymbol{k}\cdot\boldsymbol{P}_i/m-\hbar k^2/2m]^2}\exp(i\boldsymbol{k}\cdot\boldsymbol{R}_i)\,\Phi=0\quad(k<k_c). \qquad (2\cdot18)$$

The measure of the success of eliminating H_I is the smallness of the expansion parameter

$$\alpha=\langle(\boldsymbol{k}\cdot\boldsymbol{P}_i/m\tilde{\omega}_k)^2\rangle_{AV}, \qquad (2\cdot19)$$

where the average is taken over the particle momenta and the collective field wave vectors. We shall see later that the order of α is less than 0.3. We note that the new subsidiary conditions do not involve the collective variables and our collective coordinates occur only in H_{coll} and thus describe a set of uncoupled field. The frequency of these collective oscillations of the field is given by the dispersion relations (2·16).

For sufficiently small k we expand (2·16) in powers of $\boldsymbol{k}\cdot\boldsymbol{P}_i/m\tilde{\omega}_k$ and $\hbar k^2/m\tilde{\omega}_k$ and obtain a solution $\tilde{\omega}_k$. If we do this and assume an isotropic distribution of $\boldsymbol{P}_i$, we find

$$\tilde{\omega}_k^2=\omega_k^2+\frac{1}{nm^2}\left(\frac{\omega_k}{\tilde{\omega}_k}\right)^2k^2\sum_i\boldsymbol{P}_i^2+\frac{\hbar^2}{4m^2}\left(\frac{\omega_k}{\tilde{\omega}_k}\right)^2k^4+\cdots. \qquad (2\cdot20)$$

In obtaining the mean value of $\sum_i\boldsymbol{P}_i^2$, we have to use the eigenfunctions appropriate to our new Hamiltonian. Now, we write the stationary state wave function as

$$\Phi=\phi_{\text{osc}}\cdot X(\boldsymbol{R}_1,\ \ldots,\ \boldsymbol{R}_n),$$

$$\phi_{\text{osc}}=\prod_{n_k}h_{n_k}(p_k)\exp[-(|p_k|^2/2\hbar\tilde{\omega}_k)], \qquad (2\cdot21)$$

where h_n is the n-th Hermite polynomial and X represents the eigenfunction for a set of particles interacting through $H_{\text{s.r.}}$. For the lowest state, we then get

$$\Phi_0=\phi_{\text{osc}}^{(0)}X_0=[\exp-\{\sum_{k<k_c}|p_k|^2/2\hbar\tilde{\omega}_k\}]\cdot X_0(\boldsymbol{R}_1,\ \ldots,\ \boldsymbol{R}_n), \qquad (2\cdot22)$$

where we approximate for X_0 the usual Slater determinantal wave function composed of plane waves appropriate to the ground state of the individual particles. $\sum_i\boldsymbol{P}_i^2$ in (2·20) may approximately be evaluated by assuming a Fermi distribution of particles at absolute zero; we then find

$$\tilde{\omega}_k^2=\omega_k^2+\frac{3}{5}\frac{P_0^2}{m^2}\left(\frac{\omega_k}{\tilde{\omega}_k}\right)^2k^2+\frac{\hbar^2}{4m^2}\left(\frac{\omega_k}{\tilde{\omega}_k}\right)^2k^4+\cdots, \tag{2·23}$$

where P_0 is the momentum of a particle at the top of the distribution. Expanding $g(k)$ in power of k,

$$g(k)=c_0+c_2k^2+\cdots, \tag{2·24}$$

and inserting this into $\tilde{\omega}_k$ in (2·23), we get the frequencies of collective modes in powers of k;

$$\tilde{\omega}_k=\tilde{c}_1k+\tilde{c}_3k^3+\cdots, \tag{2·25}$$

where, neglecting terms of order α^2, $\tilde{c}_1$, $\tilde{c}_3$,... are given by the following relations:

$$\begin{aligned}
&\tilde{c}_1=\left(\frac{4\pi n}{m}c_0\right)^{1/2}\left\{1+\frac{3}{5}\left(\frac{P_0}{m\tilde{c}_1}\right)^2\right\}^{1/2},\\
&\tilde{c}_3=\frac{2\pi nc_0}{m\tilde{c}_1}\left\{\frac{c_2}{c_0}\left(1+\frac{3}{5}\left(\frac{P_0}{m\tilde{c}_1}\right)^2\left(1-2\frac{c_0\tilde{c}_3}{c_2\tilde{c}_1}\right)\right)+\frac{1}{4}\left(\frac{\hbar}{m\tilde{c}_1}\right)^2\left(1+2\left(\frac{P_0}{m\tilde{c}_1}\right)^2\right)\right\},\\
&\cdots
\end{aligned} \tag{2·26}$$

$\tilde{c}_1$ is the parameter corresponding to the sound velocity at low temperatures and it is determined by two parts arising from potential and kinetic energies. Unfortunately, the measurement of the sound velocity of He^3 at very low temperature has not yet been made. Mikura[6] proposed a numerical value of $\tilde{c}_1$ of 1.36×10^4cm/sec, assuming an empirical relation of M. R. Rao.[7] Taking this value, our expansion parameter $\alpha=1/5\cdot(P_0/m\tilde{c}_1)^2\langle(1+(\tilde{c}_3/\tilde{c}_1)k^2+\cdots)^{-1}\rangle_{AV}$ turns out to be ~0.3. For an adequate value of parameter c_0 or $\tilde{c}_1$, which we shall find later by comparison between our theory and experiment, α becomes less than 0.3. Thus, in our case the separability of individual and collective motions is not bad.

The second term in H_{part}, which may be interpreted as the effect of the elimination of the particles-field interaction on the particles, becomes in the approximation of small α,

$$K_2=-\mathcal{K}\sum_i P_0^2/2m, \tag{2·27}$$

$$\begin{aligned}
&\mathcal{K}=\frac{2c_0k_0^3\beta^3}{9\pi m\tilde{c}_1^2}\left(A_0+\frac{3}{4}A_1k_0\beta+\frac{3}{5}A_2k_0^2\beta^2\right),\\
&A_0=1+\tfrac{1}{5}(P_0/m\tilde{c}_1)^2,\qquad A_1=(\hbar/m\tilde{c}_1)\{1+\tfrac{2}{5}(P_0/m\tilde{c}_1)^2\},\\
&A_2=\left(\frac{c_2}{c_0}-2\frac{\tilde{c}_3}{\tilde{c}_1}\right)+\frac{1}{5}\left(\frac{P_0}{m\tilde{c}_1}\right)^2\left(\frac{c_2}{c_0}-4\frac{\tilde{c}_3}{\tilde{c}_1}\right)+\frac{3}{4}\left(\frac{\hbar}{m\tilde{c}_1}\right)^2,
\end{aligned} \tag{2·28}$$

where $\beta=k_c/k_0$; k_0 is the wave vector of a particle at the top of the Fermi distribution. In deducing (2·27) we assumed an isotropic distribution of $\boldsymbol{P}_i$ and the Fermi distribution at absolute zero. Adding (2·27) to $\sum_i(P_i^2/2m)$, we obtain

$$K_{\text{red}}=(1-\lambda)\sum_i P_i^2/2m=\sum_i P_i^2/2m^*. \tag{2·29}$$

Individual particles behave as if they had an effective mass $m^*=m/(1-\lambda)$.

$H_{\text{r.p.}}$, in the approximation of small α, may be written as

$$H_{\text{r.p.}}=-2\pi\sum_{\substack{k<k_c\\ i,j(i\neq j)}} g(k)\frac{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{P}_i)(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{P}_j)/m^2\langle V^2\rangle_{AV}}{1+\tilde{c}_1^2\left(1+2\frac{\tilde{c}_0}{\tilde{c}_1}k^2+\cdots\right)/\langle V^2\rangle_{AV}}\exp[i\boldsymbol{k}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)],$$

where $\langle V^2\rangle_{AV}=\sum_i \boldsymbol{P}_i^2/m^2n$. From the fact that the summation is only for $k<k_c$ (later we will find $k_c\sim 0.4k_0$) and $1+\tilde{c}_1^2(1+(\tilde{c}_0/\tilde{c}_1)k^2+\cdots)/\langle V^2\rangle_{AV}\sim(1/3\alpha)+1\sim 3$, we can see that $H_{\text{r.p.}}$ describes very weak long range particle-particle interactions with the potential screened by the factor $\sim(1/3\alpha)$. Then we can safely disregard $H_{\text{r.p.}}$ in comparison with $H_{\text{s.r.}}$.

We shall now evaluate approximately the ground-state energy E_0 using the Hamiltonian $H=H_{\text{part}}+H_{\text{coll}}+H_{\text{s.r.}}$ and wave function $\Psi_0=\phi_{\text{osc}}^0\cdot X_0$. We treat $H_{\text{s.r.}}$ as a perturbation in our expression for H. The short range interaction energy $\langle H_{\text{s.r.}}\rangle=E_{\text{exch}}$ is given by

$$E_{\text{exch}}=-\frac{1}{2}\sum_{k_i}\sum_{\substack{k_j\neq k_i\\ k_i-k_j>k_c\\ |k_i|,|k_j|<k_0}} 4\pi g(|\boldsymbol{k}_i-\boldsymbol{k}_j|)=\frac{1}{(2\pi)^3}\int \epsilon_k^{\text{exch}}d\boldsymbol{k}\,, \tag{2·30}$$

where ϵ_k^{exch} is the exchange energy between a particle with a wave vector $\boldsymbol{k}$ and all other particles of parallel spin in the Fermi distribution. We evalute this as follows:

$$\epsilon_k^{\text{exch}}=\begin{cases}
-\frac{2}{3}\frac{c_0}{\pi}(k_0^3-k_c^3)-\frac{c_2}{\pi}\left\{\frac{2}{5}(k_0^5-k_c^5)+\frac{2}{3}k_0^2k^2\right\}\\
\qquad\qquad (0<k<k_0-k_c),\\
-\frac{c_0}{\pi}\left\{\frac{1}{8}(k_0^2-k_c^2)^2\frac{1}{k}+\frac{1}{3}(k_0^3-k_c^3)+\frac{1}{4}(k_0^2+k_c^2)k-\frac{1}{24}k^3\right\}\\
-\frac{c_2}{\pi}\left\{\frac{1}{24}(k_0^3+2k_c^3)(k_0^2-k_c^2)^2\frac{1}{k}+\frac{1}{5}(k_0^5-k_c^5)+\frac{1}{8}(3k_0^4+k_c^4)k\right.\\
\qquad\left.+\frac{1}{3}k_0^2k^2+\frac{1}{8}k_0^2k^3-\frac{1}{120}k^5\right\}\\
\qquad\qquad (k_0-k_c<k<k_0).
\end{cases} \tag{2·31}$$

We then calculate E_{exch} by summing ϵ_k^{exch} over all $\boldsymbol{k}$ within Fermi distribution. We find

$$E_{\text{exch}}=-\frac{c_0}{9\pi^3}k_0^6\left\{1-\beta^3+\frac{9}{16}\beta^4-\frac{1}{32}\beta^6\right\}$$
$$-\frac{2c_2}{15\pi^3}k_0^8\left\{1-\frac{5}{64}\beta+\frac{25}{64}\beta^2-\frac{45}{32}\beta^3+\frac{35}{32}\beta^4+\frac{23}{64}\beta^5-\frac{15}{64}\beta^6-\frac{5}{8}\beta^7+\frac{75}{256}\beta^8\right\}. \tag{2·32}$$

Then, the ground-state energy E_0 is given by

$$E_0=\frac{3}{5}n\left(\frac{P_0^2}{2m}\right)(1-\chi)-\pi n^2\beta^3\left(c_0+\frac{3}{5}c_2k_0^2\beta^2\right)+\frac{3\hbar n}{32\pi}\beta^4\left(\tilde{c}_1k_0+\frac{4}{6}\tilde{c}_3k_0^3\beta^2\right)$$

$$-\pi n^2c_0\left(1-\beta^3+\frac{9}{16}\beta^4-\frac{1}{32}\beta^6\right)$$

$$-\frac{6}{5}\pi n^2c_2k_0^2\left(1-\frac{5}{64}\beta+\frac{25}{64}\beta^2-\frac{45}{32}\beta^3+\frac{35}{32}\beta^4+\frac{23}{64}\beta^5-\frac{15}{64}\beta^6-\frac{5}{8}\beta^7+\frac{75}{256}\beta^8\right). \tag{2·33}$$

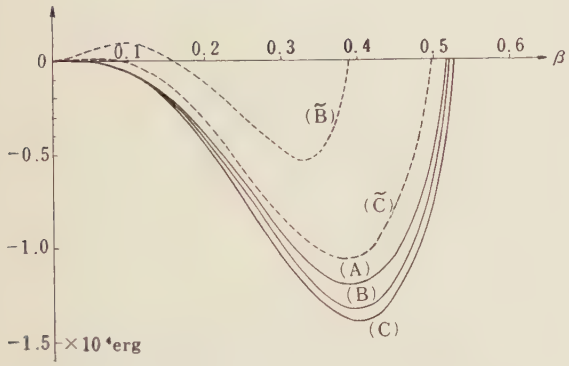

Fig. 3.

We now obtain $\beta_{\min}\equiv k_c/k_0$ for that value of β for which the ground-state energy E_0 is the minimum. In Fig. 3, we plot $E_0'=E_0-$ (the terms which are independent of β) as a function of β for several cases denoted by (A), (B) and (C) which correspond to the various values of parameter c_0 (or $\tilde{c}_1$) as illustrated in Table 1. We described the curves (A), (B) and (C) in the figure neglecting the dispersion terms. In later paragraph, the values of dispersion parameter c_2 are determined for each value of c_0 so as to coincide with the slope of $\chi T-vs-T$ curve at 0^0K. These values are tabulated in Table 1. By taking account of the dispersion terms, E_0'-vs-β curves (B) and (C) in the figure are replaced with dashed curves $(\tilde{B})$ and $(\tilde{C})$ and $\beta_{\min}$ are slightly displaced

Table 1.

Case	$c_0\times10^{-39}$	$\tilde{c}_1\times10^4$	γ_1	γ_2	$c_2\times10^{-56}$	$\tilde{c}_3\times10^{-13}$
(A)	2.34	1.36	0.53	0.14	9.86	7.80
(B)	3.12	1.50	0.71	0.07	5.02	6.03
(C)	3.72	1.60	0.84	0.02	1.66	5.02

toward smaller values. Thus we find that $\beta_{\min}$ takes a value of about $0.3\sim0.4$ in all our cases. Then the ratio of the collective degree of freedom to the total degree of freedom is

$$n'/3n=\tfrac{1}{6}\beta^3\simeq0.01.$$

§ 3. Nuclear paramagnetic susceptibility and specific heat in the collective description

a) It may be supposed that in the neighbourhood of 0°K the influence of the collective motion on χ or C_V is not remarkable considering the smallness of the number of collective modes. Nevertheless, in order to ascertain the degree of the efficiency of the collective motion for χ and C_V, we use the description given above in this paragraph. We first calculate the effect of particle-particle interactions on the nuclear paramagnetic susceptibility. As it is difficult to take into account the effect of our subsidiary condition on the exited states, our considerations are limited to $0^{\circ}K$.

The internal energy E_0^* per unit volume is expressed as follows as a function of $\overset{+}{n}$ and $\overset{-}{n}$, the number of spins per unit volume parallel and antiparallel to the external magnetic field H;

$$E_0^* = \frac{\hbar^2}{80m\pi^4}\{(6\pi^2\overset{+}{n})^{5/3}+(6\pi^2\overset{-}{n})^{5/3}\}\left\{1-\frac{2\pi c_0 n}{3m\tilde{c}_1^2}{}_1\beta^{*3}\left(A_0+\frac{3}{4}A_1 k_{0l}\beta^*\right)\right\}$$
$$-\frac{c_0 n}{3\pi}k_{0l}^3\beta^{*3}+\frac{\hbar\tilde{c}_1}{32\pi^3}k_{0l}^4\beta^{*4}-\frac{\pi c_0}{2}n^2\left\{1-{}_1\beta^{*3}+\frac{9}{16}{}_1\beta^{*4}-\frac{1}{32}{}_1\beta^{*6}\right\}$$
$$-\frac{\pi c_0}{2}\overset{-}{n}{}^2\left\{1-{}_{\bar{1}}\beta^{*3}+\frac{9}{16}{}_{\bar{1}}\beta^{*4}-\frac{1}{32}{}_{\bar{1}}\beta^{*6}\right\}-\mu(\overset{+}{n}-\overset{-}{n})\mathrm{H}, \qquad (3\cdot1)$$

here we neglect dispersion terms.

The first term is the reduced kinetic energy of the particles, the second is the self-energy, the third is the energy of collective oscillations, the fourth and fifth are the short range exchange energy and the last term is the Zeeman energy due to the presence of the magnetic field. Maximum wave vector is altered from k_c to k_c^* by a shift in spin populations. ${}_1\beta^*$, ${}_{\overset{+}{1}}\beta^*$ and ${}_{\bar{1}}\beta^*$ in (3·1) indicate respectively ${}_1\beta^*=k_c^*/k_0$, ${}_{\overset{+}{1}}\beta^*=k_c^*/\overset{+}{k}_0$, ${}_{\bar{1}}\beta^*=k_c^*/\overset{-}{k}_0$ where $\overset{+}{k}_0$, $\overset{-}{k}_0$ are respectively the wave vectors at the Fermi surfaces of the parallel and antiparallel spin systems.

If we put $\overset{+}{n}=\frac{n}{2}+p$, $\overset{-}{n}=\frac{n}{2}-p$, and express $\overset{\pm}{k}_0=k_0(1\pm 2p/n)^{1/3}$, ${}_{\overset{\pm}{1}}\beta^*={}_1\beta^*\left(1\pm\frac{2p}{n}\right)^{-1/3}$, we calculate E_0^* to order p^2 as

$$E_0^*=\frac{3}{5}n\frac{p_0^2}{2m}\left\{1-\frac{2\pi c_0 n}{3m\tilde{c}_1^2}{}_1\beta^{*3}\left(A_0+\frac{3}{4}A_1 k_{0l}\beta^*\right)\right\}-\frac{c_0 n}{3\pi}k_{0l}^3\beta^{*3}$$
$$+\frac{\hbar\tilde{c}_1}{32\pi^3}k_{0l}^4\beta^{*4}-\pi c_0 n^2\left(1-\beta^{*3}+\frac{9}{16}{}_1\beta^{*4}-\frac{1}{32}{}_1\beta^{*6}\right)$$
$$+\frac{\hbar^2}{2mk_0}\left\{1-\frac{2\pi c_0 n}{3m\tilde{c}_1^2}\beta^{*3}\left(A_0+\frac{3}{4}A_1 k_{0l}\beta^*\right)\right\}p^2$$
$$-4\pi c_0\left(1-\frac{1}{16}\beta^{*4}\right)p^2-2\mu p\mathrm{H}. \qquad (3\cdot2)$$

If we put $\beta^*=\beta+\delta$ in the conditional equation $\partial E_0^*/\partial\beta^*=f(p,\beta^*)=0$, and use the condition $f(0,\beta)=0$, we obtain

$$\delta=bp^2, \tag{3·3}$$

where

$$b=\frac{\beta}{n^2}\frac{\beta-\frac{4}{3}\left(\frac{P_0}{m\tilde{c}_1}\right)^2(A_0+A_1k_0\beta)}{\frac{6}{5}\left(\frac{P_0}{m\tilde{c}_1}\right)^2\left(A_0+\frac{3}{2}A_1k_0\beta\right)-\frac{9h\tilde{c}_1}{8\pi^2c_0n}k_0\beta+\left(\frac{27}{4}-\frac{15}{16}\beta^2\right)\beta}. \tag{3·4}$$

Then (3·2) is written as

$$E_0^*=E_0+gp^2-2\mu pH, \tag{3·5}$$

where

$$\begin{aligned} g=&\frac{k^2}{2mk_0}(1-\mathcal{K})\\ &-3\pi c_0bn^2\beta^2\left\{\frac{1}{5}\left(\frac{P_0}{m\tilde{c}_1}\right)^2(A_0+A_1k_0\beta)-\frac{h\tilde{c}_1k_0\beta}{8\pi^2nc_0}+\frac{3}{4}\beta-\frac{1}{16}\beta^3\right\}\\ &-4\pi c_0\left(1-\frac{1}{16}\beta^4\right). \end{aligned} \tag{3·6}$$

E_0^* then has a minimum at $p=\frac{\mu}{g}H$ and at this minimum

$$E_0^*=E_0-\tfrac{1}{2}\chi''H^2,\quad \chi''=2\mu^2/g,$$

χ'' being the susceptibility at 0°K. If we denote by χ_i^0 the susceptibility at 0°K in the case of ideal Fermi-Dirac gas, we obtain

$$\chi^0=\chi_i^0\frac{1}{1-\mathcal{K}-\mathcal{L}-\mathcal{M}}\simeq\chi_i^0\frac{1}{1-\gamma_1}, \tag{3·7}$$

where

$$\begin{aligned} \mathcal{L}=&\frac{6\pi mc_0k_0}{h^2}bn^2\beta^2\left\{\frac{1}{5}\left(\frac{P_0}{m\tilde{c}_1}\right)^2(A_0+A_1k_0\beta)-\frac{h\tilde{c}_1k_0\beta}{8\pi^2mc_0}+\frac{3}{4}\beta-\frac{1}{16}\beta^3\right\},\\ \mathcal{M}=&\frac{8\pi mc_0k_0}{h^2}\left(1-\frac{1}{16}\beta^4\right)\simeq\frac{8\pi mc_0k_0}{h^2}\equiv\gamma_1. \end{aligned} \tag{3·8}$$

$\mathcal{K}$ in the denominator of (3·7) represents the effect of the increase of the effective mass, $\mathcal{L}$ the effect of the fact that the energy varies by the shift of k_c due to the magnetic field, and $\mathcal{M}$ the effect of the short range exchange energy. In any one of the cases (A), (B) and (C) the order of the values of the collective terms $\mathcal{K}$ and $\mathcal{L}$ are $\mathcal{K}=0.01$, $\mathcal{L}\simeq 0.005\ \gamma_1$. It is therefore concluded that the collective motion gives only negligible effect on χ^0; the deviation of χ'' from that due to the simple free particle

theory arises mainly from the exchange contribution, so we can write as $\chi^0=\chi_f^0(1/1-\gamma_1)$.

b) We wish now to obtain an estimate of the specific heat of individual particles in the neighbourhood of 0°K. The accuracy of our estimate is limited, because in considering the specific heat we should properly take into account the effect of our subsidiary conditions on excited states of the individual particles. The effect is in the direction of reducing the effective degrees of freedom of the particles. However, since, as we have seen, $n'/3n\sim 0.01$, we might expect that the neglect of the subsidiary conditions would be a reasonably good approximation in the vicinity of 0°K. Then the specific heat of individual particles in the neighbourhood of 0°K can be written as

$$C_V^{\text{part}}=\frac{\pi^2K^2T}{3}\cdot\left(\frac{dn}{d\epsilon}\right)_{\epsilon=\varepsilon_0}=\frac{n\pi^2K^2T}{k_0}\frac{1}{(\partial\epsilon/\partial k)_{k=k_0}}, \tag{3·9}$$

where $(dn/d\epsilon)_{\epsilon=\varepsilon_0}$ is the density of energy levels at the top of the Fermi distribution and K Boltzmann's constant. Now, from (2·31) we find that the energy per particle near the top of the Fermi distribution is given by

$$\epsilon=\frac{\hbar^2k^2}{2m}-\frac{c_0}{8\pi}\left\{k_0^4(1-{}_f\beta^2)^2\frac{1}{k}+\frac{8}{3}k_0^3(1-{}_f\beta^3)+2k_0(1+{}_f\beta^2)k-\frac{1}{3}k^3\right\}. \tag{3·10}$$

Hence we obtain

$$C_V^{\text{part}}=\frac{n\pi^2K^2T}{2\varepsilon_0}\frac{1}{1-\dfrac{2\pi mc_0k_0}{\hbar^2}{}_f\beta^2\left(1-\dfrac{{}_f\beta^2}{4}\right)}\simeq C_V^f\frac{1}{1-\dfrac{{}_f\beta^2}{4}\gamma_1}, \tag{3·11}$$

where C_V^f is the specific heat of ideal Fermi gas. The effect of short range exchange energy is in the direction of increasing the specific heat but its contribution is of the order of few percent. If we compare (3·7) with (3·11), the exchange effect in C_V^{part} is very weak due to the factor ${}_f\beta^2$ appearing in the denominator of (3·11) in comparison with the case of χ^0. It seems that the experimental data on the C_V and χ support the above results qualitatively though the data do not at present pertain to the vicinity of 0°K.

Here we calculate the phonon part of the specific heat. The energy of collective oscillations is given by

$$E_{\text{osc}}=\int_0^{k_c}\left\{\frac{1}{2}\hbar\tilde{\omega}_k+\frac{\hbar\tilde{\omega}_k}{e^{\hbar\tilde{\omega}_k/KT}-1}\right\}\frac{k^2}{2\pi^2}dk, \tag{3·12}$$

and hence C_V^{osc} per mole at low temperatures is given by

$$C_V^{\text{osc}}=(N/n)\partial E/\partial T=96(NK/n)(K/\hbar\tilde{c}_1)^3T^3, \tag{3·13}$$

where N is Avogadro's number and we neglected the dispersion terms in $\tilde{\omega}_k$. We described in Fig. 3 the curves of C_V^{osc} corresponding to the cases (A), (B) and (C). It is noted that the value of $\tilde{c}_1$ was evaluated by assuming the Fermi distribution at 0°K for the kinetic part, so that our C_V^{osc} curves are approximate at finite temperatures.

4. Statistical mechanical consideration on the temperature dependence of χ and C_v

From the fact that the coupling between the individual and collective motions at 0°K is negligible, we assume that we can neglect the influence of the collective motion over a considerable range of low temperatures and apply the Fermi-Dirac statistics considering only the exchange energy, in order to explain the temperature dependence of χ and C_V.

a) In order to calculate the susceptibility we distinguish with signs + and − the two assemblies which have spins parallel and antiparallel to an external magnetic field. Further, to take account of the k-dependence of the exchange energy per particle ϵ_k^{exch}, we reserve the second term c_2k^2 in the expansion of $g(k)$. Then we have

$$\epsilon_k^{\text{exch}} = -4\pi \sum_{k'}^{(\pm)} g(|\boldsymbol{k}-\boldsymbol{k}'|) = -\frac{1}{\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int\left\{c_0+c_2\left(\frac{2m}{\hbar^2}\right)(\mathcal{E}+\mathcal{E}')\right\}\mathcal{E}'^{1/2}\overset{\pm}{f}(\mathcal{E}')\,d\mathcal{E}', \quad (4\cdot1)$$

where

$$\overset{\pm}{f}(\mathcal{E}') = \frac{1}{e^{(1/KT)(\mathcal{E}'+\overset{\pm}{\epsilon}_k^{\text{exch}}\mp\mu\mathrm{H}-\eta)}+1}. \quad (4\cdot2)$$

We assume the form:

$$\epsilon_k^{\text{exch}} = \overset{\pm}{\epsilon}_0^{\text{exch}} + \overset{\pm}{q}\mathcal{E}, \quad (4\cdot3)$$

where from (4·1)

$$\overset{\pm}{\epsilon}_0^{\text{exch}} = -\gamma_1(KT_0)^{-1/2}\left(\frac{KT}{1+\overset{\pm}{q}}\right)^{3/2}F_{1/2}(\overset{\pm}{\zeta}) - \gamma_2(KT_0)^{-3/2}\left(\frac{KT}{1+\overset{\pm}{q}}\right)^{5/2}F_{3/2}(\overset{\pm}{\zeta}), \quad (4\cdot4)$$

$$\overset{\pm}{q} = -\gamma_2(KT_0)^{-3/2}\left(\frac{KT}{1+\overset{\pm}{q}}\right)^{3/2}F_{1/2}(\overset{\pm}{\zeta}), \quad (4\cdot5)$$

with

$$F_n(\zeta) = \int_0^\infty \frac{x^n}{e^{x-\zeta}+1}dx, \quad \overset{\pm}{\zeta} = \frac{\overset{\pm}{\tilde{\varepsilon}}}{kT}, \quad \overset{\pm}{\tilde{\varepsilon}} = \eta \pm \mu\mathrm{H} - \overset{\pm}{\epsilon}_0^{\text{exch}}, \quad (4\cdot6)$$

$$\gamma_1 = \frac{1}{\pi}(2m/\hbar^2)^{3/2}c_0(KT_0)^{1/2} = 8\pi m c_0 k_0/h^2, \quad (4\cdot7)$$

$$\gamma_2 = \frac{1}{\pi}(2m/\hbar^2)^{5/2}c_2(KT_0)^{3/2} = 8\pi m c_2 k_0^3/h^2, \quad (4\cdot8)$$

$$T_0 = \mathcal{E}_0/K = \hbar^2k_i^2/2mK = \frac{\hbar^2}{2mK}(3\pi^2n)^{2/3} \simeq 5^0\mathrm{K}. \quad (4\cdot9)$$

Now, $\overset{\pm}{n}$ can be written as

$$\overset{\pm}{n} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int \mathcal{E}^{1/2}\overset{\pm}{f}(\mathcal{E})\,d\mathcal{E} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\frac{KT}{1+\overset{\pm}{q}}\right)^{3/2}F_{1/2}(\overset{\pm}{\zeta}). \quad (4\cdot10)$$

Therefore

$$\chi=\frac{\mu(\overset{+}{n}-\overset{-}{n})}{H}=\frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu^2(KT)^{1/2}\frac{1}{\left(\frac{\mu H}{KT}\right)}\left[\frac{F_{1/2}(\overset{+}{\zeta})}{(1+\overset{+}{q})^{3/2}}-\frac{F_{1/2}(\overset{-}{\zeta})}{(1+\overset{-}{q})^{3/2}}\right]. \tag{4·11}$$

On the other hand we have the condition:

$$n=\overset{+}{n}+\overset{-}{n}=\frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(KT)^{3/2}\left[\frac{F_{1/2}(\overset{+}{\zeta})}{(1+\overset{+}{q})^{3/2}}+\frac{F_{1/2}(\overset{-}{\zeta})}{(1+\overset{-}{q})^{3/2}}\right]=\frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(KT_0)^{3/2}. \tag{4·12}$$

From (4·11) and (4·12) we obtain

$$\frac{\chi T}{C}=\frac{3}{2}\left(\frac{T}{T_0}\right)^{3/2}\frac{1}{\left(\frac{\mu H}{KT}\right)}\left[\frac{F_{1/2}(\overset{+}{\zeta})}{(1+\overset{+}{q})^{3/2}}-\frac{F_{1/2}(\overset{-}{\zeta})}{(1+\overset{-}{q})^{3/2}}\right], \tag{4·13}$$

where $C=n\mu^2/K$.

Expanding (4·13) at low temperatures and taking account of the equations (4·4) and (4·5), we find

$$\frac{\chi T}{C}=\frac{3}{2}\left(\frac{T}{T_0}\right)\frac{1}{1-\gamma_1-\frac{8}{3}\gamma_2}\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_0}\right)^2\Phi(\gamma_1,\gamma_2)+\cdots\right], \tag{4·14}$$

$$\Phi(\gamma_1,\gamma_2)=\frac{1+\frac{7}{3}\gamma_2+\frac{17}{3}\gamma_2^2-\frac{50}{27}\gamma_2^3-\frac{500}{81}\gamma_2^4}{\left(1-\gamma_1-\frac{8}{3}\gamma_2\right)\left(1-\frac{2}{3}\gamma_2\right)^3\left(1-\frac{5}{3}\gamma_2\right)^2}. \tag{4·15}$$

We see that the best fit of our formula to the experiment is obtained when we take the values of our parameters to be $\gamma_1=0.22$ and $\gamma_2=0.25$. In this case we have $(1-\gamma_1-(8/3)\gamma_2)^{-1}\simeq 10$, $\Phi(\gamma_1,\gamma_2)\simeq 10^2$; so our formula (4·14) approximately coincides with that obtained by the simple particle theory of effective mass $10m$. The full curve of χT-*vs*-T can be got from (4·4), (4·5), (4·6), (4·12) and (4·13), using the numerical table of the Fermi functions $F_{1/2}$, $F_{3/2}$ calculated by McDougall and Stoner.[8] When we neglect the $\boldsymbol{k}$-dependence of $\epsilon_k^{\mathrm{exch}}$ and therefore put $\gamma_2=0$ (or $c_2=0$), as we show in Fig. 2, χT-*vs*-T curves go above the Curie line ($\chi T/C=1$). The effect of increasing the value of γ_2 is in the direction of increasing the effective mass, as we can see from (4·5), (4·3) and (4·2), so that the tendency of going above the Curie line is suppressed. The thick curve in Fig. 2 corresponds to the experiment due to Fairbank, Ard and Walters.

b) Next we calculate the temperature dependence of the specific heat. If we put likewise for $\epsilon_k^{\mathrm{exch}}$

$$\epsilon_k^{\text{exch}} = \epsilon_0^{\text{exch}} + q\mathcal{E}\,, \tag{4·16}$$

from (4·4) and (4·5), we have

$$\epsilon_0^{\text{exch}} = -\gamma_1 \mathcal{E}_0^{-1/2}\left(\frac{KT}{1+q}\right)^{3/2} F_{1/2}(\zeta) - \gamma_2 \mathcal{E}_0^{-3/2}\left(\frac{KT}{1+q}\right)^{5/2} F_{3/2}(\zeta), \tag{4·17}$$

$$q = -\gamma_2 \mathcal{E}_0^{-3/2}\left(\frac{KT}{1+q}\right)^{3/2} F_{1/2}(\zeta), \tag{4·18}$$

and have

$$E_{\text{kin}} = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)\int \frac{\mathcal{E}^{3/2}\, d\mathcal{E}}{e^{\frac{1}{KT}(\mathcal{E}-\epsilon_k^{\text{exch}}-\eta)}+1} = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\frac{KT}{1+q}\right)^{5/2} F_{3/2}(\zeta), \tag{4·19}$$

$$E_{\text{exch}} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int \frac{\mathcal{E}^{1/2}\epsilon_k^{\text{exch}}\, d\mathcal{E}}{e^{\frac{1}{KT}(\mathcal{E}+\epsilon_k^{\text{exch}}-\eta)}+1}$$

$$= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left\{\epsilon_0^{\text{exch}}\left(\frac{KT}{1+q}\right)^{3/2} F_{1/2}(\zeta) + q\left(\frac{KT}{1+q}\right)^{5/2} F_{3/2}(\zeta)\right\}, \tag{4·20}$$

where

$$\zeta = \tilde{\varepsilon}/KT = (\eta - \epsilon_0^{\text{exch}})/KT. \tag{4·21}$$

On the other hand, (4·10) becomes now

$$n = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\frac{KT}{1+q}\right)^{3/2} F_{1/2}(\zeta). \tag{4·22}$$

From (4·18) and (4·22) we see that q is independent of temperature and is given by $q_0 = -\frac{2}{3}\gamma_2$. From (4·19) and (4·20) and taking into account (4·17), (4·18) and (4·22), we obtain

$$E = E_{\text{kin}} + E_{\text{exch}} = \text{const} + \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(1-\frac{2}{3}\gamma_2\right)^{-3/2}(KT)^{5/2} F_{3/2}(\zeta). \tag{4·23}$$

By the expansion method at low temperatures we obtain

$$C_V = \frac{\partial E}{\partial T} = \frac{n\pi^2 K^2}{2\mathcal{E}_0}\left(1-\frac{2}{3}\gamma_2\right)^{-1} T\left\{1 - \frac{3\pi^2}{10(1-\frac{2}{3}\gamma_2)^2}\left(\frac{T}{T_0}\right)^2 + \cdots\right\}. \tag{4·24}$$

This formula coincides with what is obtained when we substitute $m/1-\frac{2}{3}\gamma_2$ for m in the simple theory. Here, if we take the value $\gamma_2 = 0.25$ as before, our C_V curve becomes the curve (d) in Fig. 1. It is seen that the calculated temperature dependences of χ and C_V agree with experiments at temperatures below 0.3°K, but the bending of the C_V curve found experimentally in the vicinity of 0.3°K is beyond our explanation.

In this situation not very much significance can be put on the value $\gamma_2 = 0.25$ obtained by fitting our full $\chi T - vs - T$ curve with the experiment (the value of parameter $\gamma_1 = 0.22$ corresponding to that for $\gamma_2 = 0.25$ gives too small a value for sound velocity). So we have rather determined the values of parameter c_2(or γ_2) below 0.3°K so as to agree

with the experimental result ; $(1-\gamma_1-\frac{8}{3}\gamma_2)^{-1}\simeq 10$, corresponding to the values of parameter c_0(or γ_1) which are supposed to lie in the reasonable range from Mikura's value of $\tilde{c}_1$.

§ 5. Discussion and the form of $V^*(r)$

It is supposed that the coupling between particles and phonons which we have neglected in our statistical consideration increases abruptly as the temperature exceeds about 0.3 K and has a considerable effect on χ and C_V, especially for the bending of C_V curve in the vicinity of 0.3°K. That the coupling has more remarkable effect on C_V than on χ seems to be because it directly affects C_V through the restriction of the degrees of freedom of the particles, while for χ it has two opposing effects, that is, the restriction of the degrees of freedom of the particles and the increase of the effective magnetic moment μ^* through the collective motion. For example, if we simply assume that ν particles move together, we have $n^*=n/\nu<n$, $\mu^*=\nu\mu>\mu$.

Since the subsidiary conditions are satisfied automatically at 0°K, we need not consider the restriction of the degrees of freedom of the particles provided that we do the effective mass correction. Then the system follows the simple Fermi statistics for n particles which have the effective mass $m^*\simeq m$. For excited states, however, we must reduce the number of possible eigenfunctions to satisfy the subsidiary conditions. This corresponds to the fact that the coupling between particles and phonons reduces the

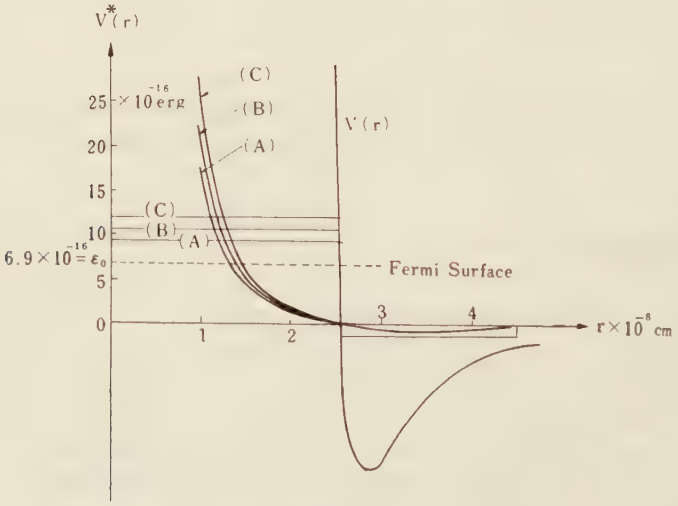

Fig. 4.

the effective degrees of freedom of individual particles. Then our Fermi statistics for individual particles collapses, and when the temperature rises sufficiently high, the system follows the Boltzmann statistics without distinction of the original character of statistics. It seems that the experimental C_V curve follows this consideration.

Finally we consider the potential $V^*(r)$ which corresponds to the above derived

values of our parameters. As shown in Table 1, we chose several sets of values of the parameters (c_0, c_2) or (r_1, r_2) which explain the experimental χ^0, corresponding to the cases (A), (B) and (C). We described in Fig. 4 the two groups of potential curve of $V^*(r)$. For one example of $V^*(r)$ we took the following type of potential:

$$V^*(r) = \alpha e^{-\lambda r}/r - \beta e^{-\mu r}/r$$

and described in Fig. 4 curves for $V^*(r)$ corresponding to several sets of values of

$$c_0 = \frac{\alpha}{\lambda^2} - \frac{\beta}{\mu^2}, \qquad c_2 = -\frac{\alpha}{\lambda^4} + \frac{\beta}{\mu^4}.$$

We made the zero point of $V^*(r)$ coincide with that of de Boer-Michels or Margenau potential $V(r)$, i.e.,

$$\alpha e^{-\lambda\sigma} = \beta e^{-\mu\sigma}\ ;\ \sigma = 2.56 \times 10^{-8}\ \text{cm}$$

and took for example $\lambda = 1.50 \times 10^8$. For another example of $V^*(r)$, we took the following one:

$$V^*(r) = \begin{cases} V_0 & (0 < r < \sigma)\ ;\ \sigma = 2.56 \times 10^{-8}\ \text{cm} \\ -V_1 & (\sigma < r < s)\ ;\ s = 4.50 \times 10^{-8}\ \text{cm}, \end{cases}$$

and showed in Fig. 4 the $V^*(r)$ curves corresponding to our sets of parameters. It is noted that in every case V_0 has the value of about two times the Fermi energy, $\varepsilon_0 = 6.9 \times 10^{-16}$ erg. The values of the parameters in case (A) give Mikura's value for sound velocity.

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Letters to the Editor

Antiproton-Nucleus and Antiproton-Proton Interactions

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Recent experiments carried out at Berkley[1] show that the attenuation cross sections σ_{att} of antiproton beams by Cu and Be are about twice those of protons.[1] Duerr and Teller,[2] to explain these large cross sections, took a nuclear model proposed by Johnson and Teller,[3] but the depth of potential used there seems to be too large.

In this note, as an alternative approach, we consider the experimental results on the basis of a simple optical model.[4] Taylor investigated the fast neutron scattering on this model[5] and took the nuclear radius $R=1.45\times10^{-13}A^{1/3}$ cm for Cu and $1.54\times10^{-13}A^{1/3}$ cm for Be. Absorption coefficient K and mean potential V which determines a refractive index are energy dependent and for about 400 Mev neutrons $K\sim0.2\times10^{13}\text{cm}^{-1}$, $V\sim10$ Mev and $k_1\sim0.04\times10^{13}\text{cm}^{-1}$, where $k_1=k[(1+V/E)^{1/2}-1]$ and k is the incident wave number outside the nucleus.

Let us take the interaction radius $R=1.5\times10^{-13}A^{1/3}$cm according to Taylor's analysis or fast neutron scattering and another parameters K and k_1 are chosen to fit the experimental results. For Cu, by the choice of $KR=3$ and $k_1/K=0.7$ (this corresponds to $V=66$ Mev), we have absorption cross section $\sigma_{ab}=0.95\pi R^2=1.085$ b and diffraction cross section $\sigma_d=1.290$ b.

After the numerical calculation the forward component ($0^\circ\sim12.7^\circ$) of total diffraction cross section comes to be about 87%, then σ_{att} turns out to be about $\sigma_{ab}+\sigma_d\times0.13=1.256$ b. This result is in fair agreement with experimental one in spite of such a simple model has been adopted. The further agreement may be obtained by two ways: one is to find a method to raise the backward component. If the backward scattering ($12.7^\circ\sim180^\circ$) increases thereby to, say 26% of total diffraction scattering, σ_{att} will be 1.43 b and the situation will be improved. Another explanation may be given by the modification of the interaction radius, that is, $R=1.6\times10^{-13}A^{1/3}$cm, $KR=3$ and $k_1/K=0.7$, then

$\sigma_{ab}=1.234$ b, $\sigma_d\times0.13=0.189$ b (the backward scattering $\theta=12.7^\circ\sim180^\circ$)

$\sigma_{att}=\sigma_{ab}+0.13\times\sigma_d=1.423$ b.

It seems to be difficult to understand the rather large interaction radius, but $\bar{p}$-nucleus potential is deeper than that of p-nucleus, so that the tail of the nuclear density is more effective and consequently the interaction radius of $\bar{p}$ becomes slightly larger than that of nucleon.

For $\bar{p}$-Be interactions, the same K and k_1 as those for $\bar{p}$-Cu are used and the results are as follows,

$R=1.5\times10^{-13}A^{1/3}$cm, $KR=1.55$,

$k_1/K=0.7$

$\sigma_{ab}=0.82\pi R^2=0.251$ b, $\sigma_d=0.294$ b

$0.15\times\sigma_d=0.046$ b

(the backward scattering $\theta=18^\circ\sim180^\circ$)

$\sigma_{att}=0.291$ b.

The same situation as in the case of $\bar{p}$-Cu interaction occurs here. Therefore if either backward scattering increases to 26% of total diffraction scattering or we take $R=1.6\times10^{-13}A^{1/3}$cm, a better agreement with the experiment can be obtained.

For comparison, the proton-nucleus interactions are calculated on the same model and the results are listed as follows,

$R=1.5\times10^{-13}A^{1/3}$cm, $K=0.2\times10^{13}$cm^{-1}

and $k_1=0.04\times10^{13}$cm^{-1}

p-Cu $\sigma_{att}=0.86$ b (cut off angle 12.7°)

p-Be $\sigma_{att}=0.162$ b (cut off angle 18°).

These results are consistent with the experiment.[1]

From the analysis of $\bar{p}$-nucleus interactions, we can estimate the interaction cross section of the elementary process. The relation between $\bar{p}-p$ interaction cross section σ in the nucleus and the absorption coefficient K is $K=(3A/4\pi R^3)\sigma$. Using this formula with $R=1.5\times10^{-13}A^{1/3}$cm and $K=0.5\times10^{-13}$cm^{-1}, we obtain that

$$\sigma=\sigma_{sc}+\sigma_{an}\sim70 \text{ mb}.$$

In the case of $\bar{p}$-Be, the annihilation cross section is 0.17 ± 0.06 b and the calculated absorption cross section is 0.251 b, therefore the ratio $\sigma_{an}/\sigma_{inel.sc}=\alpha$ has the values between 0.8 and 10 and the mean value of α is about 2. We are able to calculate σ_{an} and σ_{sc} (in nucleus) from the following two relations,

$\alpha=(P_a+0.5P_sP_a)/(0.5P_s+0.5P_sP_s)$ where

$P_a=\sigma_{an}/\sigma$ and $P_s=\sigma_{sc}/\sigma$

$\sigma=\sigma_{an}+\sigma_{sc}=70$ mb.

Because of the effect of the Pauli principle, the scattering cross section of antiproton by a nucleon in nucleus must be smaller than that by a free nucleon, and then we have σ_{sc} (in nucleus) $=2/3\ \sigma_{sc}$(free)[6]. Thus the total cross sections and the partial cross sections for the various values of α are given below;

α	$\sigma_{sc(free)}$	σ_{an}	$\sigma_{tot(free)}$
0.8	74 mb	21 mb	94 mb
2	48	38	86
4	33	48	81
10	7	65	72

According to these results, we may calculate that σ_{tot} of $\bar{p}-p$ interaction is $100\sim70$ mb and very large compared to that of $p-p$ interaction at energy about 470 Mev. This is consistent with some preliminary attenuation cross section measurements of $\bar{p}$ beams by hydrogen performed by Berkeley group.[7]

For example, if we suppose that a nucleon is a sphere with radius $R\sim\hbar/m_\pi c$ of material characterized by a complex refractive index, the absorption coefficient K is $0.6\sim0.8\times10^{13}$cm^{-1} and the mean depth of potential is $120\sim85$ Mev for $\alpha=2\sim4$. Although this simple model is very crude and may give no definite conclusions, it may be expected that the interaction of antiproton-proton is very different from the case of proton-proton.

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Remarks on the Elastic Scattering of Deuterons by Heavy Nuclei

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Recently Gove[1] measured the angular distributions of 15.2 Mev deuterons elastically scattered from heavy nuclei such as Pb and Bi. The main character of the data is the approximately exponential decrease of the cross section from that of the Coulomb scattering, beginning at a specific angle near 30°.

Porter[2] attempted to explain the deuteron data by assuming that the main effect which lowers the cross section below that of the Coulomb scattering is absorption of the deuterons from the incident beam. A similar theory was successfully applied to the elastic scattering of the alpha-particles by heavy nuclei. Unfortunately, however, the best fit for the deuteron data led to a poor result that the mean free path of the deuteron near the center of the nucleus is much longer than that of the alpha-particle. The situation will become worse if the effect of electric break-up of the deuteron is taken into account. A question arises as to the cause of this curious result. Now it will be essential to investigate the difference between the mechanisms of the anomalous Coulomb scattering of the deuteron and the alpha-particle. The aim of this preliminary note is to point out that the above curious result may disappear by introducing the loosely bound character of the deuteron into the nuclear absorption, and that the critical angle at which the deviation from the Coulomb scattering begins, can be related to the electric break-up of the deuteron. The reduced wave-length of the deuteron is much shorter than the nuclear dimensions, and $Ze^2/\hbar V_d$ (where Ze is the nuclear charge and V_d is the deuteron's velocity) is fairly larger than unity. Thus the center-of-mass motion of the deuteron can be described approximately by the classical Coulomb orbit. In the nuclei under investigation, the apsidal distance corresponding to the largest angle of observation exceeds the nuclear radius. Thus the decrease of the deuteron intensity by the nuclear absorption could be understood as a result of either neutron or proton in the deuteron being absorbed when the deuteron grazes the nucleus. Now we neglect the effects of nuclear transparency. Therefore, we may picture the target nucleus as a black sphere for the nucleon. In addition, the period of the internal motion of the deuteron requires to graze the nucleus. Hence it has an approximate meaning to speak of the instantaneous position of the nucleon during the encounter with the nucleus. On the contrary, the period of the internal motion of the alpha-particle is fairly short compared with the time which is required

by the alpha-particle to pass through the distance equal to the nuclear diameter. Thus there is good reason to discuss the nuclear absorption of the alpha-particle, as the Porter theory does, using its radius or the interaction radius. Now it will be understood that the introduction of such an "average quantity" as the deuteron radius or its interaction radius is very dangerous in the discussion of the nuclear absorption of the deuteron. Here we may use Glauber's procedure[3] to obtain the transmission coefficient with which the deuteron is capable of passing by without touching the nucleus. This angular dependence is shown by (a) or (a′) in Fig. 1.

These curves may be considered to show a satisfactory trend because there exists the possibility that the effect of electric break-up of the deuteron may lower the cross section beyond the nuclear effect. However, they are surely incapable of explaining why the cross section shows the sudden decrease from that of the Coulomb scattering at smaller angles. Now there seems to be strong evidence for attributing this to the electric break-up of the deuteron.

When the deuteron (mass $2M$) passes by the nucleus of charge $\mathcal{Z}e$ with apsidal distance q and velocity V_a, it gains the energy of approximately $\mathcal{Z}^2e^4/MV_a^2q^2$ by the Coulomb field of the nucleus. This energy will be transferred to the relative motion of the neutron and the proton in the deuteron. Thus it is necessary for this energy to exceed the binding energy of the deuteron in order that its electric break-up may be possible. This gives the critical angle of approximately 27.4° consistent with the experimental data. This angle corresponds to the apsidal distance of approximately 20.3×10^{-13}cm which is fairly longer than the radius of nuclear interaction. The close connection between the critical angle and the character of the deuteron will establish the difference between the mechanisms of the anomalous Coulomb scattering of the deuteron and the alpha-particle. It might be mentioned that a tentative calculation of the angular distributions including the electric break-up of the deuteron was tried. The cross section $\sigma(\theta)$ of the anomalous Coulomb scattering will be "classically" given by the expression: $\sigma(\theta)=T(\theta)\cdot[\sigma_c(\theta)-\sigma_b(\theta)]$. Here θ is the scattering angle, $T(\theta)$ the transmission coefficient due to the nuclear absorption, $\sigma_c(\theta)$ the cross section of the Coulomb scattering, and $\sigma_b(\theta)$ the cross section of the electric break-up of the deuteron along the Coulomb orbit. $\sigma_b(\theta)$

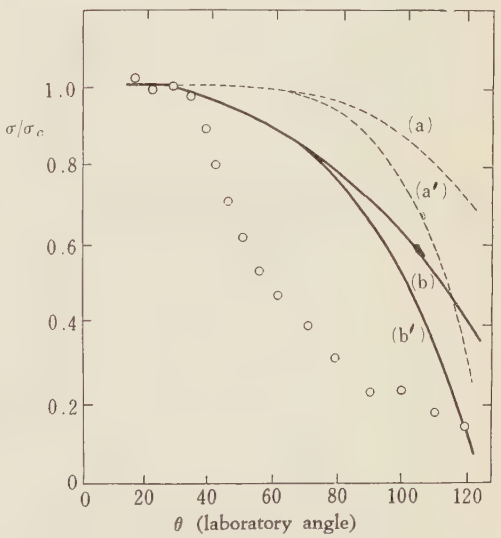

Fig. 1. Angular distributions of deuterons elastically scattered by Pb^{208}. ○○○ experimental points; (a) (a′)-------- transmission coefficients due to nuclear absorption; (b) (b′)——— required theoretical curves. (a) (b) correspond to the nuclear radius $R=1.33\times A^{1/3}\times10^{-13}$cm, and (a′) (b′) to $R=1.40\times A^{1/3}\times10^{-13}$cm.

was estimated by the semi-classical method of the virtual quanta extended to the non-relativistic case. The final result of σ/σ_c is shown by (b) or (b′) in Fig. 1.

Unfortunately there exists a considerable discrepancy in the slope of the deviation from the Coulomb scattering at smaller angles where the nuclear effect is negligible. This seems to show that in the places where the nucleon waves appear by the electric break-up of the deuteron, the classical approach of the scattering is not satisfactory, even if $Ze^2/\hbar V_d$ is fairly larger than unity. The distortion of the Coulomb wave of the deuteron due to the production of the nucleon waves will strongly contribute to the angular distributions at larger angles than the critical one. The wave mechanical approach is required there, and the curves (b) (b′) must not be taken too seriously.

The author is grateful to Professor M. Kobayasi of Kyoto University for many valuable discussions.

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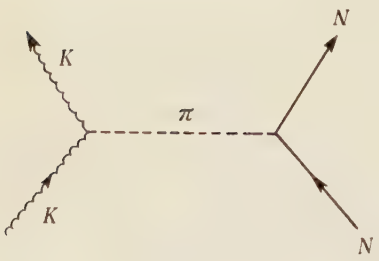

Fig 1

K-π Interaction and K^+-N Scattering

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December 3, 1956

The recent experiments on K^+-N scattering[1)2)] have shown the following results:

(i) $\sigma(K^+ + p) > \sigma(K^+ + n)$,

(ii) the K^+-mesons after scattering have practically the same branching ratio of decaying into various modes as the K^+-mesons before scattering (i.e. the incident K^+-beam)[3)4)],

(iii) the angular distribution for K^+-p scattering is larger in the forward than in the backward direction.

On the other hand, Schwinger[5)] has introduced the K-π interaction in order to explain the masses of the strange particles and has adopted the following values of coupling constants:

$$g^2_{Y\pi}/4\pi \sim 10, \qquad g^2_{K\pi}/4\pi \sim 1/10,$$
$$g^2_{KN}/4\pi \sim 1/10. \qquad (1)$$

The purpose of this paper is to examine the contributions of this K-π interaction to K^+-N scattering.

First of all let us pay our attention to

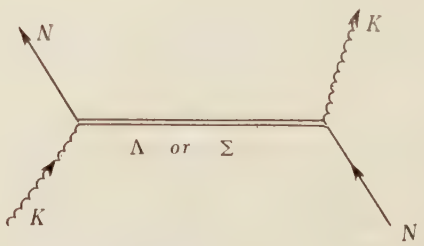

Fig 2

the ratio of the cross sections for the scatterings $K^+ + n \to K^+ + n$ and $\to K^0 + p$, $K^+ + p \to K^+ + p$. Although there are two kinds of diagrams (Fig. 1 and Fig. 2) corresponding to the lowest order perturbation, it would be sufficient to carry on our discussion by taking into account only the diagram shown in Fig. 1, if the values of coupling constants (1) are adopted.

So long as the charge independent K-π interaction

$$g_{K\pi}\phi_K^*\tau_\alpha \boldsymbol{\phi}_K \phi_\alpha + h.c. \qquad (2)^*$$

is taken into consideration, the perturbation gives the ratio

$$\sigma(K^+ + p \to K^+ + p) : \sigma(K^+ + n \to K^+ + n) : \sigma(K^+ + n \to K^0 + p) = 1 : 1 : 4, \qquad (3)$$

where ϕ_K and $\boldsymbol{\phi}_K$ are the two kinds of K-meson fields which differ in parities corresponding to the Lee-Yang model[6] and ϕ_α the pion field. To our regret, this result contradicts with the experimental one (i).

Then we will try another approach to this reaction. Since some virtual pion takes part in this process, it will be necessary to use our knowledge of π-N interaction which has been obtained by the study of pion reactions. For K-N scattering, we think it appropriate to regard the K-π interaction as a perturbation after the π-N interaction has been dealt with by means of the Chew-Low formalism[7] for instance. As such a simple approximation, let us try to describe preliminarily the following model for scattering: firstly a virtual pion is produced by $K-\pi$ interaction, secondly this pion is scattered by nucleon and finally this scattered pion is reabsorbed by K-meson. Fig. 3 and Fig. 4 are the diagrams corresponding to such scattering processes.

First of all the ratio of the cross sections for K^+-N scattering is re-examined under the crude assumption that the so-called $I=3/2$, $J=3/2$-state in pion-nucleon interaction plays also a predominant role in our process as in the cases of π-N scattering and photo-meson production. Then we can easily obtain the following result:

$$\sigma(K^+ + p \to K^+ + p) : \sigma(K^+ + n \to K^+ + n) : \sigma(K^+ + n \to K + p) = 4 : 1 : 1. \qquad (4)$$

* In this paper we adopt such a representation as: $\tau_3\phi_{K^-}{}^+ = \phi_{K^-}{}^+$, $\tau_3\boldsymbol{\phi}_{K^-}{}^+ = \boldsymbol{\phi}_{K^-}{}^+$.

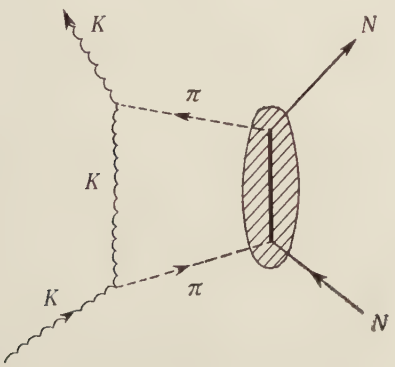

Fig. 3.

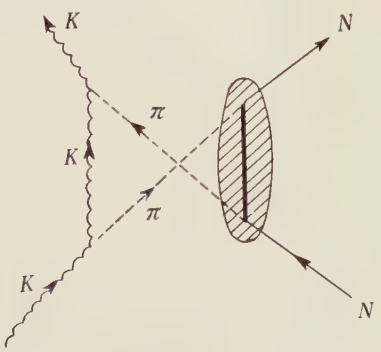

Fig. 4.

This values is consistent with the experimental result (i). So far as our model is concerned, the scattered K-meson has the same parity as that of the incident K-meson. Thus it may be said that our description of scattering is suitable to explain the experimental result (ii) with respect to the decay modes of K^+-meson.

Now let us discuss about the angular distribution. In the scattering process corresponding to Fig. 2, K-meson interacts directly with nucleon, so the theory with the conventional interaction Hamiltonian will predict the predominance of s-wave scattering in the energy region (up to 120 Mev), of which the experiments are now carried on. On the other hand, in the scattering processes corresponding to Fig. 1, Fig. 3 and Fig. 4, K-meson interacts with nucleon through the virtual pions only, so the impact parameter in the collision will be considerably large and the momentum transfer to the nucleon will be relatively small. Thus it may be expected that the angular distribution for K^+-p scattering is larger in the forward than in the backward direction.

The author would like to express his thanks to Professor G. Takeda for his kind information and also to Professor Y. Yamaguchi for his discussions.

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Quantum Numbers for Lepton Processes

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According to the experiments and the theoretical analyses there seem to exist remarkable regularities in the lepton phenomena. For instance, leptons appear partly through the "universality" of fermi interactions[1] and weak boson-fermion interactions[2]. To explain these facts many speculations have been reported, such as the introduction of the intrinsic parity[3] and of the lepton number[4], and the rule that "all lepton processes always accompany at least one neutrino"[5] is a more primitive one.

The aim of this note is, by using several assumptions, to arrange these features as follows:

A) The concept of the strong interactions[6] in baryon and meson system remains also for lepton processes.

We consider tentatively that all lepton, baryon and meson processes which are transmuted to each other by some strong interactions may have the similar interaction strengths; that is, $n \to p + e^- + \tilde{\nu}$ and $\pi^- \to e^- + \tilde{\nu}$ are equivalent for interaction strength (via $NN\pi$), and $K^+ \to \pi^0 + e^+ + \nu$ and $K^+ \to e^+ + \nu$ are same (via $KK\pi$). Both the latters of the above examples are usually not observed, but these circumstances should be explained by a more detailed theory.[7] By virtue of this assumption, all wanted and unwanted lepton processes can be classified into several groups, i.e. $\pi \to \mu + \nu$ group,

$\pi \to e + \nu$ group and so forth.

B) Neutrino is a Dirac particle, that is neutino and anti-neutrino have different properties.

C) All lepton processes conserve the lepton number N_l according to Konopinski-Mahmoud's assignment.

Assumption B) will be clarified in near future. As is known, C) is not a sufficient condition to rule out all unwanted processes, such as $K^+ \to \pi^- + \mu^+ + e^+$. Then, we need some other new quantum numbers.

D) Leptons have at least two new-quantum numbers, L_+ and L_- which change their signs by particle-antiparticle transformation.

The new quantum numbers here introduced are not identified with the strangeness s for baryons and mesons. In order to make it identified with it, we are obliged to assign the same strangeness[5] to all leptons or to introduce the three kinds of neutrinos.[8]

	N_l	L_+	L_-
μ^+	1	1	1
ν	1	0	1
e^-	1	0	0

One type of the quantum number assignment for leptons is listed in the table. Using them, we can deduce the electric charge number Q for leptons and the neutrino number n as follows:

$$Q = (L_+ + L_-) - N_l ,$$

$$n = -(L_+ - L_-) .$$

If we consider that all baryons and mesons have $L_+ = L_- = N_l = 0$ eigenvalues, while all leptons have $I = N_h$ (baryon number) $= s = 0$ eigenvalues, all elementary particles are classified under six quantum numbers. Thus we establish the selection rules for all elementary processes.

E) *The interactions of all elementary particles are characterised by*

$$\max(|\Delta s|, |\Delta L_+|, |\Delta L_-|, |\Delta L + \Delta L_-|),$$

and corresponding to its values 0, 1, 2..., we can classify them into the strong (including electromagnetic) interaction, weak interaction and very slow or unobservable interactions respectively.

According to E), all observed (principally allowed) lepton processes together with deacy interactions of baryons and mesons are classified under the same group and unwanted or unobserved processes can be ruled out.

However, there appear new strong interactions such as $\pi^0 \to e^+ + e^-$, but some of these processes are realized in principle by strong-electromagnetic interactions. The neutrino pair processes such as $\pi^0 \to \nu + \tilde{\nu}$ for strong interaction and $K \to \pi + \nu + \tilde{\nu}$ for weak interaction are not ruled out by this model, and must be removed by considering details of the theory or specifying the interaction types.

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He-He Repulsive Potential

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The problem of repulsive interaction energy between two normal helium atoms is of fundamental importance in the theory of molecular systems and yet there exist grave discrepancies between the latest experimental estimation[1] and the latest theoretical calculation[2]. In this connection we report here a simple calculation which yields rather drastically improved values of the repulsive interaction energy. It is nothing but a slight modification of the calculation of Griffing and Wehner[3]. They adopted simple Slater-type 1s-functions, χ_a and χ_b, as AO's and constructed two LCAO MO-s:

$$\varphi_1 \sim \chi_a + \chi_b \,;\, \sigma_g \,,$$

$$\varphi_2 \sim \chi_a - \chi_b \,;\, \sigma_u \,,$$

where

$$\chi_a = (Z^3/\pi)^{1/2} \exp(-Z r_a) \,,$$

and similarly for χ_b with $Z = 27/16$. Now let us try to generalize slightly their scheme of calculation by taking,

$$\varphi_1 \sim \chi_a + \chi_b \,;\, \sigma_g \,,$$

$$\varphi_2 \sim \chi_a' - \chi_b' \,;\, \sigma_u \,,$$

where

$$\chi_a' = (Z'^3/\pi)^{1/2} \exp(-Z' r_a) \,.$$

The point is that we take $Z \neq Z'$. This idea was suggested by a united-atom treatment of the He-He repulsive interaction[4]. Mathematically, this generalization is rather trivial but it bears some essential physical meanings. In short, φ_1 and φ_2 must reduce to $1s$ and $2p$ atomic orbitals, respectively, of Be in the united-atom limit and in order to cope with this physical requirement we should, at least, drop the severe restriction of $Z = Z'$. The calculation involved in the case of $Z \neq Z'$ is not tedious and can be carried out with reasonable labor. Actual calculations were performed at three values of the internuclear distance R; 1.0, 1.5 and 2.0 a.u. and the results are summarized in Table 1. E is

Table 1.

R(a.u.)	1.0	1.5	2.0
Z	2.25	2.0	1.75
Z'	1.25	2/1.5	1.5
E(a.u.)	−4.70584	−5.32781	−5.58131
$V(R)$	0.98947 a.u. (26.92 e.V.)	0.36750 a.u. (10.0 e.V.)	0.11400 a.u. (3.10 e.V.)

the total energy of the He-He system and $V(R)$ is the repulsive potential energy which is defined by $E(R) - E(\infty)$. In the above, we took $E(\infty) = -5.69531$ a.u. in accordance with Sakamoto-Ishiguro[2] and Griffing-Wehner[3]. We performed numerical calculations within the direct availability of KAIK's table[5] and this necessarily imposed restrictions on the available values of Z and Z'. The result of the present calculation may conveniently be expressed by the following approximate formula:

$$V(R) = 8.58817 \exp(-2.16097\, R) \quad \text{a.u.} \quad (R \text{ in a.u.}) \qquad (1)$$

This is compared with the corresponding formula of Sakamoto-Ishiguro[2]:

$$V(R)=15.02222\exp(-2.36478R)$$
a.u. (R in a.u.) (2)

Fig 1 shows the potential curves due to (1) and (2), together with the experimental result of Amdur and Harkness[1]. At $R=1.0$ the difference between I and II amounts to about 12 e.V. but still the experimental curve lies far below. We are attempting further improvements of theoretical calculations and the details will be reported in the near future.

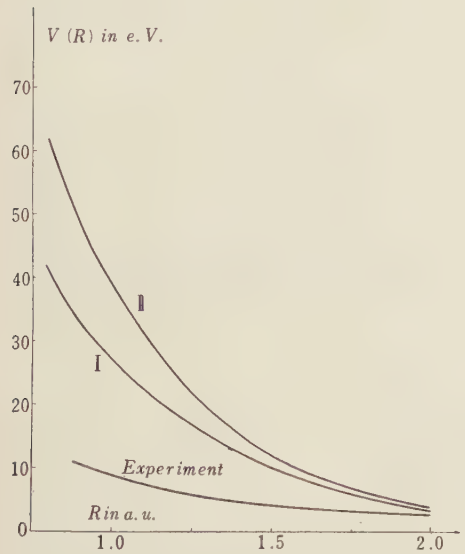

Fig. 1 He-He repulsive potential

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Equivalence between the Line Width and the Rate of Energy Relaxation in the Inversion Spectrum of Ammonia

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December 25, 1956

Based on the observation of the threshold saturation, Townes[1] argued that the rate of energy relaxation in the inversion spectrum of ammonia is of the same order of magnitude with the line width due to intermolecular collision. Bleaney and Penrose[2] claimed that $\Phi_2/\Phi_1=1.75$, where Φ_1 is the rate of energy relaxation and Φ_2 is the line width in angular frequency unit. Reanalysing the data of Bleaney and Penrose, Karplus[3] arrived at a conclusion that the same collisions interrupt the radiation process and restore the thermal equilibrium. Here we give a direct proof that the rate of energy relaxation should be identical with the line width under the experimental condition.

We make use of the relaxation function method which was originally developed for descriving magnetic resonance (Kubo and Tomita)[4]. According to this method the frequency spectrum $I(\omega)$ is given by

$$I(\omega)=\mathcal{R}m\int_0^\infty dt\,e^{-i\omega t}\phi(t) \qquad (1)$$

where $\phi(t)$ is the relaxation function of the electric dipole moment of the ammonia molecule characterized by the intermolecular collision. If we consider the resonance at frequency ω_α,

$\phi_\alpha(t)$ is written in the form

$$\phi_\alpha(t)=\exp\{i\omega_\alpha t-\psi_\alpha(t)\}\ ;\ \alpha=(J, K). \tag{2}$$

When we develop $\phi(t)$ in the ascending power of the intermolecular interaction, the zeroth and the first order terms vanish owing to the nature of inversion doublet and the dipole dipole interaction, accordingly we have

$$\psi_\alpha(t)=\psi_{\alpha 2}(t)=\sum_M \sum_\tau \langle \sigma^2_{\alpha M}(b) \times \int_0^t (t-\tau)e^{i\omega_\tau \tau} f_M\left(\frac{v\tau}{b}\right)d\tau\rangle_b \tag{3}$$

as the lowest order approximation. Here b is the closest distance of approach which characterizes a particular type of orbit, and $\langle\ \rangle_b$ denotes the summation over the number and the type of collision.

$$\langle g(b)\rangle_b = nv\int_0^\infty 2\pi b\, db g(b). \tag{4}$$

$\sigma^2_{\alpha M}(b)$ is the second moment of the intermolecular interaction

$$\sigma^2_{\alpha M}(b)=\frac{\mu^4 C_M}{\hbar^2 b^6}\frac{K^2}{J(J+1)}\frac{K'^2}{J'(J'+1)} \times \int_{-\infty}^\infty R^2_M\left(\frac{vx}{b}\right)dx \tag{5}$$

and $f_M(v\tau/b)$ is the correlation function of the radial part of the interaction

$$f_M\left(\frac{v\tau}{b}\right)=\frac{\int_{-\infty}^\infty R_M\left(\frac{vx}{b}\right)R_M\left(\frac{v(x+\tau)}{b}\right)dx}{\int_{-\infty}^\infty R^2_M\left(\frac{vx}{b}\right)dx}. \tag{6}$$

μ is the electric dipole moment and v is the mean velocity and (J, K, M) are the rotational quantum numbers of the molecule under consideration. $\hbar\omega_\tau$ is the change in energy which is associated with the second order effect of the interaction and is given by $\hbar(\omega_\alpha \pm \omega_{\alpha'})$, where $\omega_{\alpha'}$ is the inversion frequency of the colliding molecule. This is due to the fact that the dipole moment is purely off-diagonal with respect to the inversion doublet.

When the pressure is not very high, say below 10 mm Hg, we have a condition

$$\sigma_{\alpha M}(b) \ll \frac{v}{b}. \tag{7}$$

In this case $\psi_\alpha(t)$ may be written as

$$\psi_\alpha(t)=t\sum_\tau \sum_M \langle \sigma^2_{\alpha M}(b)\left(\frac{b}{v}\right)F_M\left(\frac{b\omega_\tau}{v}\right)\rangle_b$$

$$=t\sum_\tau \langle \Psi(\omega_\tau)_b\rangle = t\Phi_2 \tag{8}$$

where

$$F_M(b\omega_\tau/v)=\int_0^\infty d\tau' e^{i(b\omega_\tau/v)\tau'} f_M(\tau'). \tag{9}$$

In the equation (8) the quantity Φ_2 stands for the line width and is explicitly given by

$$\Phi_2=\sum_{\alpha'}\{\Psi(\omega_\alpha+\omega_{\alpha'})+\Psi(\omega_\alpha-\omega_{\alpha'})\}, \tag{10}$$

where $\Psi(\omega_\tau)$ is the number of transition per unit time accompanied by an energy change $\hbar\omega_\tau$.

Now the rate of energy relaxation Φ_1 is equivalent to the number of energy quanta transferred per unit time, and is given by

$$\Phi_1=\frac{1}{\omega_\alpha}\sum_{\alpha'}\{(\omega_\alpha+\omega_{\alpha'})\Psi(\omega_\alpha+\omega_{\alpha'}) + (\omega_\alpha-\omega_{\alpha'})\Psi(\omega_\alpha-\omega_{\alpha'})\}. \tag{11}$$

Suppose in addition we have a condition

$$\omega_\tau \ll \frac{v}{b}, \tag{12}$$

then there is a relation

$$\Psi(\omega_\alpha+\omega_{\alpha'}) \doteqdot \Psi(\omega_\alpha-\omega_{\alpha'}) \doteqdot \Psi(0) \tag{13}$$

and it follows immediately that

$$\Phi_1 = \Phi_2. \tag{14}$$

Therefore the rate of energy relaxation is equivalent to the line width in this particular case, which corresponds to the extremely narrowed case in magnetic resonance.[5] That the condition (12) holds in usual cases was demonstrated by Anderson,[6] therefore the proof we desired has now been completed.

This note is the result of a discussion which the author enjoyed with Professor C. H. Townes in Tokyo University during the summer of 1956.

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The Coupling Constants of Strong Interactions

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From the recent experiments about the strange particles it is known that there are several strong interactions. In this paper we shall estimate the order of the magnitude of these coupling constants by the perturbation methods. For this purpose the experimental data on the K^+-nucleon scattering, the π^--proton scattering, and the strange particle production by the π^--proton collision are used.

If we assume the spin $\frac{1}{2}$ for Λ and $\sum$ particles, there are four possible assignments of their parities; namely $\alpha)$: Λ is $\frac{1}{2}^-$ and $\sum$ is $\frac{1}{2}^-$, $\beta)$: Λ is $\frac{1}{2}^+$ and $\sum$ is $\frac{1}{2}^-$, $\gamma)$: Λ is $\frac{1}{2}^-$ and $\sum$ is $\frac{1}{2}^+$, $\delta)$: Λ is $\frac{1}{2}^+$ and $\sum$ is $\frac{1}{2}^+$.

We assume furthermore that the nucleon is $\frac{1}{2}^+$ and the K particle is 0^+, then we can take the simplest inteaaction Hamiltonian densities as follows,

$$H_{NN\pi} = ig_1\bar{\psi}_N\gamma_5\tau_\alpha\psi_N\varphi_\alpha + H\cdot C$$

$$H_{N\Lambda K} = G_1\bar{\psi}_\Lambda\Gamma'\psi_N\phi_K + H\cdot C$$

$$H_{N\Sigma K} = G_2\bar{\psi}_{\Sigma\alpha}\Gamma'\tau_\alpha\psi_N\phi_K + H\cdot C$$

$$H_{\Lambda\Sigma\pi} = g_2\bar{\psi}_\Lambda\Gamma'\psi_{\Sigma\alpha}\varphi_\alpha + H\cdot C$$

$$H_{\Sigma\Sigma\pi} = g_2\bar{\psi}_\Sigma\Gamma' I_\alpha\psi_\Sigma\varphi_\alpha + H\cdot C$$

where Γ' is either $i\gamma_5$ or 1.

The value of the $NN\pi$ coupling constant, $g_1^2/4\pi \doteqdot 10$, for $ps(ps)$ interaction has been estimated from low energy phenomena, but if we adopt here this value, it leads to the unfavorable results. Assuming that $g_1^2/4\pi$ should be adjusted at each energy in the case of calculation by the lowest order perturbation, we determine the the value of effective coupling constant $g_1^2/4\pi$ at the higher energy ~ 1.3 BeV (kinetic energy of π mesons in the laboratory system). According to Walker,[1] the elastic scattering cross section at 1.3 BeV is $2\sim 3$ mb extracting the shadow scattering ($\theta < 60°$). From this data we obtain the value of the $NN\pi$ coupling constant, $1.6 \lesssim g_1^2/4\pi \lesssim 2.0$.

Lannutti et al.[2] observed the nuclear interactions of K^+ mesons in flight in energy region of $30\sim 120$ MeV. These

cross sections are 6 mb per nucleon. Furthermore there are 4 charge exchange events and 36 non-exchange ones, namely $36/4=R$. While, according to the Göttingen group[3] the cross section of the elementary process, $K^+ + p \to p + K^+$ is 14 $\sim$18 mb. To see how to choose the values of the coupling constants whens we adopt various experimental data, we use 6 mb as the lower limit for the cross section, while 18 mb as the upper limit from the perturbational expressions for various cross sections, and take $R=36/4 \pm \sqrt{9}=5\sim15$. Then we obtain the two possible sets of values for the $N\Lambda K$ coupling constant $G_1^2/4\pi$ and the $N\Sigma K$ coupling constant $G_2^2/4\pi$ at 50 MeV in the cases α) and δ). The results are given in Table 1. But in the cases γ) and β) we find no solutions.

We have no experimental data about K^+-nucleon scattering at high energy $\gtrsim$ 500 MeV. But we assume that the order of magnitude of $G_1^2/4\pi$, $G_2^2/4\pi$ at high energy is the same as that of the ones at 50 MeV.

By Forler et al.[4] or Steinberger et al.,[5] the cross sections of the strange particle production in $\pi^- + p$ collision are $\sim$1 mb at 1.3 BeV. Moreover Steinberger et al.[5] observed 37 neutral particle production. From these data and above mentioned values of $g_1^2/4\pi$, $G_1^2/4\pi$ and $G_2^2/4\pi$, we can obtain the four possible values for the $\Lambda\sum\pi$ coupling constant $g_2^2/4\pi$ and the $\sum\sum\pi$ coupling constant $g_3^2/4\pi$ which are listed in Table 1.

We have no reason to exclude any of these eight types of solutions in Table 1. However if we take the (B) solution for the assignment α) which contains γ_5 in interaction Hamiltonians, the values of these five coupling constants are of the same order of magnitude. Thus it seems to be possible to assume universality of coupling constants. This possibility for the strong interaction, however, is based on the results obtained by the lowest order perturbation. Therefore, in future, with more accumulated experiments and improved methods of calculation, we shall again discuss this attractive possibility.

The author is deeply grateful to Dr. T. Nakano for his leadership, to Professor Y. Yamaguchi and Dr. S. Minami for their valuable advice.

1) Walker, Fifth Rochester Conference, 1955.
2) J. E. Lannutti, Phys. Rev. **101** (1956), 1617.
3) Dallapota, Sixth Rochester Conference, 1956.
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5) T. Steinberger et al, Phys. Rev. **103** (1956), 1827.

Table 1

case α)				case δ)			
$1.6\lesssim g_1^2/4\pi\lesssim 2.0$	$1.9\lesssim G_1^2/4\pi\lesssim 3.4$ $0.8\lesssim G_2^2/4\pi\lesssim 1.7$	$2.0\gtrsim g_2^2/4\pi\gtrsim 0.6$ $0\lesssim g_3^2/4\pi\lesssim 0.4$	(A)	$1.2\lesssim G_1^2/4\pi\lesssim 2.3$ $0.3\lesssim G_2^2/4\pi\lesssim 0.4$	$36\gtrsim g_2^2/4\pi\gtrsim 30$ $9\lesssim g_3^2/4\pi\lesssim 14$	(A)
		$0.4\lesssim g_2^2/4\pi\lesssim 0.4$ $1.8\gtrsim g_3^2/4\pi\gtrsim 1.2$	(B)		$4\gtrsim g_2^2/4\pi\gtrsim 1$ $46\gtrsim g_3^2/4\pi\gtrsim 41$	(B)
	$0\lesssim G_1^2/4\pi\lesssim 0.6$ $1.5\lesssim G_2^2/4\pi\lesssim 2.0$	$0.2\gtrsim g_2^2/4\pi\gtrsim 0$ $1.5\gtrsim g_3^2/4\pi\gtrsim 1.2$	(C)	$0\lesssim G_1^2/4\pi\lesssim 0.5$ $0.6\lesssim G_2^2/4\pi\lesssim 1.8$	$2\gtrsim g_2^2/4\pi\gtrsim 0.5$ $0\lesssim g_3^2/4\pi\lesssim 0.04$	(C)
		$0.2\gtrsim g_2^2/4\pi\gtrsim 1.4$ $1.6\gtrsim g_3^2/4\pi\gtrsim 1.0$	(D)		$2\gtrsim g_2^2/4\pi\gtrsim 0.8$ $0\lesssim g_3^2/4\pi\lesssim 0.5$	(D)

A Remark on the Lepton Processes

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As is well known, the charge independence hypothesis proposed by Nishijima[1] and Gell-Mann[2] has succeeded in explaining the curious natures of the strange particles as far as the strong interactions are concerned. But such a theory, as it stands, seems to be unsatisfactory for understanding the lepton processes.[3),4)]

In the customary scheme the leptons are not assigned any iso-spin and strangeness.[4)] This view-point is usually accepted to be supported by the following fact that the leptons always interact, except for electromagnetic interactions, with a strength about 10^{-14} weaker than the strong interctions. According to the current scheme, we cannot summarize the lepton processes in a unified way.[5)] Moreover in that case we are obliged to allow such unwanted processes ;[6)]

$$K^0 \to \mu^+ + \mu^- \qquad (1)$$

$$K^0 \to e^+ + e^- \qquad (2)$$

$$K^+ \to \pi^+ + \mu^+ + \mu^- \qquad (3)$$

$$K^+ \to \pi^+ + e^+ + e^- \qquad (4)$$

$$K^+ \to \pi^+ + \nu + \tilde{\nu} \qquad (5)$$

etc.

However, as far as the charge is to be considered as a fundamental property of a particle, it seems that the leptons should be assigned iso-spin and strangeness, since the leptons as well as the baryons and bosons have the elementary charge whose conservation can be related to invariance under a gauge transformation.[7)]

In this note we assign an appropriate iso-spin and strangeness for leptons and assume, for each lepton having strangeness s, the following relation between the third component I_3 and charge Q

$$Q = I_3 + \tfrac{1}{2}u, \quad u = n + s, \qquad (6)$$

where n means lepton number. This relation is an extention of that introduced by Nishijima and Gell-Mann for each baryon and boson with strangeness S :

$$Q = I_3 + \tfrac{1}{2}U, \quad U = N + S. \qquad (7)$$

N in (7) is baryon number. We assume in this paper, as shown in Table 1, any iso-spin of leptons is zero. μ^+ and e^- are taken as normal,[8)] so μ^- and e^+ are considered to be anti-particles. The iso-spin assignment for baryons and bosons is assumed to be the same as that of the paper (I).

Since we have introduced the lepton number n into the lepton family independently of the baryon number N, we

Table 1. Iso-spin assignment for leptons

Particle	I	I_3	$Q=I_3+\frac{1}{2}u$		$u=n+s$	s	Transformation properties
μ^+	0	0	1	I_3+1	2	1	iso-pseudoscalar
e^-	0	0	-1	I_3-1	-2	-3	iso-pseudoscalar
ν	0	0	0	I_3	0	-1	iso-scalar

can sharply distinguish the allowed lepton interactions from the strong and electromagnetic ones by the following selection rules :

(i) *lepton*$\rightarrow$*leptons* ($\mu-e$ *decay*)

$$\Delta n=0,\ \Delta I=0,\ \Delta u=0,\ N=0,\quad (\Delta Q=0) \qquad \text{(a)}\ *$$

(ii) *strongly*$\rightarrow$*stronglys*$+$*leptons*

$$\Delta N=\Delta n=0,\ \Delta Q=0,\ |\Delta I|=\tfrac{1}{2}\ \text{or}\ |\Delta I|=1\ (\Delta I_3\neq 0),\ |\Delta u|=2. \qquad \text{(b)}$$

Here "strongly" refers to baryon and boson. $N=0$ in (a) means that a baryon is not concerned to the $\mu-e$ decay. It is remarkable here that such unwanted processes as (1) $\sim$ (5) are all forbidden by the selection rule (b).

Now, if we assume that primary interactions in the strong ones are of Yukawa type and primary weak interactions of Fermi type, we can consistently deduce the weak interactions as the secondary processes from the intermediary of some of the strong interactions of Yukawa type and one of the weak interactions of Fermi type.

Finally we mention the energy spectrum of the secondary π^+ in the decay process $K^+\rightarrow\pi^++\nu+\tilde{\nu}$ which is allowed by the selection rule (α) in the paper (I), while it is forbidden by (b). If we assume for this process an interaction Hamiltonian like (7·1) in (I), we get the transition probability :

$$w=\frac{g^2}{3\pi}M_K^5\int_{\alpha_\pi}^{\alpha^0}(5\epsilon^2-6\epsilon+3-2\alpha_\pi^2)\sqrt{\epsilon^2-\alpha_\pi^2}\,d\epsilon \qquad (9)$$

$$\alpha^0=\tfrac{1}{2}+\tfrac{1}{2}\alpha_\pi^2,\ \alpha_\pi=m_\pi/M_K$$

* Dr. S. Hanawa informed to the author that introduction of "neutrino number" is useful to summarize the lepton processes, but in our model it is not necessary to introduce the neutrino number.

where $M_K\epsilon$ means the energy (including the rest energy) of the secondary π^+. If (5) is realized in nature as an allowed process, we should certainly find out, among the processes $K^+\rightarrow\pi^++2$ neutrals, the secondary π^+ which has the kinetic energy greater than the maximum one ($\sim$50 Mev) of the π^+ in the τ'^+-decays ($\tau'^+\rightarrow\pi^++2\pi^0$), since the transition rate of $K^+\rightarrow\pi^++\nu+\tilde{\nu}$ may be nearly equal to that of $K_{e3}^+\rightarrow e^++\pi^0+\nu$. But such an event as above has not been identified up till now, so the process (5) may belong to the forbidden one, as was shown in this note.

Detailed discussions will be published in a later issue of this journal.

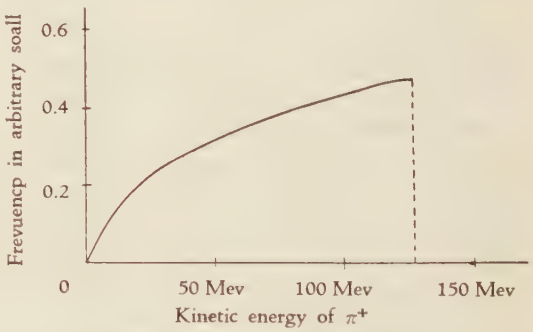

Fig. 1 Energy spectrum of the secondary π^+ in the decay process : $K^+\rightarrow\pi^++\nu+\tilde{\nu}$

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2) M. Gell-Mann, Phys. Rev. **92** (1953), 833.
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6) Cf. (3).
7) W. Pauli, Rev. Mod. Phys. **13** (1941), 203.
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8) E. J. Konopinski and H. M. Mahmoud, Phys, Rev. **92** (1953), 1045.

On a Classification of Light Particles

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January 4, 1957

Nishijima and Gell-Mann[1] have presented a classification for the strange particles under the hypothesis of charge independence. With this model the phenomena among these particles, and especially the discrimination whether the interactions are strong or not, have been successfully explained, as long as the particles belong to the baryon-meson families.

However, the N-G theory, which was constructed basing upon the concept of isotopic-spin, has not taken into account the decay processes in which lepton (μ, e, ν) takes part. The simplest way to extend this theory to lepton processes may be the application of the charge independence hypothesis for lepton. Though this may produces good selection rules, we are subjected to the introduction of the redundant neutral particles (μ_0, e_0), as investigated by some authors.[2]

From the experimental observation for the phenomena involving lepton, there seems to exist the following characteristic features: 1) The lepton processes involve at least one neutrino, 2) they take place always in the pair ($\mu\nu$) or ($e\nu$), and 3) in two body decay only, μ and e take place in an asymmetrical way.

The purpose of this note is to give a scheme for classifying the lepton family without introducing the charge independence hypothesis into leptons. For this purpose, we introduce the η_l-charge into lepton in an analogous manner to the Nishijima model, in addition to the lepton number (n) conservation proposed by Konopinski-Mahmoud.[3] Then, there should be an intimate relationship between the charge Q and η_l-charge for each particle.*

We may write this relation as

$$Q=\tfrac{1}{2}(n+\eta_l). \qquad (1)$$

We assign these quantities to each particle as the following table:

Table Types of fields

name	Lorentz space	Isotopic spin space	Q	n	η_l
μ^+	spinor	pseudo-scalar	1	1	1
e^-	spinor	pseudo-scalar	-1	1	-3
ν	spinor	scalar	0	1	-1

An anti-particle, i.e. charge conjugate particle, has an opposite η_l-charge to that of the normal particle as is the case with the electric charge, and we have

$$\eta_l C=-C\eta_l\,, \quad QC=-CQ \qquad (2)$$

where C is the charge conjugation operator.

In applying this idea, we assume that 1) $\Delta\eta_l=\pm 2$ for the transitions, baryon$\rightarrow$baryon+lepton, 2) $\Delta\eta_l=0$ for the transition, lepton$\rightarrow$lepton.

On the other hand, the estrblished lepton processes are written as

$$\begin{aligned}
&(\beta\text{-decay}) && N\rightarrow P+e^-+\bar{\nu}\\
&(\mu\text{-capture}) && P+\mu^-\rightarrow N+\bar{\nu}\\
&(\mu\text{-decay}) && \mu^+\rightarrow e^++\nu+\nu\\
&(\pi\text{-decay}) && \pi^+\rightarrow\mu^++\bar{\nu}\\
&(K_{\mu 2}\text{-decay}) && K^+_{\mu 2}\rightarrow\mu^++\bar{\nu}
\end{aligned} \qquad (3)$$

* Prof. S. Goto has presented a similar scheme independently of us (private communication).

($K_{\mu 3}$-decay) $K_{\mu 3}^{+}\to\pi^{0}+\mu^{+}+\bar{\nu}$

(K_{e3}-decay) $K_{e3}^{+}\to\pi^{0}+e^{+}+\nu$.

Although the lepton number conservation is sufficient to make the established processes (3) "allowed", further restriction will be required for the unobserved proceses $K^{+}\to\pi^{-}+e^{+}+\mu^{+}$, $K^{+}\to\pi^{+}+e^{+}+e^{-}$, etc, to be excluded. For this purpose, η_l-charge introduced in our model is useful to prohibit these unwanted processes, and in this case there appear no difficulties of the neutral counter particles which were encountered in the charge independence model.

Details of the present note will be published in near future.

The author would like to thank Professor K. Sakuma, Drs. S. Ogawa, T. Ouchi and M. Yonezawa for their helpful discussions.

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Some Identities for Renormalization Constants in $S(S)$ and $PS(PS)$ Meson Theories

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Using the approach to a solution of the quantum field theory presented by Landau, Abrikosov and Khalatnikov,[1] renormalized coupling constant g_c^2 and renormalization constants Z_i are expressed in terms of unrenormalized coupling constant g_0^2, renormalized Fermion mass m and cut off momentum λ as follows:[1)2)3)]

$$g_c^2=g_0^2\,Q^{-1},\quad Z_i=Q^{b_i}\quad Q=1+\alpha g_0^2\log\frac{\lambda^2}{m^2},\tag{1}$$

where $\infty>\alpha>0$, $0>(b_2,\ b_3)>-\infty$ and $-2b_1+2b_2+b_3=-1$. In this equation the terms

$$g_0^{2m}\,(g_0^2\log\lambda^2/m^2)^n\quad(m,\ n=1,\ 2,\ldots)\tag{2}$$

have been neglected. In this note we shall, using the approach, show that there exist some identities between the renormalization constants Z_i for $S(S)$ and $PS(PS)$ meson theories.

Eq. (1), in principle, can be obtained with the aid of the Dyson-Schwinger integral equations. For simplicity the method presented by reference 3) will be used. According to their method, this equation can be obtained directly without solving the integral equations. There we use the renormalizability conditions,[4] and the asymptotic forms, for large momentum, of the one particle Green's functions and the vertex part in g_0^2 order.

When the main part of interaction Hamiltonian H_i in pure neutral $S(S)$ and $PS(PS)$ meson theories are given by

$$H_i=g_0\bar{\psi}\psi\phi\quad\text{and}\quad H_i=ig_0\bar{\psi}\gamma_5\psi\phi\tag{3}$$

respectively, the vertex part and the one particle Green's functions, in both $S(S)$ and $PS(PS)$ cases, have the following approximate expressions up to g_0^2 order for the case $-p^2\gg m^2$:

$$\Gamma_5/\gamma_5\simeq\Gamma'\simeq 1+\frac{g_0^2}{8\pi}\log\lambda^2/p^2,$$

$$S'_F(p)/S_F(p) \sim 1 - \frac{g_0^2}{8\pi}\log \lambda^2/p^2, \quad (4)^*$$

$$J'_F(p)/J_F(p) \simeq 1 - \frac{g_0^2}{2\pi}\log \lambda^2/p^2.$$

Making use of the renormalizability conditions[4] and eq. (4), simple calculations yield :

$$Q = 1 + \frac{g_0^2}{2\pi}\log\frac{\lambda^2}{m^2} \quad b_1 = b_2 = -\tfrac{1}{4}$$

$$\text{and} \quad b_3 = -1. \qquad (5)^*$$

Let us consider the most general case where the main parts of interaction Hamiltonian are given by

$$H_i = \sum_{j=1}^{4} g_{0j}\bar{\psi}\tau_j\psi\phi_j \quad \text{and}$$

$$H_i = i\sum_{j=1}^{1} g_{0j}\bar{\psi}\gamma_5\tau_j\psi\phi_j, \qquad (6)$$

where

$$g_{0j} = a_j g_0, \quad \tau_4 = 1 \quad \text{and} \quad a_1 = a_2. \qquad (7)$$

Due to the invariance under the charge conjugation, a_1 must be equal to a_2, and any two of three factors a_1, a_3 and a_4 can take independent values with each other. In this general case the vertex part and the self energy parts differ only from the corresponding ones in pure neutral case by isotopic spin parts alone. Using the same method as that in the pure neutral case, we get, in this general case, instead of eq. (1)

$$g_{c,j}^2 = g_0^2 Q_j^{-1}, \qquad Z_{i,j} = Q_j^{b_{i,j}},$$

$$Q_j > 0 \text{ and } 0 > (b_{2,j}, b_{3,j}) > -\infty, \qquad (8)$$

where suffix j specifies the type of the external mesons, and $Q-1$ and $b_{i,j}$ are listed in Table 1.

Table 1 gives us the following two significant informations about the identities for renormalization constants in $S(S)$ and $PS(PS)$ meson theories :

i) Renormalization constants for $S(S)$ case equal to those for $PS(PS)$ case.

ii) Especially in both neutral ($j=3$ and $a_1=0$) and pure neutral ($j=4$) cases, there hold Ward's identities for renormalization constants, as in the case of quantum electrodynamics.

In the next note we shall compare these results with those obtained by usual perturbation method.

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3) Pometranchuk, Sudakov and Ter-martirosyan, Phys. Rev. **103** (1956), 783 : J. C. Taylor. Proc. Roy. Soc. **234** (1956), 296.
4) M. Gell-Mann and F. E. Low, Phys. Rev. **95** (1954), 1300.

* These results differ from those of reference 3 by a constant factor.

Table 1.

	$Q_j - 1$	$b_{1,j}$	$b_{2,j}$	$b_{3,j}$
$j=1$ or 2	$(3a_1^2+a_3^2)\frac{g_0^2}{2\pi}\log\lambda^2/m^2$	$\frac{(a_3^2-a_4^2)}{4(3a_1^2+a_3^2)}$	$-\frac{(2a_1^2+a_3^2+a_4^2)}{4(3a_1^2+a_3^2)}$	$-\frac{2a_1^2}{(3a_1^2+a_3^2)}$
$j=3$	$(a_1^2+a_3^2)\frac{g_0^2}{\pi}\log\lambda^2/m^2$	$\frac{(2a_1^2-a_3^2-a_4^2)}{8(a_1^2+a_3^2)}$	$-\frac{(2a_1^2+a_3^2+a_4^2)}{8(a_1^2+a_3^2)}$	$-\frac{a_3^2}{(a_1^2+a_3^2)}$
$j=4$	$a_4^2\frac{g_0^2}{2\pi}\log\lambda^2/m^2$	$-\frac{(2a_1^2+a_3^2+a_4^2)}{4a_4^2}$	$-\frac{(2a_1^2+a_3^2+a_4^2)}{4a_4^2}$	-1

ERRATA

On the Quantum Mechanics-like Description of the Theories of the Brownian Motion and Quantum Statistical Mechanics

N. SAITÔ and M. NAMIKI

(Prog. Theor. Phys. **16** (1956), 71)

1. In the following Table A should read B.

Page	line or equation	A	B
75	18	(p. 2)	(p. 72)
80	8	τ	$-\tau$
83	Eq. (4.24)	$2Dt$	Dt
83	Eq. (4.26)	p_i	$\dot{p}_i$
84	Eqs. (4.31), (4.32), (4.33)	m	m^0
	last term of Eq. (4.33)	$(1/m\beta^3)$	$(1/m^2\beta^2)$
91	10 from bottom	system of described	system is described
	7 from bottom	composed is	composed of
93	Eqs. (A.12), (A.12′)	t_2	t

2. From dimension-analytical point of view, it may be convenient to rewrite eqs. (3.3) and (3.5) as follows;

$$<X(t_1)X(t_2)>=\frac{<X^2>}{\beta}\delta(t_1-t_2), \tag{3.3}$$

$$B_t(\Delta t)=\beta\int_t^{t+\Delta t}X(t)\,dt\simeq X(t)\beta\Delta t, \tag{3.5}$$

where β is defined in eq. (3.23), $\beta=\zeta/m$. This requires slight changes in eqs. (3.4), (3.6), (3.7) and the middle number of (3.8), that is,

$$W(X)\propto\exp(-X^2/4\beta\zeta kT), \tag{3.4}$$

$$<B_t^2(\Delta t)>=\beta^2\int_t^{t+\Delta t}\!\!\int<X(t)X(s)>dtds=<X^2>\beta\Delta t, \tag{3.6}$$

$$W\{B_t(\Delta t)\}\propto\exp\{-B_t^2(\Delta t)/4\beta^2\zeta kT\Delta t\}, \tag{3.7}$$

$$\exp\{-(1/4\beta^2\zeta kT\Delta t)\Sigma B_t^2(\Delta t)\}. \tag{3.8}$$

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On the Vibration of Disordered Linear Lattice

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(Received November 26, 1956)

The method of transfer matrix which was recently developed by Kerner to treat the problem of electronic band structure of mixed linear lattices was applied to the problem of vibration of atomic linear chain. It was shown that this method affords effective means of calculating the eigenfrequency-distribution of several kinds of lattices, including disordered ones.

The well-known results for regular monatomic and diatomic lattices, and the results which were obtained by Montroll and Potts for linear lattices containing a few impurity atoms, were rederived by the matrix method in a much simpler way. The eigenvalue equations for the lattices containing several impurities at arbitrarily given positions were then derived. In cases where the force-constant is not altered by impurities the formulas become so simple that we can obtain the eigenfrequencies without laborious calculations. It is also possible to derive in such cases a simple approximate eigenvalue equation for lattices containing impurities distributed completely at random. This equation indicates that the mode of change in the distribution of eigenfrequencies is qualitatively the same as in the case of one or two impurities.

Monatomic lattice with a single hole and diatomic lattice with a single impurity were treated as simple examples. An attempt to extend the matrix method to the case in which the next-nearest interaction between atoms is taken into account was also mentioned.

§ 1. Introduction

Recently Kerner[1] treated the problem of the electronic band structure of periodic or random linear lattices containing impurity atoms by the method of transfer matrices. In this paper we show that if we treat a problem of vibration of linear atomic lattice by a similar method, we can obtain not only the same results as hitherto obtained for a regular periodic lattices or lattices containing a few impurity atoms, but also, in a comparatively simple way, a considerable amount of information as to the vibration of lattices which have more complicated structures.

The problem of vibration of a lattice containing a few impurities was recently investigated by Montroll and Potts[2], using the method of Green functions. Though their treatment is very elegant and several important results are included therein, the calculations seem to come out considerably complicated if we attempt to apply it to lattices with more complicated structures, e.g., those containing several impurities at arbitrarily given positions or at irregular unknown positions. Especially it will prove difficult to apply this method effectively to the latter. It is true, on the other hand, that for such random lattices there is a rigorous mathematical theory formulated by Dyson and Bellman[3,4], which provides us with a general procedure of calculating the distribu-

tion of eigenfrequencies of random lattices. Practically, however, this method will not be of great use, inasmuch as it involves mathematical formulas which are hardly workable, such as an integral equation whose solution cannot be obtained without formidable numerical calculations. The case treated by Dyson is perhaps the only one example in which the equation can be solved in a closed form, which however is not physically very significant. Our method, which furnishes much new information as to the vibrational modes of lattices with several impurities situated at given positions or at random, will make up, to a certain extent, for the drawbacks of these theories.*

In § 2 and § 3, we explain our method and show that the well-known eigenfrequency-distribution for regular monatomic and diatomic lattices can be derived therefrom in a very simple way. In § 4, the lattice with one impurity atom arbitrarily located is treated. For the case in which the force-constant between impurity and regular atom is the same as that between regular atoms (Case M), we obtain the result which is very similar to that of Montroll and Potts. In § 5, the spatial wave forms of the normal vibrations are discussed for the Case M with one impurity atom, and the localized character of the impurity mode is demonstrated.

In § 6 we treat the case of two or more impurities and show that in Case M it is especially easy to derive the eigenvalue equation for lattice with several impurities. In § 7 the effect of randomly distributed impurities is discussed also for Case M. An approximate eigenvalue equation is given for the case where a comparatively small number of impurities are distributed completely at random, the case which is physically most important. It is concluded that also for such a random case the mode of change in eigenfrequency-distribution due to impurities is essentially the same as in the case of one or two impurities.

In § 8 we consider, as a simple application, a case in which one "hole," i.e., one impurity atom whose mass is zero, exists. It is argued that here we are driven to take into account the force between the "next-nearest" atoms neighboring the hole, so as not to break the lattice into two independent portions. In § 9 we consider, as the second application, a diatomic lattice with one impurity atom. It is shown that under certain conditions there appear additional impurity modes from the top of the acoustical branch or from the bottom of the optical branch. Finally in § 10, we attempt to take into account the force between next-nearest atoms in a general way. It is shown that it is possible to treat such cases by a similar matrix method. For regular lattices we reach the same result as hitherto obtained. The eigenvalue equation for the case of one impurity is derived, which shows that the manner of the influence of impurity atom on the eigenfrequency-distribution is the same as in the case of the nearest neighbor approximation.

*) While preparing the manuscript we became aware that a considerable extension of Montroll's theory had already been made by Lifschitz and others[3], which includes a general formula giving the frequency-spectrum of 3-dimensional random lattices. We think however that our treatment is still of value in its methodological aspect and in its simplicity.

§ 2. Regular monatomic lattice

Consider a linear atomic lattice composed of equally spaced N atoms of the same kind. Let the mass of an atom be m, and the force-constant between adjacent atoms be λ. Then the equation of motion is

$$m\,d^2x_i/dt^2=-\lambda\{(x_i-x_{i-1})+(x_i-x_{i+1})\},\quad i=1,2,\cdots,N, \tag{2·1}$$

where x_i is the displacement of the i-th atom. If we introduce a new variable $x_{i+1}-x_i=y_i$ and put the time factor as $\exp(i\omega t)$, (2·1) becomes

$$-m\omega^2x_i=\lambda(y_i-y_{i-1}). \tag{2·2}$$

Combining this with the definition of y_i, we obtain a system of equations which can be written in the vectorial form

$$\begin{pmatrix}1 & 0\\ m\omega^2 & \lambda\end{pmatrix}\begin{pmatrix}x_{i+1}\\ y_{i+1}\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}x_i\\ y_i\end{pmatrix}.$$

Multiplying both sides by the inverse matrix of that appearing on the left-hand side, we have

$$\boldsymbol{X}_{i+1}=\begin{pmatrix}x_{i+1}\\ y_{i+1}\end{pmatrix}=\begin{pmatrix}1 & 1\\ -m\omega^2/\lambda & 1-m\omega^2/\lambda\end{pmatrix}\begin{pmatrix}x_i\\ y_i\end{pmatrix}=(\boldsymbol{I}+\boldsymbol{T})\boldsymbol{X}_i, \tag{2·3}$$

where $\boldsymbol{I}$ is the unit matrix. Thus it is seen that the motion of the lattice can be described in terms of a "transfer matrix" $\boldsymbol{I}+\boldsymbol{T}$, which transforms the "state vector" $\boldsymbol{X}_i$ of an atom into that of the next atom $\boldsymbol{X}_{i+1}$.

Putting $\omega^2=(4\lambda/m)\sin^2\beta$, we obtain for the eigenvalues of the matrix $\boldsymbol{I}+\boldsymbol{T}$:

$$1+\lambda_{\frac{1}{2}}=\exp(\pm 2i\beta). \tag{2·4}$$

The matrix $\boldsymbol{U}$ and its inverse $\boldsymbol{U}^{-1}$ which diagonalize $\boldsymbol{T}$ by a similar transformation ($\boldsymbol{U}^{-1}\boldsymbol{T}\boldsymbol{U}=\boldsymbol{\Lambda}\equiv\mathrm{diag}\ (\lambda_1,\lambda_2)$) are

$$\boldsymbol{U}=\begin{pmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{pmatrix}\text{ and }\boldsymbol{U}^{-1}=\frac{1}{\lambda_2-\lambda_1}\begin{pmatrix}\lambda_2 & -1\\ -\lambda_1 & 1\end{pmatrix}, \tag{2·5}$$

respectively. It is to be noted here that $\boldsymbol{U}^{-1}$ does not exist when $\lambda_2=\lambda_1$.

We now let the lattice be subject to the cyclic boundary condition $\boldsymbol{X}_{N+1}=\boldsymbol{X}_1$. Then if $\boldsymbol{X}_{N+1}=\boldsymbol{H}\boldsymbol{X}_1$ it is necessary that

$$|\boldsymbol{H}-\boldsymbol{I}|=0. \tag{2·6}$$

In the present case $\boldsymbol{H}=(\boldsymbol{I}+\boldsymbol{T})^N$. Since $|\boldsymbol{I}+\boldsymbol{T}|=1$, (2·6) is equivalent to the equation

$$\mathrm{Trace}\,\boldsymbol{H}=2. \tag{2·7}$$

This gives the eigenvalue equation for the frequency ω.

Using (2·4) we get

$$\mathrm{Trace}\,\boldsymbol{H}=\mathrm{Trace}\{\boldsymbol{U}(\boldsymbol{I}+\boldsymbol{\Lambda})^N\boldsymbol{U}^{-1}\}=\mathrm{Trace}(\boldsymbol{I}+\boldsymbol{\Lambda})^N$$

$$=(1+\lambda_1)^N+(1+\lambda_2)^N=2\cos(2N_i\beta), \tag{2·8}$$

and the eigenvalue equation (2·7) becomes

$$\cos(2N_i\beta)=1. \tag{2·9}$$

Hence the eigenfrequencies are given by

$$\omega^2=(4\lambda/m)\sin^2(n\pi/N),\ n=1,2,\cdots,\ N/2 \text{ for even } N,$$
$$(N-1)/2 \text{ for odd } N. \tag{2·10}$$

This is the well-known result.

§ 3. Regular diatomic lattice

For regular diatomic lattice, there appear two different matrices. Let the mass of the atom in odd positions be M and that in even positions be m, and the number of each atom be N. Denoting the two different force-constants by λ and μ, we obtain for the equation of motion

$$\left.\begin{aligned} -m\omega^2x_{2i}&=-\lambda(x_{2i}-x_{2i+1})-\mu(x_{2i}-x_{2i-1}),\\ -M\omega^2x_{2i+1}&=-\lambda(x_{2i+1}-x_{2i})-\mu(x_{2i+1}-x_{2i-2}).\end{aligned}\right\} \tag{3·1}$$

If we introduce a new variable $x_{2i+1}-x_{2i}=y_{2i}$ and rewrite (3·1) in the form of vector equation, we have:

$$\boldsymbol{X}_{2i}\equiv\begin{pmatrix}x_{2i}\\ y_{2i}\end{pmatrix}=\begin{pmatrix}1 & 1\\ (-m/\lambda)\omega^2 & (\mu/\lambda)-(m\omega^2/\lambda)\end{pmatrix}\begin{pmatrix}x_{2i-1}\\ y_{2i-1}\end{pmatrix}=(\boldsymbol{I}+\boldsymbol{T}_1)\boldsymbol{X}_{2i-1}, \tag{3·2}$$

and

$$\boldsymbol{X}_{2i+1}\equiv\begin{pmatrix}x_{2i+1}\\ y_{2i+1}\end{pmatrix}=\begin{pmatrix}1 & 1\\ -(M/\mu)\omega^2 & -(M/\mu)\omega^2+(\lambda/\mu)\end{pmatrix}\begin{pmatrix}x_{2i}\\ y_{2i}\end{pmatrix}=(\boldsymbol{I}+\boldsymbol{T}_2)\boldsymbol{X}_{2i}. \tag{3·3}$$

Consequently the transfer matrix which connects the state vectors of two atoms in the nearest odd positions is

$$\boldsymbol{I}+\boldsymbol{T}'=\begin{vmatrix}1-\dfrac{m}{\lambda}\omega^2 & 1+\dfrac{\mu}{\lambda}-\dfrac{m}{\lambda}\omega^2\\ -\dfrac{1}{\mu}(M+m)\omega^2+\dfrac{mM\omega^4}{\lambda\mu} & 1-\dfrac{1}{\mu}(M+m)\omega^2-\dfrac{M}{\lambda}\omega^2+\dfrac{Mm}{\lambda\mu}\omega^4\end{vmatrix}. \tag{3·4}$$

We have now only to use this matrix in place of $\boldsymbol{I}+\boldsymbol{T}$ in the preceding section. Putting

$$4\sin^2\beta=(M+m)\frac{\lambda+\mu}{\lambda\mu}\omega^2-\frac{Mm}{\lambda\mu}\omega^4, \tag{3·5}$$

and utilizing the fact that $|\boldsymbol{I}+\boldsymbol{T}'|=1$, we obtain the same expressions as in § 2 for the

eigenvalues of $\boldsymbol{I}+\boldsymbol{T}$ and the eigenvalue equation for ω. Eigenfrequencies are obtained by solving (3·5) for $\beta=(n\pi)/N$:

$$\omega^2=[(M+m)(\lambda+\mu)\pm\{(M+m)^2(\lambda+\mu)^2-16\lambda\mu Mm\sin^2(n\pi/N)\}^{1/2}]/2Mm,$$

$$n=1,2,\cdots\cdots,\ N/2 \text{ for even } N,$$

$$(N-1)/2 \text{ for odd } N. \qquad (3\cdot6)$$

This is again the well-known result, showing that there appear two (acoustical and optical) branches of eigenfrequencies.

§ 4. One impurity atom

Let us next consider the case in which one (say i-th) atom in the regular monatomic lattice as treated in § 2 is replaced by an impurity atom with mass M. The equation of motion of the atoms near the impurity then becomes:

$$-m\omega^2x_{i-1}=-\lambda(x_{i-1}-x_{i-2})-\mu(x_{i-1}-x_i), \qquad (4\cdot1.1)$$

$$-M\omega^2x_i=-\mu(x_i-x_{i-1})-\mu(x_i-x_{i+1}), \qquad (4\cdot1.2)$$

$$-m\omega^2x_{i+1}=-\mu(x_{i+1}-x_i)-\lambda(x_{i+1}-x_{i+2}). \qquad (4\cdot1.3)$$

Again putting $x_i-x_{i-1}=y_{i-1}$, we can write these equations in the form of vector equations, as before. Only the transfer matrices corresponding to the above equations come out to be different from that of the regular lattice $\boldsymbol{I}+\boldsymbol{T}$. They are, respectively,

$$\boldsymbol{I}+\boldsymbol{T}_1=\begin{pmatrix} 1 & 1 \\ -\frac{m}{\mu}\omega^2 & \frac{\lambda}{\mu}-\frac{m}{\mu}\omega^2 \end{pmatrix}, \qquad (4\cdot2.1)$$

$$\boldsymbol{I}+\boldsymbol{T}_2=\begin{pmatrix} 1 & 1 \\ -\frac{M}{\mu}\omega^2 & 1-\frac{M}{\mu}\omega^2 \end{pmatrix}, \qquad (4\cdot2.2)$$

$$\boldsymbol{I}+\boldsymbol{T}_3=\begin{pmatrix} 1 & 1 \\ -\frac{m}{\lambda}\omega^2 & \frac{\mu}{\lambda}-\frac{m}{\lambda}\omega^2 \end{pmatrix}. \qquad (4\cdot2.3)$$

We shall call them "impurity matrices" in the following.

Writing $\boldsymbol{P}=(\boldsymbol{I}+\boldsymbol{T}_3)(\boldsymbol{I}+\boldsymbol{T}_2)(\boldsymbol{I}+\boldsymbol{T}_1)$, we obtain for $\boldsymbol{H}$:

$$\boldsymbol{H}=(\boldsymbol{I}+\boldsymbol{T})^{N-i}\boldsymbol{P}(\boldsymbol{I}+\boldsymbol{T})^{i-1}.$$

Since $|\boldsymbol{P}|=1$ and consequently $|\boldsymbol{H}|=1$, the eigenvalue equation under the cyclic boundary condition is again given by Trace $\boldsymbol{H}=2$. Trace $\boldsymbol{H}$ can be calculated as follows:

$$\text{Trace } \boldsymbol{H}=\text{Trace}\{\boldsymbol{U}(\boldsymbol{I}+\Lambda)\boldsymbol{U}^{-1}\boldsymbol{U}(\boldsymbol{I}+\Lambda)\boldsymbol{U}^{-}\boldsymbol{U}\cdots\boldsymbol{U}^{-1}\boldsymbol{U}(\boldsymbol{I}+\Lambda)\times$$

$$\times \boldsymbol{U}^{-1}\boldsymbol{P}\boldsymbol{U}(\boldsymbol{I}+\boldsymbol{\Lambda})\boldsymbol{U}^{-1}\cdots\cdots\boldsymbol{U}(\boldsymbol{I}+\boldsymbol{\Lambda})\boldsymbol{U}^{-1}\}$$

$$=\text{Trace}\ \{(\boldsymbol{I}+\boldsymbol{\Lambda})^{N-3}\boldsymbol{U}^{-1}\boldsymbol{P}\boldsymbol{U}\}. \tag{4·3}$$

Putting $M/m=f=1+\varepsilon$, $\lambda/\mu=g=1+\delta$ and $\omega^2=(4\lambda/m)\sin^2\beta$, we get, after some calculations, for the elements of $\boldsymbol{P}$:

$$\begin{aligned}
p_{11}&=1-12\sin^2\beta+16\sin^4\beta-(4\sin^2\beta-16\sin^4\beta)\varepsilon\\
&\quad-(12\sin^2\beta-32\sin^4\beta)\delta-(4\sin^2\beta-32\sin^4\beta)\varepsilon\delta\\
&\quad+16\sin^4\beta\cdot\delta^2(1+\varepsilon),\\
p_{22}&=1-24\sin^2\beta+80\sin^4\beta-64\sin^6\beta\\
&\quad-(8\sin^2\beta-48\sin^4\beta+64\sin^6\beta)\varepsilon\\
&\quad-(12\sin^2\beta-96\sin^4\beta+128\sin^6\beta)\delta\\
&\quad-(4\sin^2\beta-64\sin^4\beta+128\sin^6\beta)\varepsilon\delta\\
&\quad+(16\sin^4\beta-64\sin^6\beta)\delta^2+(16\sin^4\beta-64\sin^6\beta)\varepsilon\delta^2,\\
p_{21}&=-12\sin^2\beta+64\sin^4\beta-64\sin^6\beta-(4\sin^2\beta\\
&\quad-32\sin^4\beta+64\sin^6\beta)\varepsilon+(64\sin^4\beta-128\sin^6\beta)\delta\\
&\quad+(32\sin^4\beta-128\sin^6\beta)\varepsilon\delta-64\sin^6\beta\cdot\delta^2-64\sin^6\beta\cdot\varepsilon\delta^2,\\
p_{12}&=3-16\sin^2\beta+16\sin^4\beta-(8\sin^2\beta-16\sin^4\beta)\varepsilon\\
&\quad+(2-20\sin^2\beta+32\sin^4\beta)\delta-(12\sin^2\beta-32\sin^4\beta)\varepsilon\delta\\
&\quad-(4\sin^2\beta-16\sin^4\beta)\delta^2-(4\sin^2\beta-16\sin^4\beta)\varepsilon\delta^2.
\end{aligned} \tag{4·4}$$

From this and (2·4) and (2·5) we obtain, after somewhat lengthy calculations, for the elements of $\boldsymbol{Q}=\boldsymbol{U}^{-1}\boldsymbol{P}\boldsymbol{U}$

$$\begin{aligned}
Q_{11}&=\overline{Q}_{22}=e^{6i\beta}(1+i\varepsilon\tan\beta+2i\delta\tan\beta+2i\delta\varepsilon\tan\beta)\\
&\quad-e^{4i\beta}\{2i\delta\varepsilon\tan\beta+4i\delta^2(1+\varepsilon)\sin^2\beta\tan\beta\},\\
Q_{12}&=Q_{21}=e^{-2i\beta}\{i(\varepsilon-2\delta)\tan\beta+4i(\delta+\varepsilon\delta+\delta^2+\varepsilon\delta^2)\tan\beta\sin^2\beta\}.
\end{aligned} \tag{4·5}$$

Consequently,

$$\begin{aligned}
\text{Trace}\ \boldsymbol{H}&=(1+\lambda_1)^{N-3}Q_{11}+(1+\lambda_2)^{N-3}Q_{22}\\
&=\cos\{2(N-3)\beta\}\,(Q_{11}+Q_{22})+i\sin\{2(N-3)\beta\}\,(Q_{11}-Q_{22})\\
&=2\cos(2N\beta)-2(\varepsilon+2\delta+2\delta\varepsilon)\tan\beta\sin(2N\beta)\\
&\quad+4\delta\varepsilon\tan\beta\sin\{2(N-1)\beta\}\\
&\quad+8\delta^2(1+\varepsilon)\sin^2\beta\tan\beta\sin\{2(N-1)\beta\}.
\end{aligned} \tag{4·6}$$

For regular lattices ($f=g=1$) (4·6) reduces to $2\cos(2N\beta)$, just as it ought to be.

For simplicity let us consider the case $\delta=0$, i.e., Case M mentioned in the introduc-

tion. In this case the eigenvalue equation becomes

$$f(\beta)\equiv\cos(2N\beta)-\varepsilon\tan\beta\sin(2N\beta)=1. \tag{4·7}$$

The eigenfrequencies are obtained from the points of intersection of the curve

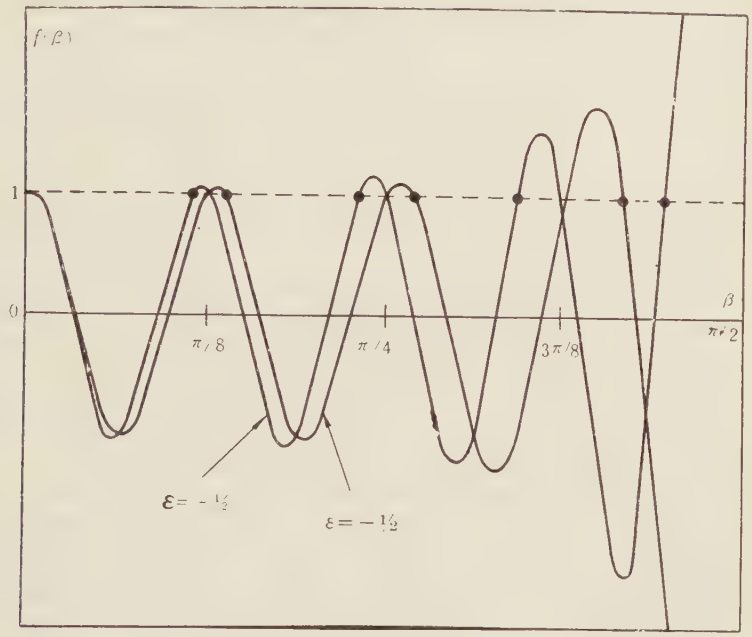

Fig. 1.

representing the function $f(\beta)$ and the line representing the constant 1. Fig. 1 shows this for the case $N=8$ and $\varepsilon=\pm\frac{1}{2}$. As is seen, primarily doubly degenerated eigenfrequencies are all split, one set of components decreasing or increasing according as $\varepsilon>0$ or $\varepsilon<0$, and the other set remaining unchanged. Moreover in the case of $\varepsilon<0$, i.e., when the impurity is lighter than the regular atom, the maximum eigenfrequency runs out of the frequency band. (In Fig. 1 we took N as even. As can readily be seen, the situation remains unchanged also in case N is odd.) Since this extra-band frequency corresponds to the "impurity level" which appears between the allowed electronic energy bands of a metal when it contains impurities, we shall call it "impurity frequency" or "impurity mode."

In order to evaluate the impurity frequency, we have merely to put $\beta=\pi/2+i\gamma$. Then (4·7) becomes

$$\cosh(2N\gamma)+\varepsilon\coth\gamma\sinh(2N\gamma)=\pm1, \tag{4·8}$$

or

$$h(\gamma)\equiv\coth(2N\gamma)+\varepsilon\coth\gamma=\pm\operatorname{cosech}(2N\gamma), \tag{4·9}$$

where the plus sign corresponds to even N, and the minus sign to odd N. If we plot

three hyperbolic functions appearing in (4·9), we can see that if $\varepsilon > 0$ (4·9) has no root and if $\varepsilon < 0$ it has one, irrespective of the odd- or evenness of N. This is in accordance with the above result.

Though the above result and that obtained by Montroll and Potts are very similar to each other, they are not identical. This is due to our adoption of the cyclic boundary condition, in contrast to the condition of fixed ends used by Montroll and Potts. It can easily be shown that their results are regained if the condition of fixed ends is used (see Appendix I). The similarity of their results and ours shows that qualitative nature of the change in distribution of eigenfrequencies is not so sensitive to the kind of boundary condition.

As can be seen from (4·7), the distribution of eigenfrequencies is independent of the position of the impurity. This is due to the fact that Trace $\boldsymbol{H}$ does not depend on the position of the impurity. This situation does not change if the number of the impurities is increased, i.e., the eigenfrequency-distribution does not in general depend on the absolute positions of the impurities, but only on their relative configuration. When the condition of fixed ends is assumed, on the contrary, the distribution of eigenfrequencies depends on the absolute positions of impurities, since there appears then in the eigenvalue equation not the trace but a particular element of $\boldsymbol{H}$ (Appendix I).

It should be noted here that in Case M, the matrices $\boldsymbol{I}+\boldsymbol{T}_2$ and $\boldsymbol{I}+\boldsymbol{T}_1$ reduce to the regular transfer matrix $\boldsymbol{I}+\boldsymbol{T}$, and there remains only one impurity matrix $\boldsymbol{I}+\boldsymbol{T}_3$. Therefore in such a case only one transfer matrix must be replaced by the impurity matrix $\boldsymbol{P}'$, which is $\boldsymbol{I}+\boldsymbol{T}_3$. In the representation in which $\boldsymbol{T}$ is diagonalized, this means that one transfer matrix $\boldsymbol{I}+\boldsymbol{\Lambda}$ is replaced by an impurity matrix $\boldsymbol{Q}'=\boldsymbol{U}^{-1}\boldsymbol{P}'\boldsymbol{U}$. Since $\boldsymbol{Q}=(\boldsymbol{I}+\boldsymbol{\Lambda})\boldsymbol{Q}'(\boldsymbol{I}+\boldsymbol{\Lambda})$, $\boldsymbol{Q}'$ is given by

$$\boldsymbol{Q}'=\begin{pmatrix} Q_{11}\,e^{-4\beta\delta} & Q_{12} \\ Q_{21} & Q_{22}\,e^{4\beta\delta} \end{pmatrix}$$

$$=\begin{pmatrix} e^{2\beta\delta}(1+i\varepsilon\tan\beta\delta) & i\varepsilon\tan\beta\delta\, e^{-2\beta\delta} \\ -i\varepsilon\tan\beta\delta\, e^{2\beta\delta} & e^{-2\beta\delta}(1-i\varepsilon\tan\beta\delta) \end{pmatrix}. \qquad (4\cdot10)$$

From now on we shall use $\boldsymbol{Q}'$ instead of $\boldsymbol{Q}$ whenever the Case M is concerned, in order to simplify the calculation. By doing so, we can easily derive, e.g., the eigenvalue equation (4·7).

§ 5. Wave forms of normal vibrations

Let us next investigate the spatial wave forms of the normal vibrations for Case M. First consider a regular lattice. In this case the equation determining the allowed initial vector is, in the representation in which $\boldsymbol{T}$ is diagonalized,

$$\{(\boldsymbol{I}+\boldsymbol{\Lambda})^N-\boldsymbol{I}\}\,\boldsymbol{X}_1'=\begin{pmatrix} e^{2N\beta\delta}-1 & 0 \\ 0 & e^{-2N\beta\delta}-1 \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=0, \qquad (5\cdot1)$$

where $\boldsymbol{X}_1' \equiv (x_1', y_1')^{\mathrm{T}} = \boldsymbol{U}^{-1}\boldsymbol{X}_1$. Since for the eigenvalues $\beta = n\pi/N$ the matrix in the above formula is null, the initial vector is here arbitrary, except for $\beta = \pi/2$, when $\boldsymbol{U}^{-1}$ does not exist. Let it be $(x_1', y_1')^{\mathrm{T}}$, then the state of the $(n+1)$-th atom is given by

$$\begin{pmatrix} e^{2ni\beta} & 0 \\ 0 & e^{-2ni\beta} \end{pmatrix} \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} e^{2ni\beta}x_1' \\ e^{-2ni\beta}y_1' \end{pmatrix}. \tag{5·2}$$

If we return to the original representation, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \{\cos(2n\beta) + \tan\beta \sin(2n\beta)\}x_1 + \sin(2n\beta)y_1/\sin 2\beta \\ -2\tan\beta \sin(2n\beta)x_1 + \{\cos(2n\beta) - \tan\beta \sin(2n\beta)\}y_1 \end{pmatrix}. \tag{5·3}$$

It is seen that the normal vibration consists of stationary harmonic waves, just as expected.

Next consider the case where the M-th atom is replaced by an impurity atom with the mass M. The equation determining the initial vector becomes

$$[(\boldsymbol{I}+\boldsymbol{A})^{N-M+1}\boldsymbol{Q}'(\boldsymbol{I}+\boldsymbol{A})^{M-2} - \boldsymbol{I}]\boldsymbol{X}_1' = 0. \tag{5·4}$$

By introducing (4·10) and using (2·4), we get

$$\{e^{2Ni\beta}(1+i\varepsilon\tan\beta) - 1\}x_1' + i\varepsilon\tan\beta e^{2(N-2M+2)i\beta}y_1' = 0, \tag{5·5}$$

$$-i\varepsilon\tan\beta e^{-2(N-2M+2)i\beta}x_1' + \{e^{-2Ni\beta}(1-i\varepsilon\tan\beta) - 1\}y_1' = 0. \tag{5·6}$$

From (5·5) and (5·6) we obtain the relations

$$y_1' = \frac{1-e^{2Ni\beta}(1+i\varepsilon\tan\beta)}{i\varepsilon\tan\beta e^{2Ni\beta}e^{-4(M-1)i\beta}}x_1', \tag{5·7}$$

and

$$y_1' = \frac{i\varepsilon\tan\beta e^{-2Ni\beta}e^{4(M-1)i\beta}}{e^{-2Ni\beta}(1-i\varepsilon\tan\beta) - 1}x_1', \tag{5·8}$$

respectively. It is easy to verify, by aid of the eigenvalue equation (4·7), that these two relations are identical. Moreover we can assert that

$$i\varepsilon\tan\beta/\{e^{-2Ni\beta}(1-i\varepsilon\tan\beta) - 1\} = -1, \text{ for } \beta = (n\pi/N)$$
$$= +1, \text{ for other roots of (4·7)} \tag{5·9}$$

Hence

$$\boldsymbol{X}_1' = C\begin{pmatrix} e^{Ni\beta}e^{-2(M-1)i\beta} \\ \pm e^{-Ni\beta}e^{2(M-1)i\beta} \end{pmatrix}, \tag{5·10}$$

where C must be chosen so that we may obtain the real vector $\boldsymbol{X}_1$ in the original representation. Thus in the presence of impurity we cannot arbitrarily choose the initial vector $\boldsymbol{X}_1$. It depends not only on the eigenfrequency but also on the absolute positions of the impurities.

The state of the atoms up to the $(M-1)$-th is given by

$$\begin{pmatrix} x'_{k+1} \\ y'_{k+1} \end{pmatrix} = (\boldsymbol{I}+\boldsymbol{A})^k \boldsymbol{X}_1' = C \begin{pmatrix} e^{(N-2M+2+2k)i\beta} \\ \pm e^{-(N-2M+2+2k)i\beta} \end{pmatrix}. \tag{5·11}$$

The states of the impurity atom itself and of the atoms that follow are given by

$$\begin{pmatrix} x'_M \\ y'_M \end{pmatrix} = \boldsymbol{Q}' \boldsymbol{X}'_{M-1} = C \begin{pmatrix} e^{Ni\beta}\ (1+i\varepsilon \tan i\beta) \pm i\varepsilon \tan i\beta e^{-Ni\beta} \\ -e^{Ni\beta}\, i\varepsilon \tan i\beta \pm e^{-Ni\beta}(1-i\varepsilon \tan i\beta) \end{pmatrix} \tag{5·12}$$

and

$$\begin{pmatrix} x'_{M+l} \\ y'_{M+l} \end{pmatrix} = (\boldsymbol{I}+\boldsymbol{A})^l \boldsymbol{X}'_M = C \begin{pmatrix} e^{(N-2l)i\beta}(1+i\varepsilon \tan i\beta) \pm i\varepsilon \tan i\beta e^{-(N-2l)i\beta} \\ -e^{(N-2l)i\beta} i\varepsilon \tan i\beta \pm e^{-(N-2l)i\beta}(1-i\varepsilon \tan i\beta) \end{pmatrix}, \tag{5·13}$$

respectively. In particular, $\boldsymbol{X}'_{N+1}$ is

$$\boldsymbol{X}'_{N+1} = C \begin{pmatrix} e^{(N-2M+2)i\beta}\left[e^{2Ni\beta}(1+i\varepsilon \tan i\beta) \pm i\varepsilon \tan i\beta\right] \\ e^{-(N-2M+2)i\beta}\left[-i\varepsilon \tan i\beta \pm (1-i\varepsilon \tan i\beta)e^{-2Ni\beta}\right] \end{pmatrix}. \tag{5·14}$$

From (4·7) and (5·9) we see that the expression in the square-bracket of the upper component is equal to 1 and that of the lower component is equal to $+1$ or -1 according to whether the sign of the last term is positive or negative. The cyclic condition is thus in fact satisfied.

The amplitude of vibration of each atom can easily be obtained from (5·10), (5·11), (5·12) and (5·13). If, for example, we put $C=1$, we obtain, again by aid of (5·9),

$$\begin{aligned}
&x_1 = 2\cos\,(N-2M+2)i\beta, \\
&\cdots\cdots\cdots\cdots \\
&x_{k+1} = 2\cos(N-2M+2+2k)i\beta, \\
&\cdots\cdots\cdots\cdots \\
&x_{M-1} = 2\cos\,(N-2)i\beta, \\
&x_M = 2\cos N i\beta, \\
&x_{M+1} = 2\cos(N-2)i\beta, \\
&\cdots\cdots\cdots\cdots \\
&x_{M+l} = 2\cos\,(N-2l)i\beta, \\
&\cdots\cdots\cdots\cdots \\
&x_{N+1} = 2\cos(N-2M+2)i\beta.
\end{aligned} \tag{5·15}$$

It is seen that a regular sinusoidal vibration continues up to the position of the impurity,

where an abrupt change in phase occurs, beyond which there continues again a sinusoidal vibration, so that X_{N+1} may coincide with X_1.

To investigate the wave form of impurity vibration corresponding to the impurity frequency, we have only to put $\beta=\pi/2+i\gamma$. Then we have, for $C=1$,

$$
\begin{aligned}
&x_1=\pm 2\cosh(N-2M+2)\gamma,\\
&x_2=\mp 2\cosh(N-2M+2+2)\gamma,\\
&\cdots\cdots\cdots\cdots\\
&x_{k+1}=\pm(-1)^k\,2\cosh(N-2M+2+2k)\gamma,\\
&\cdots\cdots\cdots\cdots\\
&x_{M-1}=\pm(-1)^{M-2}\,2\cosh(N-2)\gamma,\\
&x_M=\pm(-1)^{M-1}\,2\cosh N\gamma,\\
&x_{M+1}=\pm(-1)^M\,2\cosh(N-2)\gamma,\\
&\cdots\cdots\cdots\cdots\\
&x_{M+l}=\pm(-1)^{M+l-1}\,2\cosh(N-2l)\gamma,\\
&\cdots\cdots\cdots\cdots\\
&x_{N+1}=\pm 2\cosh(N-2M+2)\gamma. \qquad (5\cdot16)
\end{aligned}
$$

This clearly demonstrates the localized character of the impurity mode, which was already found by Montroll and Potts. The alternation of sign of the amplitude clearly indicates that the impurity vibration corresponds to the normal vibration with maximum frequency.

§ 6. Several impurities

In exactly the same manner as in § 4 we can treat the case in which several impurities are present. Here we restrict ourselves to Case M for simplicity. First let us imagine that there are two impurities with the interval $M+1$. In order to obtain the eigenvalue equation, we need only to calculate

$$\mathrm{Trace}\,\boldsymbol{H}=\mathrm{Trace}\{(\boldsymbol{I}+\boldsymbol{\Lambda})^{N-M-2}\boldsymbol{Q}'(\boldsymbol{I}+\boldsymbol{\Lambda})^{M}\boldsymbol{Q}'\}. \qquad (6\cdot1)$$

Using (4·10) and (2·4) we get

$$
\begin{aligned}
\mathrm{Trace}\,\boldsymbol{H}&=(1+\lambda_1)^{N-2}Q'^2_{11}+Q'_{12}Q'_{21}\{(1+\lambda_1)^{N-M-2}(1+\lambda_2)^{M}\\
&\quad+(1+\lambda_2)^{N-M-2}(1+\lambda_1)^{M}\}+(1+\lambda_2)^{N-2}Q'^2_{22}\\
&=2\cos(2N\beta)-4\varepsilon\tan\beta\sin(2N\beta)\\
&\quad-2\varepsilon^2\tan^2\beta[\cos(2N\beta)-\cos\{2(N-2M-2)\beta\}]=2h(\beta). \qquad (6\cdot2)
\end{aligned}
$$

Eigenfrequencies can be obtained from the points of intersection of the curve $h(\beta)$ and the line representing the constant 1. This is illustrated in Fig. 2. We see that in the present case both of the degenerated frequencies are influenced by the impurities. The

one set of them decrease or increase according as $\varepsilon > 0$ or $\varepsilon < 0$, and in the latter case the maximum frequency is pushed out of the band. The other set decreases in both

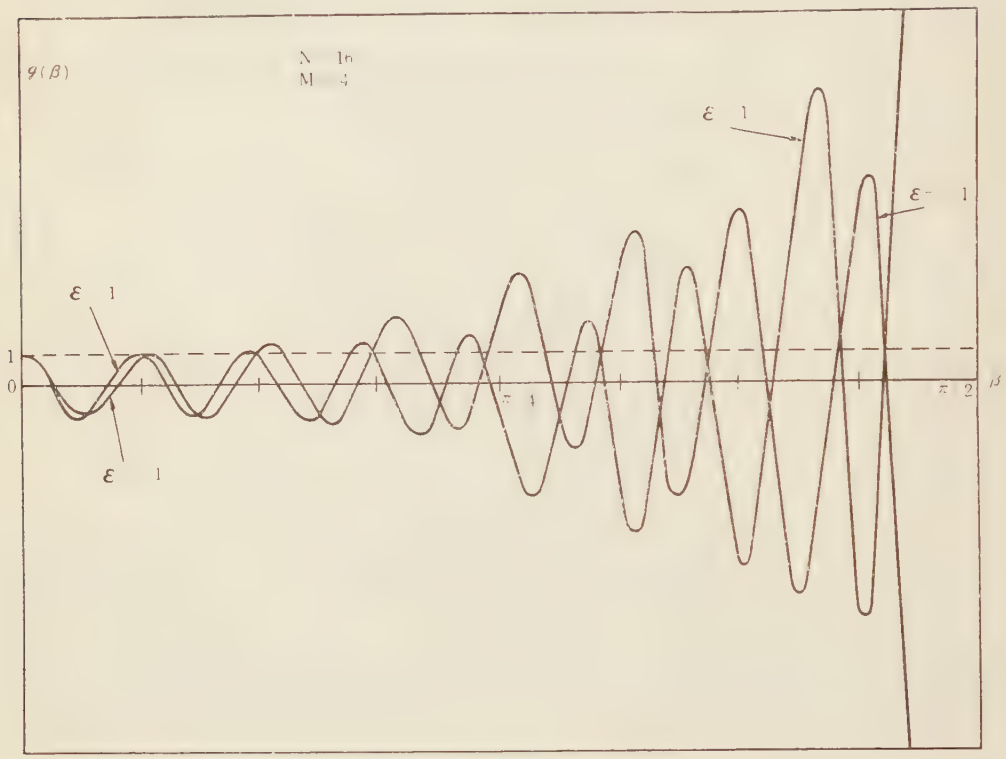

Fig. 2.

cases, the general trend of the shift being however positive when $\varepsilon < 0$ and negative when $\varepsilon > 0$, as in the case of one impurity.

If there are three impurities whose intervals are $M+1$ and $L+1$, we have, in a similar manner,

$$\begin{aligned}\text{Trace } \boldsymbol{H} = \text{Trace}\{(\boldsymbol{I}+\boldsymbol{\Lambda})^{N-M-L-2}\boldsymbol{Q}'(\boldsymbol{I}+\boldsymbol{\Lambda})^{L}\boldsymbol{Q}'(\boldsymbol{I}+\boldsymbol{\Lambda})^{M}\boldsymbol{Q}'\} \\ = 2\cos(2N_i\beta) - 6\varepsilon\tan_i\beta\sin(2N_i\beta) \\ -2\varepsilon^2\tan^2{}_i\beta\{3\cos(2N_i\beta) - \cos\{2(N-2M-2)_i\beta\} - \cos\{2(N-2L-2)_i\beta\} \\ -\cos\{2(N-2L-2M-4)_i\beta\}\} - 2\varepsilon^3\tan^3{}_i\beta\{\sin(2N_i\beta) \\ -\sin\{2(N-2M-2)_i\beta\} - \sin\{2(N-2L-2)_i\beta\} \\ -\sin\{2(N-2L-2M-4)_i\beta\}\}. \qquad (6\cdot3)\end{aligned}$$

Following the same procedure, we can obtain the eigenvalue equation for arbitrary number and arbitrary relative configuration of impurities. Since, however, in most of the practical cases we cannot locate the impurities, and it is as much as we can do to

know their statistical distribution, it would be futile to pursue the above-mentioned line of argument. We now therefore turn directly to the statistical case.

§ 7. Random lattices

By a random lattice we mean here one in which each atom in a regular lattice is replaced with a definite probability by an impurity atom of fixed kind, independently of each other. In Case M which we treat here this means that each transfer matrix $\boldsymbol{I}+\boldsymbol{T}$ composing the matrix $\boldsymbol{H}$ is replaced randomly by the impurity matrix $\boldsymbol{P}'$. It is clear that also in such a case the eigenvalue equation is given by Trace $\boldsymbol{H}=2$.

At first let us calculate the average value of Trace $\boldsymbol{H}$. In order to do this, it is necessary first to average the expression such as (6·2) over all possible relative configurations of the impurities, i.e., ever all possible values of e.g., M or L, and then to average the obtained expressions over all the possible numbers of the impurities.

In averaging for example (6·2) or (6·3) over all values of M and L, we must notice that M and L enter the formula only as the arguments of sine or cosine functions. Except when β is very close to zero or $\pi/2$, these terms will oscillate rapidly as L or M varies, and vanish on the average. When β is very close to zero or $\pi/2$, cosine functions do not vary appreciably even if M or L changes considerably, and consequently do not vanish in the averaging operation, while sine functions are nearly zero themselves and do not contribute to the average. The distance between zero or $\pi/2$ and the β-value at which the average value of the cosine functions becomes negligible compared with unity depends on the range of variation of M and L. If the probability of occurrence of the impurity is sufficiently small, the average value of M or L will be of the order of N. Then cosine function will vanish on the average when β is apart from zero or $\pi/2$ by the distance of the order π/N. Thus when N is large, the domains in which the average value does not vanish will be very narrow. Excepting these narrow domains at the two extremities of the band, we obtain therefore the approximate formulas

$$\langle \text{Trace } \boldsymbol{H} \rangle_{AV} = (1+\lambda_1)^{N-2} Q'^{2}_{11} + (1+\lambda_2)^{N-2} Q'^{2}_{22}, \tag{7·1}$$

and

$$\langle \text{Trace } \boldsymbol{H} \rangle_{AV} = (1+\lambda_1)^{N-3} Q'^{3}_{11} + (1+\lambda_2)^{N-3} Q'^{3}_{22}, \tag{7·2}$$

respectively for (6·2) and (6·3), since the terms which contain M and L originate exclusively from the terms containing the non-diagonal elements of $\boldsymbol{Q}'$. The same reasoning will lead in general to the formula

$$\langle \text{Trace } \boldsymbol{H} \rangle_{AV} = (1+\lambda_1)^{N-n} Q'^{n}_{11} + (1+\lambda_2)^{N-n} Q'^{n}_{22}, \tag{7·3}$$

for n impurities.

Next, consider the average over the number of the impurities. Let the probability that an atom of the lattice is a regular one be p and the probability that it is an impurity be q. Then the probability that there are n impurities is given by the binomial distribution. Hence we have

$$\langle \text{Trace } \boldsymbol{H} \rangle_{AV} = \sum_{n=0}^{N} {}_{N}C_{n}\, p^{N-n} q^{n} [\,(1+\lambda_1)^{N-n}\, Q'_{11}{}^{n} + (1+\lambda_2)^{N-n}\, Q'_{22}{}^{n}\,]. \tag{7·4}$$

In the limiting case where $N \to \infty$ but the proportion of the impurities remains finite, the binomial distribution can be approximated by the Poisson distribution. If we let the average number of the impurities be $N\eta$, this is given by $P(n) = \exp(-N\eta) \cdot (\eta N)^n / n!$. (7·4) then becomes, by aid of (2·4) and (4·10),

$$\langle \text{Trace } \boldsymbol{H} \rangle_{AV} = \sum_{n=0}^{N} e^{-N\eta} \frac{(\eta N)^n}{n!} [\,(1+\lambda_1)^{N-n}\, Q'_{11}{}^{n} + (1+\lambda_2)^{N-n}\, Q'_{22}{}^{n}\,]$$

$$\simeq 2 \cos\{2N\beta + N\eta\varepsilon \tan \beta\}. \tag{7·5}$$

Consequently, the eigenvalue equation determining the average eigenvalues is

$$k(\beta) \equiv 2N\beta + N\eta\varepsilon \tan \beta = 2n\pi, \quad n = 1, 2, \cdots, N/2 \text{ or } (N-1)/2, \tag{7·6}$$

whose roots are given by the points of intersection of the curve $k(\beta)$ and the parallel lines representing the constants $2n\pi$. It may be seen that all eigenvalues decrease or increase according as $\varepsilon > 0$ or $\varepsilon < 0$. Since, however, the equation (7·5) is not valid even approximately near the ends of the band, we cannot infer from this whether there appears any impurity frequency. Except for this point, (7·6) shows that in the random lattice with a few impurities the average distribution of the eigenfrequencies is essentially the same as in the lattice with one or two impurities.

We have thus clarified to some extent the average behavior of the random lattice. The behavior of the actual lattice will exhibit some fortuitous deviation from this average behavior. We observe here, however, that we can regard the distribution of the impurities on the lattice as forming a stationary stochastic process. Then, through the ergodic nature of the stationary stochastic process and the invariance of Trace $\boldsymbol{H}$ under translation, we can conclude that the average value of Trace $\boldsymbol{H}$ obtained above and the distribution of eigenfrequencies given by it give the exact value of Trace $\boldsymbol{H}$ and the exact distribution, respectively, for at least almost all actual lattices. Thus the conclusion obtained above is valid not only on the average but also for almost all individual samples of the lattice.

Such a simple situation is of course peculiar to the cyclic boundary condition. However, since the qualitative feature of the eigenfrequency-distribution seems not likely to depend so crucially on the type of the boundary condition, as mentioned in § 4, the above conclusion may be considered to be valid at least qualitatively irrespective of the kind of the boundary condition.

§ 8. A hole in a regular lattice

In this section we treat, as a simple and somewhat realistic application of our method, the case in which one "hole," i.e., one impurity atom whose mass is zero, exists. First let us provisionally derive the eigenvalue equation for this case formally putting $M = 0$, or $\varepsilon = -1$, in equation (4·6). We have (somewhat modifying the form of the equation

for the sake of later discussion),

$$\left[\tfrac{1}{2}\,\mathrm{Trace}\,\boldsymbol{H}\right]_{\varepsilon=-1}=2\cos(2N\beta)+\tan\beta\left[\sin(2N\beta)-2\delta\sin\{2(N-1)\beta\}\right]$$
$$=\cos\{2(N-1)\beta\}-(2g-1)\tan\beta\sin\{2(N-1)\beta\}=1. \qquad (8\cdot1)$$

This procedure is, however, physically unjustifiable, because it is unreasonable that there remains finite force-constant between impurity and its neighbors even after the impurity has vanished. It will be more realistic to consider that this force-constant becomes also zero. We then have to take into account the force between the "next-nearest" atoms neighboring the hole, in order not to break the lattice into two independent portions. Denote this "next-nearest" force-constant by λ'. Then, as can easily be inferred, we have here to replace the matrix $\boldsymbol{P}$ by a new matrix

$$\boldsymbol{T}'=\begin{pmatrix} 1 & & 1 \\ -\dfrac{m\omega^2}{\lambda} & & \dfrac{\lambda'}{\lambda}-\dfrac{m\omega^2}{\lambda} \end{pmatrix}\begin{pmatrix} 1 & & 1 \\ -\dfrac{m\omega^2}{\lambda'} & & \dfrac{\lambda}{\lambda'}-\dfrac{m\omega^2}{\lambda'} \end{pmatrix}. \qquad (8\cdot2)$$

By putting $\lambda/\lambda'=g'$, this can be written as

$$\boldsymbol{T}'=(\boldsymbol{I}+\boldsymbol{T})^2+(g'-1)(\boldsymbol{I}+\boldsymbol{T})\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}(\boldsymbol{I}+\boldsymbol{T}). \qquad (8\cdot3)$$

Calculating $\mathrm{Trace}\,\boldsymbol{H}=\mathrm{Trace}\left|(\boldsymbol{I}+\boldsymbol{T})^{N-2}\boldsymbol{T}'\right|$ and putting it equal to 2, we obtain the eigenvalue equation

$$\cos\{2(N-1)\beta\}-(g'-1)\tan\beta\sin\{2(N-1)\beta\}=1. \qquad (8\cdot4)$$

Comparing this with (8·1), we see that in both treatment the parameter representing the change of the force-constant plays just the same role. That is, we can obtain the same result by the first treatment which is physically unjustifiable as by the second reasonable treatment if we only use twice the next-nearest force-constant formally as the force-constant between the hole and the neighboring atoms. We also see from (8·4) that we obtain the same result if we consider an impurity of mass $M=m\lambda/\lambda'$ instead of the hole, leaving the force-constant unchanged. This means that the relaxation of coupling and the introduction of larger mass give the same influence on the eigenfrequencies, just as expected.

§ 9. One impurity atom in diatomic lattice

We consider in this section, as the second example of the application of our method, a diatomic lattice with one impurity atom. In the diatomic lattice considered in § 3, we replace one regular atom in the odd position by an impurity atom with the mass $M'=M(1+\varepsilon)$. Then it is seen that we have to replace a transfer matrix $\boldsymbol{I}+\boldsymbol{T}'$ in $\boldsymbol{H}$ by a matrix

$$\boldsymbol{I}+\boldsymbol{T}''=\boldsymbol{I}+\boldsymbol{T}'+\varepsilon\begin{pmatrix}0 & 0\\ -\frac{M\omega^2}{\mu}+\frac{mM\omega^4}{\lambda\mu} & -\frac{M\omega^2}{\mu}-\frac{M\omega^2}{\lambda}+\frac{mM\omega^4}{\lambda\mu}\end{pmatrix}$$
$$=\boldsymbol{I}+\boldsymbol{T}'+\Delta\boldsymbol{T}'. \tag{9.1}$$

Trace $\boldsymbol{H}$ then becomes

$$\text{Trace}\,\boldsymbol{H}=\text{Trace}[(\boldsymbol{I}+\Lambda')^N+(\boldsymbol{I}+\Lambda')^{N-1}\boldsymbol{U}'^{-1}\Delta\boldsymbol{T}'\boldsymbol{U}'], \tag{9·2}$$

where $\boldsymbol{U}'$ is the matrix which diagonalizes $\boldsymbol{T}'$, and Λ' is the matrix so diagonalized.

For simplicity, let us consider the case in which M and m, on the one hand, and μ and λ, on the other, do not differ appreciably from each other, that is, putting $M=(1+\tilde{\xi})m$ and $\mu=(1+\eta)\lambda$, we neglect the terms of higher powers of $\tilde{\xi}$ and η than the first. With ${}_i\beta$ given by (3·5), we have

$$m\omega^2\simeq 2\lambda(2+\tilde{\xi}+\eta)/(1+\tilde{\xi})\cdot\sin^2({}_i\beta/2), \tag{9·3}$$

and hence

$$(\boldsymbol{U}'^{-1}\Delta\boldsymbol{T}'\boldsymbol{U}')_{11}=(\boldsymbol{U}'^{-1}\Delta\boldsymbol{T}'\boldsymbol{U}')_{22}$$
$$\simeq-(\varepsilon/2i)(2+\tilde{\xi}\sec{}_i\beta)\cot({}_i\beta/2)\exp(2i{}_i\beta), \tag{9·4}$$

for the acoustical branch, and

$$m\omega^2\simeq 2\lambda(2+\tilde{\xi}+\eta)/(1+\tilde{\xi})\cdot\cos^2({}_i\beta/2), \tag{9·5}$$

and hence

$$(\boldsymbol{U}'^{-1}\Delta\boldsymbol{T}'\boldsymbol{U}')_{11}=(\boldsymbol{U}'^{-1}\Delta\boldsymbol{T}'\boldsymbol{U}')_{22}$$
$$\simeq-(\varepsilon/2i)(-2+\tilde{\xi}\sec{}_i\beta)\cot({}_i\beta/2)\exp(2i{}_i\beta) \tag{9·6}$$

for the optical branch. The eigenvalue equation thus becomes

$$\cos(2N{}_i\beta)-\varepsilon(1+(\tilde{\xi}/2)\sec{}_i\beta)\tan({}_i\beta/2)\cdot\sin(2N{}_i\beta)=1, \tag{9·7}$$

and

$$\cos(2N{}_i\beta)+\varepsilon(1-(\tilde{\xi}/2)\sec{}_i\beta)\cot({}_i\beta/2)\cdot\sin(2N{}_i\beta)=1, \tag{9·8}$$

for acoustical and optical branches, respectively. The interval of the variable ${}_i\beta$ is $(0,\pi/2)$ for both of these equations. Since, however, if we replace the variable ${}_i\beta$ in (9·8) by $\pi-{}_i\beta$ we obtain as the function of ${}_i\beta$ exactly the same formula as (9·7), we may use (9·7) in the extended interval $(0,\pi)$ as the eigenvalue equation for both branches. ω is then given, according to (9·3) and (9·5), by

$$\omega^2=(\lambda+\mu)(1/m+1/M)\sin^2({}_i\beta/2). \tag{9·9}$$

Let us use N instead of $2N$ and 2β instead of β, in order to compare the above results with that for monatomic case. Then (9·8) and (9·9) become

$$\cos(2N\beta) - \varepsilon(1 + (\xi/2)\sec 2\beta)\tan\beta\sin(2N\beta) = 1, \qquad (9\cdot10)$$

and

$$\omega^2 = (\lambda+\mu)(1/m+1/M)\sin^2\beta, \quad 0 \leq \beta \leq \pi/2, \qquad (9\cdot11)$$

respectively. This is the formula that corresponds to (4·7) obtained previously.

From (9·10) we see that, owing to the term containing $\sec 2\beta$, it becomes possible for impurity frequencies split from the top of lower branch or from the bottom of the upper branch. The situations are tabulated below. In all cases two degenerated frequencies split owing to the impurity, and one of them decreases or increases, sometimes being pressed out of the band. The direction of its shift is indicated by + or − according as it increases or decreases. $\pi/4$ corresponds or does not correspond to a non-perturbed eigenvalue according as $N/2$ is even or odd. The two cases are tabulated separately. The symbols ↑ and ↓ mean respectively that the last and the first eigenfrequency in the band is pressed out of the band. One of the two frequencies at $\pi/4$ is regarded as belonging to the lower band and the other to the upper band. We see that the eigenfrequency corresponding to $\pi/2$ becomes an impurity frequency when $\varepsilon \cdot 0$, just as expected from the result obtained in monatomic case.

Branch→			acoustical branch		optical branch	
Case ↓			$0 \leq \beta < \pi/4$	$\beta = \pi/4$	$\beta = \pi/4$	$\pi/4 < \beta \leq \pi/2$
$\varepsilon > 0$	$\xi > 0$	$N/2$ odd	−			+ −
		$N/2$ even	−	−	+	+ −
	$\xi < 0$	$N/2$ odd	− + ↑			↓ −
		$N/2$ even	− +	↑	↓	−
$\varepsilon < 0$	$\xi > 0$	$N/2$ odd	+ ↑			↓ − + ↑
		$N/2$ even	+	↑	↓	− + ↑
	$\xi < 0$	$N/2$ odd	+ −			+ ↑
		$N/2$ even	+ −	−	+	+ ↑

§ 10. Influence of the next-nearest interaction

Finally we attempt to generalize our method by taking the force between next-nearest atoms into account.

Let the force-constant between the $(i-1)$-th and the i-th atoms be λ_i, and that between the $(i-2)$-th and the i-th atoms be μ_i. Then, putting the time factor as exp $(i\omega t)$, we have the equation of motion:

$$-m_i\omega^2 x_i = -\lambda_i(x_i - x_{i-1}) + \lambda_{i+1}(x_{i+1} - x_i)$$
$$-\mu_i(x_i - x_{i-2}) + \mu_{i+2}(x_{i+2} - x_i) \qquad (10\cdot1)$$

By introducing the variables $y_i=x_{i-1}-x_i$, $z_i=x_{i-1}-x_i$ and $v_i=z_{i-1}+z_i$, this can be written as

$$\boldsymbol{X}_{i+1}\equiv\begin{pmatrix} -y_{i+1} \\ x_{i+1} \\ z_{i+1} \\ v_{i+1} \end{pmatrix}=\begin{pmatrix} -1-\dfrac{\lambda_{i+1}}{\mu_{i+2}} & \dfrac{m_i\omega^2}{\mu_{i+2}} & \dfrac{\lambda_i}{\mu_{i+2}} & \dfrac{\mu_i}{\mu_{i+2}} \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{X}_i\equiv\boldsymbol{S}_i\boldsymbol{X}_i. \tag{10·2}$$

First consider a regular lattice. Putting $m_i=m$, $\lambda_i=\lambda$, $\mu_i=\mu$, and $\lambda/\mu=f$, we have

$$\boldsymbol{S}_i=\boldsymbol{S}=\begin{pmatrix} -(1+f) & m\omega^2/\mu & f & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{10·3}$$

Putting $m\omega^2=4\lambda\sin^2\beta+4\mu\sin^2 2\beta$, we get $\exp(\pm 2i\beta)$ and $-\exp(\pm 2i\gamma)$ as the eigenvalues of $\boldsymbol{S}$, where $\cos 2\gamma=f/2+\cos 2\beta$. The eigenvalue equation for ω is given by $|\boldsymbol{S}^N-1|=0$, which is equivalent to $|\boldsymbol{D}^N-1|=0$, where $\boldsymbol{D}$ is the diagonal form of $\boldsymbol{S}$. It may be supposed that $f>4$ in ordinary cases in which next-nearest interaction is sufficiently weak compared with the nearest neighbor interaction. Then γ becomes imaginary, and the eigenvalue equation reduces to $\cos(2N_i\beta)=1$. This is the well-known result.

If an impurity which does not affect the force-constants is introduced, a transfer matrix (say $(j+1)$-th) comes out to be different from the regular one $\boldsymbol{S}$. Putting $M=m(1+\varepsilon)$, this can be written as

$$\boldsymbol{S}'\equiv\boldsymbol{S}+\Delta\boldsymbol{S}\equiv\boldsymbol{S}+4\varepsilon(f\sin^2\beta+\sin^2 2\beta)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10·4}$$

The eigenvalue equation is then given by $|\boldsymbol{D}^N+\boldsymbol{D}^{N-(j+1)}\boldsymbol{W}^{-1}\Delta\boldsymbol{S}\boldsymbol{W}\boldsymbol{D}^j-1|=0$, which reduces to

$$\begin{aligned}&(\cos(2N_i\beta)-1)\,(\cos(2N_i\gamma)\pm 1)\\ &-\varepsilon\frac{f\sin^2\beta+\sin^2 2\beta}{\cos 2\beta+\cos 2\gamma}\Big\{\frac{\sin 2N_i\gamma}{\sin 2\gamma}(\cos(2N_i\beta)-1)\\ &\qquad+\frac{\sin 2N_i\beta}{\sin 2\beta}(\cos(2N_i\gamma)\pm 1)\Big\}=0,\end{aligned} \tag{10·5}$$

where $\boldsymbol{W}$ is the matrix which diagonalizes $\boldsymbol{S}$ into $\boldsymbol{D}$. Plus sign corresponds to odd N, and minus sign to even N. From this equation we can conclude that the manner of influence of the impurity on the eigenfrequencies is similar to that in the nearest-neighbor approximation already studied, as far as the next-nearest interaction is sufficiently

weak compared with the nearest-neighbor interaction.

If we take into account the change of force-constants due to impurity, the calculation will become considerably complicated, so that we shall not proceed further. The above argument suggests however the possibility of applying our method effectively to the problems which require more delicate treatment.

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Appendix I

Let us derive the eigenvalue equation under the condition of fixed ends. First consider the case of regular lattice. Since the displacement of the first atom x_1 is always zero, the component of the vector $\boldsymbol{X}_{N+1}$ is given by

$$\left.\begin{aligned} x_{N+1} &= h_{12}\ y_1, \\ y_{N+1} &= h_{22}\ y_1, \end{aligned}\right\} \tag{A1}$$

where h_{ij}'s are the elements of $\boldsymbol{H}$. We should have $x_{N+1}=0$ irrespective of the value of y_1. Hence it is required that $h_{12}=0$. h_{12} can be calculated with the aid of (2·4) and (2·5) as

$$h_{12}=\sin(2N\beta)/\sin 2\beta. \tag{A2}$$

The eigenvalue equation is therefore given by

$$\sin(2N\beta)=0, \tag{A3}$$

whose roots are $\beta=n\pi/2N$ $(n=1, 2, \cdots, N)$. Now the number of the eigenfrequencies is twice as large as that in the case of cyclic boundary condition, each frequency being however non-degenerate.

Next consider the case in which one regular atom (say the M-th) is replaced by an impurity with mass M. Then

$$\boldsymbol{H}=\boldsymbol{U}(\boldsymbol{I}+\Lambda)^{N-M+1}\boldsymbol{Q}'(\boldsymbol{I}+\Lambda)^{M-2}\boldsymbol{U}^{-1}, \tag{A4}$$

whose element h_{12} is calculated, by using (2·5), as

$$\begin{aligned} h_{12}=(\lambda_2-\lambda_1)^{-1}[&-Q'_{11}(1+\lambda_1)^{N-1}+Q'_{12}(1+\lambda_1)^{N-M+1}(1+\lambda_2)^{M-2} \\ &-Q'_{21}(1+\lambda_1)^{M-2}(1+\lambda_2)^{N-M+1}+Q'_{22}(1+\lambda_2)^{N-1}. \end{aligned}$$

From this we see that the eigenvalue equation depends on the position of the impurity atom.

If we consider the case treated by Montroll and Potts, we have only to replace N in the above equation anew by $2N$, and put $M=N+1$. Then we obtain

$$h_{12}=-4\sin 2\beta\sin(4N\beta)+8\varepsilon\sin^2\beta\{1-\cos(4N\beta)\}. \tag{A5}$$

Consequently, the eigenvalue equation is given by

$$\sin(4N_i\tilde{\beta})\cot\tilde{\beta}=\varepsilon\{1-\cos(4N_i\tilde{\beta})\}. \tag{A6}$$

The roots of this equation can be obtained from the points of intersection of two curves representing the functions on both sides of (A6). The result comes out to be just the same as that Montroll and Potts have obtained.

In the same manner, one may obtain the eigenfrequency-distribution for the case of several impurities. Since, however, in the present case we have to do with a particular matrix element of $\boldsymbol{H}$ instead of its trace, calculations become more complicated, and moreover the result depends on the absolute positions of the impurities, so that we cannot secure such a perspective result as was obtained in the text for the case of cyclic boundary condition.

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Note added in proof

In the course of publication one of the authors (H) found that in Case M an exact eigenvalue equation for the random monatomic lattice can be obtained. It is

$$\cos[N\cos^{-1}(\cos 2\tilde{\beta}-\eta\varepsilon\tan\tilde{\beta}\sin 2\tilde{\beta})]=1.$$

In the limit $N\to\infty$ this gives for the density $\rho(\omega)$ of eigenfrequencies

$$\rho(\omega)=2/\pi(\omega_0^2-\omega^2)^{1/2} \quad \text{for} \quad \omega\leqq\omega_0,$$

$$=0 \quad \text{for} \quad \omega>\omega_0,$$

where $\omega_0=\{4\lambda/m(1+\eta\varepsilon)\}^{1/2}$, a reasonable result inasmuch as $m(1+\eta\varepsilon)$ is the average mass of the atom. When $\varepsilon<0$ there appear the modes whose frequencies lie outside the original band $(0<\omega\leqq(4\lambda/m)^{1/2})$. It can be shown that all of them have more or less localized character.

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Cluster Development Method in the Quantum Mechanics of Many Particle System, I

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In order to investigate the many particle system by the variational principle, the cluster development method proposed by Jastrow is systematically formulated. It is a direct generalization of the Hartree method, in which two particle functions are included in the trial function to represent the correlations between particles. The energy expectation value is expanded in a series, each term of which is expressed in cluster integrals. It is found that the series converges rapidly when the correlations are marked only for small particle separations, so that this method is expected to be useful especially for the system of particles interacting through short range potentials with repulsive cores.

§ 1. Introduction

The Hartree method for a many particle system, in which each particle is supposed to move independently and to be influenced by other particles only through their average field of force, has been an important approximation and successful in various problems. However, in some cases where the correlations between particles play an important role, this simple approximation is no longer valid.

In one case, when the range of the interaction between particles is long compared with the mean distance between them, many particles may be coupled together to move in some organized mode of motion as a whole, as is seen in the typical example of the plasma oscillation in a dense electron gas. The recent study by Tomonaga[1] made the mechanism of this so-called collective motion quite clear.

In another case, when the interaction is strong but of short range, the actual potential determined by the instantaneous configuration of the other particles fluctuates from the average one considerably, and the image of the independent particle motion becomes far from the truth. Such a situation occurs especially when the interaction potential has a repulsive core at short distance as in liquid helium or in the atomic nucleus. (In the latter case the existence of the repulsive core seems very likely from the analysis of the nucleon-nucleon scattering,[2] the binding energy difference between H^3 and He^3,[3] and so forth.) In fact, if we apply the Hartree method to these systems, the resulting average potential diverges to infinity. Actually, the wave function of the system must vanish in the region of the configuration space where at least two particles are found within the hard core range. In such cases we have to take into account this strong short range correlation in some way.

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If we can construct a theory for this picture, it is expected that the saturation property of nuclear forces will be explained. Of course it is not the case that the saturation occurs with the close packing of hard cores, because the core itself is much smaller than the observed mean distance between nucleons. However, as the wave function must vanish in the cores, it is bent strongly in the neighborhood of the cores. If the particles tend to collapse under the attraction of the outer part of the potential, the curvature of the wave function becomes quite large and the kinetic energy of the system much increases, which will prevent the collapse.

It has been another mystery in the fundamental properties of nuclei that the shell model based on the independent particle picture is, in some respects, a good approximation in spite of the strong interactions between the nucleons. We know already that the potential effective to the internal motion is reduced from the original one at the long range part after the separation of the collective motion. In the same manner, if we can eliminate the effect of the strong parts of the potentials that cause the short range correlations, the residual potentials become weak, and this may be an important clue to solve the problem.

To construct the theory along this line, as opposed to the long range correlations separated as the collective motions, it seems to be a good approximation to take into account the quantities referring successively to the single particle, two particles, three particles, and so forth, because the probability of finding particles in such a small range at the same time is expected to decrease rapidly with their number.

Among the investigations of this kind Brueckner and his collaborators[4] built up an interesting method which was critically developed by Bethe.[5] In their method all the two particle correlations, as well as some average effects of more than two particles, are taken into account. This may be a good approximation and seems to be a promising approach.

The more orthodox approach based on the variational principle was considered by Drell and Huang.[6] They took the trial function of the system as the usual Slater determinant, composed of single particle functions, multiplied by a product of two particle functions which vanishes in the cores so as to prevent the particles from entering into the cores. Also Dingle[7] tried a similar approach to liquid helium. Since this is a Bose system, only the product of two particle functions was taken as the trial function for the ground state. However, the calculation of the energy expectation value with these trial functions became very difficult, since the integrals did not separate. To overcome these difficulties Jastrow[8] proposed to apply the cluster development method well-known in the classical theory of an imperfect gas. However, he overlooked several terms and calculated only the kinetic energy of a hard sphere gas. In the present series of papers starting from the systematic formulation of this method we shall attempt to develop the theory further and discuss various interesting problems including the above mentioned ones.

To illustrate the method most clearly we take here a simple trial function for a fermion system. Its generalization will be given in the later part of this series. In § 2

we shall show that the calculation of the energy expectation value with this trial function reduces to the evaluation of an integral, called a (generalized) normalization integral. In § 3 we shall break down this complicated integral into cluster integrals in a manner similar to the decomposition of the partition function of an imperfect gas. In § 4, after rearrangement of the terms, the normalization integral will be expressed by a series each term of which is a product of the cluster integrals. The ratio of any term to the preceding one is of the order of $N\omega/\Omega$, where ω is the volume within which the correlation of two particles is strong, Ω is the volume of the system, and N is the total number of particles. When correlations are short compared with the mean distance between particles, $N\omega/\Omega$ is small and the series will rapidly converge. Using the technique given in § 2 the final formula for the energy expectation value will be derived in § 5. Discussion is given in § 6. In the appendix we shall show that the formula for a boson system can easily be derived by making certain modifications to the fermion results.

§ 2. The generalized normalization integral

Let us consider a system of N Fermi particles of mass m and denote their space, spin and isotopic spin coordinates by x_i, $(i=1, \cdots, N)$, their space coordinates alone by $\boldsymbol{r}_i$. The Hamiltonian of the system is assumed to be

$$H=-\hbar^2/2m \cdot \sum_i \nabla_i^2+\sum_{(i,j)} V(x_i, x_j), \tag{1}$$

where the interaction potential $V(x_i, x_j)$ may have a repulsive core for small value of the separation $r_{ij}=|r_i-r_j|$ and may also have an exchange character. We shall use the notation $\sum_{(i,j)}$, $\sum_{(i,j,k)}$, $\cdots$, and $\prod_{(i,j)}$, $\prod_{(i,j,k)}$, $\cdots$, for the sums and the products over all the combinations and $\sum_{i,j}$, $\sum_{i,j,k}$, $\cdots$, for the sums over all the permutations from N objects.

As the trial function for the ground state of the system, we first construct the usual Slater determinant with single particle functions $\varphi_i(x)$, $(i=1, \cdots, N)$ and then multiply a product of two particle functions $f(r_{ij})$ to represent the correlation between particles:

$$\Psi(x_1, \cdots, x_N)=\Big[\prod_{(i,j)} f(r_{ij})\Big] \cdot (N!)^{-1/2} \sum_{\binom{1 \cdots N}{p_1 \cdots p_N}} \varepsilon_P \prod_i \varphi_{p_i}(x_i), \tag{2}$$

where the summation is taken over all the permutations $P=\binom{1 \cdots N}{p_1 \cdots p_N}$. ε_P is $+1$ or -1 according as P is even or odd. $f(r_{ij})$ tends to unity when the distance between particles r_{ij} becomes large, and to zero when the two particles are within the core. For simplicity we have assumed that the two particle function $f(r_{ij})$ depends only on the particle distance irrespective of the relevant single particle states. Further we restrict the single particle functions $\varphi_i(x)$ to be orthonormal.

Our purpose is to obtain the energy expectation value under the trial function Ψ. For this purpose we consider a generalized normalization integral

$$I(\beta) = \sum_{\binom{1\cdots N}{p_1\cdots p_N}} \varepsilon_P \int \prod_i \varphi^*_{p_i}(x_i) \cdot \prod_{(i,j)} f^*(r_{ij}) f(r_{ij}) \, e^{\beta H_{ij}(x_i, x_j)} \cdot \prod_{(i,j,k)} e^{\beta H_{ijk}(x_i, x_j, x_k)} \cdot \prod_i \varphi_i(x_i) \, e^{\beta H_i(x_i)} \cdot \prod_i dx_i \,. \tag{3}$$

Evidently $I(0)$ is the usual normalization integral (Ψ, Ψ), and

$$\partial I(\beta)/\partial\beta|_{\beta=0} = ((N!)^{1/2}\Psi, \{\sum_i H_i(x_i) + \sum_{(i,j)} H_{ij}(x_i, x_j) + \sum_{(i,j,k)} H_{ijk}(x_i, x_j, x_k)\} \cdot \prod_{(i,j)} f(r_{ij}) \cdot \prod_i \varphi_i(x_i)) \,.$$

If we take

$$H_i(x) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \varphi_i(x)}{\varphi_i(x)} \,,$$

$$H_{ij}(x, x') = -\frac{\hbar^2}{m} \left\{ \frac{\nabla'^2 f(|\boldsymbol{r} - \boldsymbol{r}'|)}{f(|\boldsymbol{r} - \boldsymbol{r}'|)} - \frac{\nabla\varphi_i(x) \cdot \nabla f(|\boldsymbol{r} - \boldsymbol{r}'|)}{\varphi_i(x) f(|\boldsymbol{r} - \boldsymbol{r}'|)} + \frac{\nabla'\varphi_j(x') \cdot \nabla' f(|\boldsymbol{r} - \boldsymbol{r}'|)}{\varphi_j(x') f(|\boldsymbol{r} - \boldsymbol{r}'|)} \right\} + V(x, x') \,,$$

$$H_{ijk}(x, x', x'') = -\frac{\hbar^2}{m} \sum_{x, x', x'' \text{ cyclic}} \frac{\nabla' f(|\boldsymbol{r} - \boldsymbol{r}'|) \cdot \nabla' f(|\boldsymbol{r}' - \boldsymbol{r}''|)}{f(|\boldsymbol{r} - \boldsymbol{r}'|) f(|\boldsymbol{r}' - \boldsymbol{r}''|)} \,, \tag{4}$$

where the prime on ∇ means that it operates on the coordinate $\boldsymbol{r}'$, then the energy expectation value is given by

$$\langle H \rangle = \frac{(\Psi, H\Psi)}{(\Psi, \Psi)} = \frac{\partial}{\partial\beta} \ln I(\beta)|_{\beta=0} \,. \tag{5}$$

Note that all we need are the value and the derivative of $I(\beta)$ for $\beta = 0$, so that we have only to calculate in the arbitrary small neighborhood of $\beta = 0$. Hence, the order of the product in (3) is not important even if the $V(x, x')$'s have an exchange character so that the $H_{ij}(x, x')$'s do not commute. Also no ambiguity arises in the definition of $I(\beta)$ from the product over the combinations, since we have taken in (4)

$$H_{ij}(x, x') = H_{ji}(x', x) \,,$$

$$H_{ijk}(x, x', x'') = H_{jik}(x', x, x'') = \cdots \,. \tag{6}$$

For convenience, we suppress β by putting

$$\psi_i(x) = \varphi_i(x) \, e^{\beta H_i(x)} \,,$$

$$g_{ij}(x, x') = f^*(r_{ij}) f(r_{ij}) \, e^{\beta H_{ij}(x, x')} \,,$$

$$g_{ijk}(x, x', x'') = e^{\beta H_{ijk}(x_i, x_j, x_k)} \,. \tag{7}$$

Then, (3) can be written as

$$I = \sum_{\binom{1\cdots N}{p_1\cdots p_N}} \varepsilon_P \int \prod_i \varphi^*_{p_i}(x_i) \cdot \prod_{(i,j)} g_{ij}(x_i, x_j) \cdot \prod_{(i,j,k)} g_{ijk}(x_i, x_j, x_k) \cdot \prod_i \psi_i(x_i) \cdot \prod_i dx_i \,. \tag{8}$$

All the problems are now reduced to the calculation of I which we shall simply call the normalization integral.

§ 3. Decomposition of the normalization integral to cluster integrals

In contrast to the Hartree approximation, the integral (8) does not separate because of the appearances of the g's. But we see from (4) and (7) that, if the $f(r_{ij})$'s are different from unity only for small r_{ij}'s, so are the function g's, too. Therefore, we first take the integral as if g's are equal to unity everywhere, and then add successively the terms due to the deviations of g's from unity. In this way we can obtain a series which converges rapidly. This is the cluster development method originally used by Ursell[9] in the classical theory of an imperfect gas.

To carry out this procedure explicitly we pick up the parts of the integrand of (8) referring only to some subsets of states and construct integrals similar to (8). They are

$$\begin{aligned}
&I_i = \int \varphi_i^*(x)\,\psi_i(x)\,dx,\\
&I_{ij} = (ij-ji\,|\,g(1,2)\,|\,ij),\\
&I_{ijk} = \Big(\sum_{\substack{i\ j\ k\\ p_i\ p_j\ p_k}} \varepsilon_P p_i p_j p_k\,|\,g(1,2)\,g(2,3)\,g(3,1)\,g(1,2,3)\,|\,ijk\Big),\\
&\cdots\cdots,
\end{aligned} \tag{9}$$

where the suffixes represent the subsets of states, and we have used the notation:

$$\begin{aligned}
(j_1\cdots j_n\,|\,A(1,\cdots,n)\,|\,i_1\cdots i_n) &= \int \varphi_{j_1}^*(x_1)\cdots\varphi_{j_n}^*(x_n)\cdot A_{i_1\cdots i_n}(x_1,\cdots,x_n)\\
&\qquad\cdot\psi_{i_1}(x_1)\cdots\psi_{i_n}(x_n)\,dx_1\cdots dx_n,\\
(j_1\cdots j_n \pm j_1'\cdots j_n'\,|\,A(1,\cdots,n)\,|\,i_1\cdots i_n)&\\
= (j_1\cdots j_n\,|\,A(1,\cdots,n)\,|\,i_1\cdots i_n) &\pm (j_1'\cdots j_n'\,|\,A(1,\cdots,n)\,|\,i_1\cdots i_n).
\end{aligned} \tag{10}$$

Especially

$$I = I_{12\cdots N}.$$

Note that from (6)

$$\begin{aligned}
&I_{ij} = I_{ji},\\
&I_{ijk} = I_{pqr} = \cdots,\\
&\cdots,
\end{aligned}$$

so that we need not discriminate the orders of suffixes.

From these I_i, I_{ij}, I_{ijk}, $\cdots$ we define the cluster integrals X_i, X_{ij}, X_{ijk}, $\cdots$ in the following way:

$$I_i = X_i\,,$$

$$I_{ij} = X_i X_j + X_{ij}\,,$$

$$I_{ijk} = X_i X_j X_k + X_i X_{jk} + X_j X_{ki} + X_k X_{ij} + X_{ijk}\,,$$

$$\cdots, \tag{11}$$

and finally we write the normalization integral as

$$I = \sum_{\nu}{}' X_{i_1 \cdots i_n} \cdots X_{j_1 \cdots j_m}\,, \tag{12}$$

where the sum is taken over all the partitions of N states.

We can now write down the explicit formulas of the cluster integrals. In doing this, complication arises from the fact that φ's and φ''s are not mutually orthonormal even if we take φ's to be orthonormal. But, as was mentioned before, we can consider β to be so small that we write

$$(\varphi_i, \varphi'_j) = \delta_{ij} + O(\beta)$$

and we have only to retain the terms up to the first order in β. From (9) and (11) neglecting the higher order terms in β,

$$X_i = (\varphi_i, \varphi'_i)\,, \tag{13}$$

$$X_{ij} = (ij - ji \,|\, h(1,2) \,|\, ij)\,, \tag{14}$$

$$X_{ijk} = X^{(0)}_{ijk} + X^{(1)}_{ijk}\,, \tag{15}$$

with

$$X^{(0)}_{ijk} = \Big(\sum_{\substack{i\,j\,k \\ p_i p_j p_k}}{}' \varepsilon_{p_i p_j p_k} \,|\, h(1,2)h(2,3) + h(2,3)h(3,1) + h(3,1)h(1,2)$$
$$+ h(1,2)h(2,3)h(3,1) + \eta(1,2)\eta(2,3)\eta(3,1)h(1,2,3) \,|\, ijk\Big)\,, \tag{15·1}$$

$$X^{(1)}_{ijk} = (jki + kij - jik - kji - ikj \,|\, h(2,3) \,|\, ijk)$$
$$+ (kij + jki - jik - ikj \,|\, h(3,1) \,|\, ijk)$$
$$+ (jki + kij - kji - ikj \,|\, h(1,2) \,|\, ijk)\,, \tag{15·2}$$
$$\cdots\cdots.$$

where we have split X_{ijk} according to the order in β and we have put

$$h_{ij}(x, x') = \eta_{ij}(x, x') - 1,$$

$$h_{ijk}(x, x', x'') = \eta_{ijk}(x, x', x'') - 1\,. \tag{16}$$

Now let us consider the orders of magnitudes of the cluster integrals. Obviously X_i is nearly equal to unity. When the correlations are of short range, the above defined h's may be appreciably different from zero only when the relevant particles come near each other. We shall denote by ω the volume within this small range or, speaking a little more precisely, the weighted volume with h. It is easily seen that X_{ij} is of the

order of ω/Ω. Here Ω is the total volume of the system where φ's are normalized. $X^{(\omega)}_{ijk}$, in which three particles are all connected by h's, is of the order of $(\omega/\Omega)^2$. The order of $X^{(1)}_{ijk}$, in which one particle is connected to the other two by exchanges, is $\beta(\omega/\Omega)$. If we want to consider up to these orders, there appears one more term X'_{ijkl} of the order of $(\omega/\Omega)^2$ as a part of X_{ijkl}:

$$X'_{ijkl}=\Big(\sum_{\substack{i\ j\ k\ l\\ p_i p_j p_k p_l}}{}' \varepsilon_P p_i p_j p_k p_l \,|\, h(1,2)h(3,4)+h(1,3)h(2,4) +h(1,4)h(2,3)\,|\,ijkl\Big) - (X_{ij}X_{kl}+X_{ik}X_{jl}+X_{il}X_{jk}), \tag{17}$$

where two separate h's are connected by an exchange.

If we further introduce the normalized cluster integrals,

$$\begin{aligned} x_{ij}&=X_{ij}/X_iX_j\,,\\ x_{ijk}&=X_{ijk}/X_iX_jX_k\,,\\ &\cdots\,, \end{aligned} \tag{18}$$

into (12), the normalization integral becomes

$$I=\prod_i X_i\cdot J(\cdots, x_{ij}, \cdots, x_{ijk}, \cdots), \tag{19}$$

where J is given by

$$J=1+\sum_{n+\cdots+m\le N}{}' x_{i_1\cdots i_n}\cdots x_{j_1\cdots j_m}. \tag{20}$$

If we write (20) explicitly,

$$J=1+\sum_{(i,j)}{}' x_{ij}+\sum_{(i,j,k)}{}' x_{ij}x_{kl}+\sum_{(i,j,k,l)}{}' x_{ij}x_{kl}+\cdots \tag{21}$$

The first sum is taken over $N(N-1)/2$ pairs and its order of magnitude is $N(N\omega/\Omega)$. Similarly the second and the third sums are of the orders of $N(N\omega/\Omega)^2$ and $N^2(N\omega/\Omega)^2$ respectively. Thus each term in (21) increases with the particle number N. Therefore, we can not approximate J by taking the first few terms in (21). How to arrange all terms neatly will be seen in the next section.

§ 4. Transformation to the converging series

In this section we shall show that the normalization integral decomposed into cluster integrals in the previous section can be expressed with a converging series after applying the transformation due to Ursell.[9]

To begin with we shall try to make a simple and plausible explanation. If we neglect the cluster integrals of more than two particles in (21),

$$J\approx\prod_{(i,j)}(1+x_{ij}).$$

Then, if we take all x_{ij}'s to be equal to ω/Ω, the above formula becomes

$$J \approx (1+\omega/\Omega)^{N(N-1)/2}.$$

Noting that ω, Ω tends to zero when the particle number N becomes large and replacing $N(N-1)/2 \cdot (\omega/\Omega)$ by $\sum_{(i,j)}' x_{ij}$, J can be approximated as

$$J \approx e^{\sum_{(i,j)}' x_{ij}}. \tag{22}$$

In this way we can gather x_{ij}'s appearing in (21) into the exponent when N is large.

To carry out this procedure more systematically we note that $J(\cdots, x_{ij}, \cdots, x_{ijk}, \cdots)$ expressed in (20) or (21) is a linear function in each variable x, and the coefficient of x is just that part of J consisting of all terms whose suffixes do not overlap those of x. Therefore J satisfies the following differential equation:

$$\frac{\partial J}{\partial x_{ij}} = J - \left\{ x_{ij} \frac{\partial}{\partial x_{ij}} + \sum_{k\neq i,j}' \left(x_{ik} \frac{\partial}{\partial x_{ik}} + x_{jk} \frac{\partial}{\partial x_{jk}} \right) + \cdots \right\} J, \tag{23}$$

where in the right-hand side all terms referring to i or j are subtracted from J, while the left-hand side is the coefficient of x_{ij}. Similarly,

$$\frac{\partial J}{\partial x_{ijk}} = J - \left\{ x_{ij} \frac{\partial}{\partial x_{ij}} + \cdots + \sum_{l\neq i,j,k}' \left(x_{il} \frac{\partial}{\partial x_{il}} + x_{jl} \frac{\partial}{\partial x_{jl}} + x_{kl} \frac{\partial}{\partial x_{kl}} \right) + \cdots \right\} J,$$

$$\frac{\partial J}{\partial x_{ijkl}} = J - \left\{ x_{ij} \frac{\partial}{\partial x_{ij}} + \cdots + \sum_{m\neq i,j,k,l}' \left(x_{im} \frac{\partial}{\partial x_{im}} + x_{jm} \frac{\partial}{\partial x_{jm}} + x_{km} \frac{\partial}{\partial x_{km}} + x_{lm} \frac{\partial}{\partial x_{lm}} \right) + \cdots \right\} J,$$

$\cdots$.

Putting

$$J = e^{NG(\cdots, x_{ij}, \cdots, x_{ijk}, \cdots)}, \tag{24}$$

the differential equations are transformed into

$$\frac{\partial G}{\partial x_{ij}} = \frac{1}{N} - \left\{ x_{ij} \frac{\partial}{\partial x_{ij}} + \sum_{k\neq i,j}' \left(x_{ik} \frac{\partial}{\partial x_{ik}} + x_{jk} \frac{\partial}{\partial x_{jk}} \right) + \cdots \right\} G,$$

$$\frac{\partial G}{\partial x_{ijk}} = \frac{1}{N} - \left\{ x_{ij} \frac{\partial}{\partial x_{ij}} + \cdots + \sum_{l\neq i,j,k}' \left(x_{il} \frac{\partial}{\partial x_{il}} + x_{jl} \frac{\partial}{\partial x_{jl}} + x_{kl} \frac{\partial}{\partial x_{kl}} \right) + \cdots \right\} G,$$

$$\frac{\partial G}{\partial x_{ijkl}} = \frac{1}{N} - \left\{ x_{ij} \frac{\partial}{\partial x_{ij}} + \cdots + \sum_{m\neq i,j,k,l}' \left(x_{im} \frac{\partial}{\partial x_{im}} + x_{jm} \frac{\partial}{\partial x_{jm}} + x_{km} \frac{\partial}{\partial x_{km}} + x_{lm} \frac{\partial}{\partial x_{lm}} \right) + \cdots \right\} G,$$

$\cdots$. (25)

Let us solve these equations by successive approximations starting from the initial value

$$G(\cdots, 0, \cdots, 0, \cdots) = 0\,, \tag{26}$$

which is easily seen from (20) and (24). Remembering the orders of magnitudes of x's we put (26) into the right-hand side of (25), which gives the equation in the first approximation,

$$\partial G^{(1)}/\partial x_{ij} = 1/N\,.$$

Here the index of G means the order of approximation. From this equation we obtain the first approximation of G,

$$G^{(1)} = 1/N \cdot \sum_{(i,j)} x_{ij}\,, \tag{27}$$

which corresponds to the previous formula (22) inferred from the rather crude argument. Substituting (27) again into the right-hand side of (25), we get

$$\frac{\partial G^{(2)}}{\partial x_{ij}} = \frac{1}{N} - \frac{1}{N}\left\{x_{ij} + \sum_{k \neq i,j} (x_{ik} + x_{jk})\right\},$$

$$\partial G^{(2)}/\partial x_{ijk} = 1/N\,,$$

$$\partial G^{(2)}/\partial x_{ijkl} = 1/N\,.$$

Then the second approximation of G is

$$G^{(2)} = \frac{1}{N}\sum_{(i,j)} x_{ij} - \frac{1}{2N}\sum_{(i,j)} x_{ij}^2 - \frac{1}{2N}\sum_{i,j,k} x_{ij}\,x_{jk}$$

$$+ \frac{1}{N}\sum_{(i,j,k)} x_{ijk} + \frac{1}{N}\sum_{(i,j,k,l)} x_{ijkl}\,, \tag{28}$$

and further we can repeat the same process.

Now we consider the order of magnitude of each term in (28). The first term is clearly of the order of $N\omega/\Omega$. The second is of $N^{-1}(N\omega/\Omega)$, which can be neglected when N is large. The third and the $x_{ijk}^{(0)}$ part in the fourth term are of the order of $(N\omega/\Omega)^2$. At first sight the orders of the $x_{ijk}^{(1)}$ part and the last term which contains x'_{ijkl} seem to be $N\beta(N\omega/\Omega)$ and $N(N\omega/\Omega)^2$ respectively, which increase with the particle number N. However it appears that this is not the case from the following reasoning. If we take φ's to be plane waves there is a momentum conservation among i, j, k states or among i, j, k, l states since they are connected by an exchange. In this case i, j, k or i, j, k, l are no longer independent, so that the orders of magnitudes of $x_{ijk}^{(1)}$ and x'_{ijkl} terms are $\beta(N\omega/\Omega)$ and $(N\omega/\Omega)^2$ respectively. These relations may continue to hold at least in the case when the system is fairly large and the φ's are the solutions in an almost uniform field of force. The remaining part of x_{ijkl} and the more complicated terms arising from the repeated iterations are of the higher orders.

We can write I from (19) and (24) as

$$I = /\!/ X_i \cdot e^{NG}\,, \tag{29}$$

and G is given as the series in which the ratio of any term to the preceding one is of the order of $(N\omega/\Omega)$.

§ 5. The energy expectation value in the cluster-developed form

Now we can derive the energy expectation value in the cluster developed form following the procedure stated in § 2.

From (5) and (29) the energy expectation value is given by

$$\langle H \rangle = (\partial/\partial\beta)\,\{\sum_i \ln X_i(\beta) + NG(\beta)\}\,|_{\beta=0}\,. \tag{30}$$

After a simple manipulation of this formula we obtain the expectation value of the single particle Hamiltonian $\sum_i' H_i(x_i)$ in (4) representing the kinetic energy of the single particle motions as

$$\begin{aligned}
&\sum_i' (i|H(1)|i) \\
&+\sum_{i,j}' \{(ij-ji|H(1)h(1,2)|ij) - (i|H(1)|i)\,(ij-ji|h(1,2)|ij)\} \\
&+\sum_{i,j,k}' (j|H(1)|i)\,(ki-ik|h(1,2)|jk) \\
&+\cdots\,.
\end{aligned} \tag{31}$$

The expectation value of $\sum_{(i,j)}' H_{ij}(x_i, x_j)$ which contains the potential energy and some kind of kinetic energy due to the particle correlations is written as

$$\begin{aligned}
&\tfrac{1}{2}\sum_{i,j}' (ij-ji|H(1,2)g(1,2)|ij) \\
&-\sum_{i,j,k}' (ij-ji|H(1,2)g(1,2)|ij)\,(jk-kj|h(1,2)|jk) \\
&+\tfrac{1}{2}\sum_{i,j,k}' \Big(\sum_{\binom{i\,j\,k}{p_i p_j p_k}}' \varepsilon_P p_i p_j p_k|H(1,2)g(1,2)h(2,3)\{2+h(3,1)\}|ijk\Big) \\
&+\tfrac{1}{4}\sum_{i,j,k,l}' \Big(\sum_P{}' \varepsilon_P p_i p_j p_k p_l|H(1,2)g(1,2)h(3,4)|ijkl\Big) \\
&+\cdots\,,
\end{aligned} \tag{32}$$

where $\sum_P'$ means the sum over the permutations $\binom{i\,j\,k\,l}{p_i p_j p_k p_l}$ excluding 1, (ij), (kl), and $(ij)\cdot(kl)$. Finally the expectation value of $\sum_{(i,j,k)}' H_{ijk}(x_i, x_j, x_k)$ arising from $\Gamma f \cdot \Gamma f$ terms is

$$\begin{aligned}
&\tfrac{1}{6}\sum_{i,j,k}' \Big(\sum_{\binom{i\,j\,k}{p_i p_j p_k}}' \varepsilon_P p_i p_j p_k|g(1,2)g(2,3)g(3,1)H(1,2,3)|ijk\Big) \\
&+\cdots\,.
\end{aligned} \tag{33}$$

This term is not zero as was stated by Dingle[7] and Jastrow.[8] The sum of these expressions (31), (32) and (33) gives the energy expectation value $\langle H \rangle$. Note that β should be taken zero in these formulas although we have used the same notation as before. Therefore, $\psi=\varphi$, $g=f^*f$ and $h=f^*f-1$.

We made the classification of cluster integrals in the normalization integral according to their magnitudes. We may need some slight change of this classification in the energy expectation value, because the differentiation by β may change the relations between the orders of magnitudes of the cluster integrals. This change generally depends on the shapes of the potentials and the trial function. However, when the potentials are of short range and the φ's are the solutions in an almost uniform field of force, there occurs little change. For this case all the zeroth and the first order terms in $N\omega/\Omega$ are explicitly written in (31), (32) and (33). Otherwise, we must pay attention to the above mentioned changes of the orders, and must keep sufficient terms in the normalization integral.

§ 6. Discussions

We have generalized the Hartree approximation to include the two particle functions representing the correlations between particles. In this case the problem becomes much more difficult and we have to develop the energy expectation value into cluster integrals. In order to get the ground state wave function we can take the variations of the functions φ and f and seek the minimum of the energy expectation value. However, we can not take the variations unrestrictedly. We must confine the functions within the domain where the cluster development converges rapidly. For this reason the function f can be appreciably different from unity only in the short range. Therefore, we can not take account of long range correlations in this method.

To avoid the complications, we took here a simple f function neglecting possible dependence on single particle orbits, momenta, spin orientations, and so forth. Also we assumed that f is the function of the particle distance only, but the orientation of $\boldsymbol{r}_i-\boldsymbol{r}_j$ will be important especially to take into account the effect of the tensor force. Further it may be necessary to introduce three particle functions, four particle functions, and so forth, to represent the correlations which can not be included in the product of the two particle functions. The general theory including these effects will be found in another part of this series.

Nevertheless the total spin and the parity of the system are determined by the single particle functions, because the product of all the correlation functions must include a scalar part, i. e. it becomes unity when all the particles are far apart from each other. In order to make the spin and the parity of the ground state definite, we must take them the same as those of the Slater determinant. Accordingly we hope our theory will partly explain the fact that the simple shell model predicts nuclear spins and parities fairly well, but does not generally give, for example, correct transition probabilities. Of course, it is clear that long range correlations, such as deformation as a whole, rotation,

oscillation, and so forth, also play an important role in these problems.

The saturation property of nuclear forces will be discussed in the next part and thereby we can see the weakening of the effective potential for the single particle motion.

The authors would like to express their sincere thanks to Professors T. Yamanouchi and M. Nogami for their kind guidance and continual encouragement and to Dr. K. Way for many valuable discussions.

Appendix. Cluster method for boson system

Let the trial function be

$$\Psi(x_1, \cdots, x_N) = \Omega^{-N/2} \cdot \prod_{(i,j)} f(r_{ij}) \tag{A1}$$

for the ground state of a system of N Bose particles interacting through the potentials $V(r_{ij})$ in the volume Ω.

In this case the problem is much easier because of the constancy of the single particle functions and the absence of exchanges. We have only to replace the $\varphi_i(x)$'s by $\Omega^{-1/2}$ and need not sum over permutations in all formulas in the fermion case.

Then

$$H_i(x) = 0\,,$$

$$H_{ij}(x, x') = -\frac{\hbar^2}{m}\frac{\Gamma^2 f(|\boldsymbol{r}-\boldsymbol{r}'|)}{f(|\boldsymbol{r}-\boldsymbol{r}'|)} + V(|\boldsymbol{r}-\boldsymbol{r}'|)\,,$$

and $H_{ijk}(x, x', x'')$'s are the same as before. The cluster integrals are

$$\begin{aligned} X_i &= 1\,, \\ X_{ij} &= \Omega^{-2}\int h(x, x')\,dx\,dx', \\ X_{ijk} &= \Omega^{-3}\int \{3h(x, x')\,h(x', x'') + h(x, x')\,h(x', x'')\,h(x'', x) \\ &\quad + g(x, x')\,g(x', x'')\,g(x'', x)\,h(x, x', x'')\}\,dx\,dx'\,dx'', \\ &\cdots . \end{aligned} \tag{A2}$$

Since they are independent of their suffixes, we put in (28)

$$\begin{aligned} x_{ij} &= X_{ij} = X_{(2)}\,, \\ x_{ijk} &= X_{ijk} = X_{(3)}\,, \\ &\cdots . \end{aligned}$$

Then neglecting $O(N^{-1})$, the normalization integral is given by

$$I = e^{\lambda G},$$

where

$$G=\tfrac{1}{2}NX_{(2)}-\tfrac{1}{2}(NX_{(2)})^2+\tfrac{1}{6}N^2X_{(\,)}+\cdots . \tag{A3}$$

From these formulas we can derive the final formula for the energy expectation value :

$$\begin{aligned}\langle H\rangle = {} & \frac{N^2}{2\Omega}\int f^*(r)\left\{V(r)f(r)-\frac{\hbar^2}{m}\nabla^2 f(r)\right\}4\pi r^2 dr \\ & +\frac{N^3}{2\Omega^2}\int f^*(r_{12})\left\{V(r_{12})f(r_{12})-\frac{\hbar^2}{m}\nabla^2 f(r_{12})\right\} \\ & \qquad h(r_{23})h(r_{31})\,8\pi^2 r_{12}r_{23}r_{31}dr_{12}dr_{23}dr_{31} \\ & -\frac{N^3}{2\Omega^2}\frac{\hbar^2}{m}\int f^*(r_{12})f^*(r_{23})f^*(r_{31})\nabla_2 f(r_{12})\cdot\nabla_2 f(r_{23})f(r_{31}) \\ & \qquad 8\pi^2 r_{12}r_{23}r_{31}dr_{12}dr_{23}dr_{31} \\ & +\cdots,\end{aligned} \tag{A4}$$

with

$$h(r)=f^*(r)f(r)-1 .$$

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On the Elastic Scattering of Charged Particles by Nuclei in the Intermediate Energy Region

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An expression has been obtained for the differential cross-section for the elastic scattering of intermediate energy charged particles by heavy nuclei, using a complex potential and employing the variational method due to Montroll and Greenberg. The potential considered is of the type $V=-(V_0+iW_0)$ inside the nucleus and zZe^2/r outside.

The expression obtained for the differential elastic scattering cross-section has been used in the case of scattering of 32Mev protons from gold. These have been compared with the experimentally determined cross-sections. It has also been shown that the present method of calculation gives results which are in good agreement with those obtained by using the partial wave method.

Theoretical calculations of the angular distribution of the elastic scattering of protons using a complex square-well potential have been carried out by a number of workers,[1] upto an energy of 18.5 Mev. In all these, partial wave method of calculation has been employed which becomes somewhat tedious for higher proton energies and heavier scatterers.

For the calculation of the scattering cross-sections of protons of higher energy from heavier scatterers we have employed the Montroll-Greenberg[2] variational method with complex square-well potential. This method has recently been used by Izumo[3] for investigating alpha-scattering and by Mohr and Robson[4] in the case of neutron-scattering.

The potential assumed in the present calculations is

$$\left.\begin{aligned} V(r) &= -(V_0+iW_0) \quad \text{for} \quad r<a \\ &= \frac{Ze^2}{r} \qquad\quad \text{for} \quad r>a \end{aligned}\right\}. \tag{1}$$

The scattered wave function for a uniform isotopic spherical scatterer can be written as an integral in the following manner.[2]

$$\psi_s(R) = \frac{e^{ik_0R}}{R} A_s \int_0^\infty \{k^2(r)-k_0^2\} r^2 \left\{\frac{\sin \alpha r}{\alpha r} - Ke^{2ik_1a} \frac{\sin \gamma r}{\gamma r}\right\} dr \tag{2}$$

where

$$A_s = \frac{2k_1 e^{ia(k_1-k_0)}}{(k_1+k_0)\,(1-K^2e^{4ik_1a})},$$

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$$K=\frac{k_1-k_0}{k_1+k_0}, \quad k^2(r)=\frac{2M}{\hbar^2}[E_p-V(r)], \quad k_0^2=\frac{2M}{\hbar^2}E_p,$$

$$\omega^2=k_1^2+k_0^2-2k_1k_0\cos\theta \quad \text{and} \quad v^2=k_1^2+k_0^2+2k_1k_0\cos\theta.$$

k_1 is the internal wave number of the scattered particle. For a potential of arbitrary shape k_1 and a have to be determined by applying the variational method.

The Montroll and Greenberg method gives the following expression for the scattered wave function for the type of potential given by equation (1).

$$\psi_s(\vec{R})=\frac{e^{ik_0R}}{R}\left\{A_s(V_0+iW_0)\frac{2M}{\hbar^2}a^3\sqrt{\pi/2}\times\right.$$

$$\left.\times\left[\frac{J_{3/2}(\omega a)}{(\omega a)^{3/2}}-Ke^{2iak_1}\frac{J_{3/2}(va)}{(va)^{3/2}}\right]-\frac{2M}{\hbar^2}\frac{Ze^2}{\omega'^2}\cos(\omega' a)\right\} \qquad (3)$$

where

$$\omega'=4k_0^2\sin^2\theta/2.$$

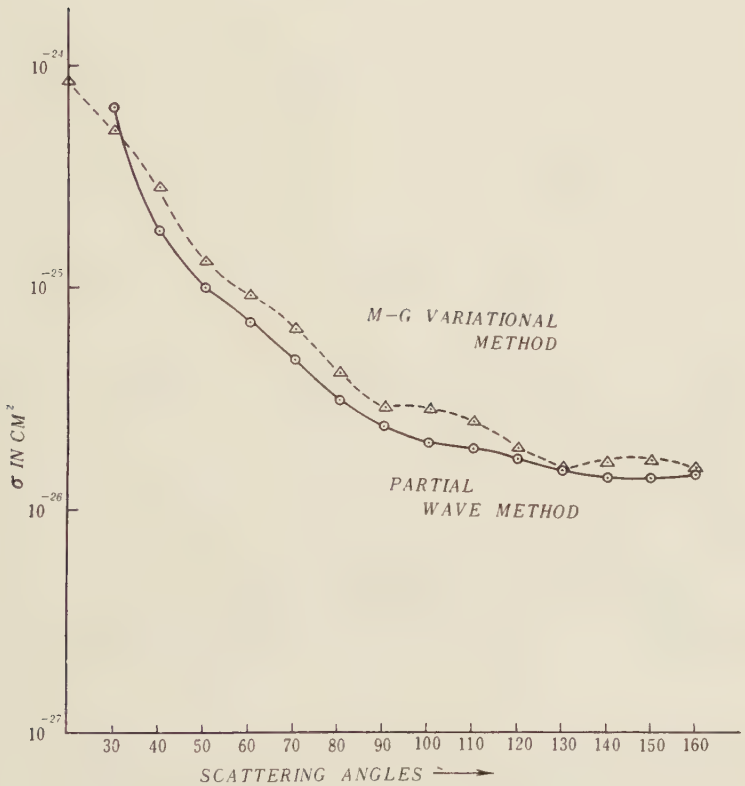

Fig. 1. Comparison of differential elastic scattering cross-sections calculated by two methods for 18.3 Mev protons scattered from copper. The results of partial wave calculations are due to Chase and Röhrlich. The square-well type of nuclear potential is used in both the methods.

The differential scattering cross-section is given by

$$\sigma_s(\vartheta) = R^2 |\psi_s(\vec{R})|^2 d\Omega.$$

In equation (3), A_s, K, k_1, ωa and νa are complex quantities. In order to obtain the scattering cross-section, it is necessary to separate real and imaginary parts in equation (3).

To do this, we first note that both $|\omega a|$ and $|\nu a|$ are large compared to 3/2 for the values of the parameters employed here. Hence it is possible to write the Bessel functions in the asymptotic forms[5]

$$J_{n+1/2}(x) = \sqrt{2/\pi x} \sin(x - n\pi/2).$$

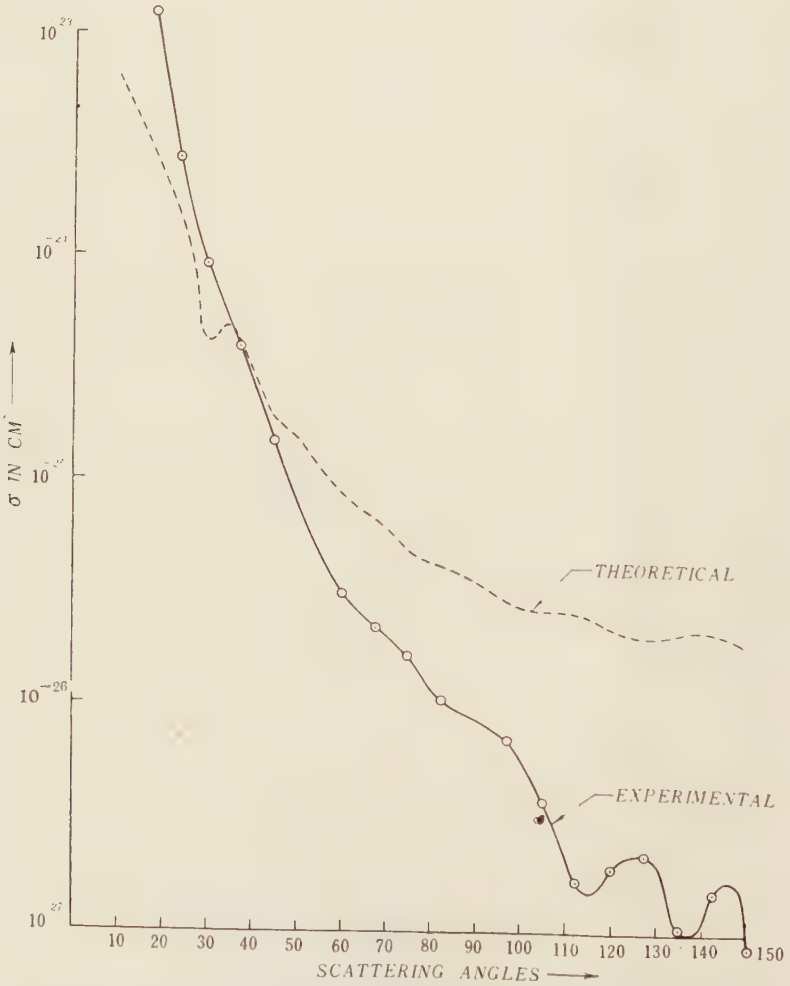

Fig. 2. Comparison of theoretical differential elastic scattering cross-sections for 32 Mev protons from gold obtained by using M-G variational method with experimentally determined ones.

The separation of real and imaginary parts of equation (3) is now fairly simple.

We also make the assumption that the absorptive part of the complex potential W_0 is small compared to (E_p+V_0). Then if we write $k_1=c_1+id_1$, the following approximate expressions for c_1 and d_1 can be easily obtained:

$$c_1=\frac{\sqrt{2M(E_p+V_0)}}{\hbar} \quad \text{and} \quad d_1=\frac{W_0}{\hbar}\sqrt{\frac{M}{2(E_p+V_0)}}.$$

The evaluation of real and imaginary parts of K then follows simply.

Using similar approximations the real and imaginary parts of ω and v can be evaluated.

The following values were used in carrying out the calculations: $V_0=45$ Mev, $W_0=20$ Mev and $a=1.42\times A^{1/3}\times 10^{-13}$ cm. The above method has been used to calculate the differential elastic scattering cross-section for the two elements, gold and copper for the proton energies of 32.1 Mev and 18.3 Mev respectively. The latter have been compared with the cross-sections calculated by Chase and Röhrlich employing the partial wave method. These are plotted in Figure 1. The qualitative agreement between the two curves justifies the use of this approximation method in calculating the differential elastic scattering cross-section of intermediate energy protons. From Figure 1, it is found that the cross-sections calculated by the M-G method are in general higher than those calculated by the exact method. This has also been observed by Mohr and Robson[4] who point out that the M-G cross-sections in the case of a complex potential are expected to be higher than the exact ones because of the fact that the former include a factor equal to the square of the modulus of the complex potential. For scattering angles below about 40°, however, M-G cross-sections in the present case turn out to be lower than the exact ones.

In Figure 2, the cross-sections calculated from the above considerations have been compared with the experimentally determined cross-sections for the scattering of 32 Mev protons from gold*. The poor agreement is probably due to the use of the square-well type of potential in the present calculations. It is known that the square-well type complex potential gives results which are in agreement with experimental observations only for scatterers of low nuclear charge, *e.g.* aluminium. For heavier nuclei where back scattering is relatively low this model does not give satisfactory agreement[6]. It has been shown by Wood and Saxon[7] that better agreement with experimental results can be obtained by using a diffused boundary model of the nucleus. Calculations are in progress using this type of potential.

* A report of this experimental work has been communicated elsewhere.

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Saturation of Nuclear Forces.....Neutron Proton Scattering

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With the help of a potential so chosen as to, in a sense, just satisfy the saturation requirements and the various experimental data on the deuteron, and nucleon-nucleon scattering at low energies, the neutron proton differential scattering cross-section $\sigma(\theta)$ is calculated at 100Mev and 260Mev in Born's approximation. The error due to the use of Born's approximation at 100Mev is estimated. The calculated cross-sections are in fair agreement with experiment and indicate the desirability of making an exact phase shift calculation of $\sigma(\theta)$ for this potential.

§ 1. Introduction

It is well known that the meson theory of nuclear forces does not enable one to arrive at an unambiguous expression for the nucleon nucleon potential. One, therefore, usually proceeds by writing down the most general form of the potential with some radial dependence (Yukawa, exponential or square well type) and sets out to determine by trial and error the various constants in the general expression for nucleon nucleon interaction from the diverse experimental data on the deuteron, and low and high energy p–p and p–n scattering. Many attempts of this kind have been made in the past: we refer here to the work of Christian and Hart[1] whose work, as far as is known to us, has not since been superseded. They found that the high energy (up to about 300 Mev) n–p scattering data available up to the time their work was written (1950) could be best explained by assuming that the interaction vanishes in states of odd orbital angular momentum (Serber's potential). This conclusion was due to the fact that at high energies ($\gtrsim$ 100 Mev) the n–p differential scattering cross-section $\sigma(\theta)$ in the center of mass system exhibited, within the limits of experimental error, almost complete symmetry around 90°. Later experiments seem to show[2)3)] that $\sigma(\theta)$ is not quite symmetrical around 90°. These newer experiments may remove an unsatisfactory situation existing previously, viz, a charge independent* potential with Serber exchange leading to complete symmetry of $\sigma(\theta)$ around 90° does not lead to saturation of nuclear forces.

The purpose of the present note is to report a calculation of $\sigma(\theta)$ for n–p scattering at 100 Mev and 260 Mev in Born's approximation with the help of a potential

*) We shall assume here charge independence of nuclear forces as all available experimental data is consistent with this hypothesis.

so chosen as to just (see below) satisfy the saturation conditions. The error involved in using Born's approximation at 100 Mev is estimated by making a rough calculation of the correction to σ in the second Born approximation. The agreement between theory and experiment is reasonable and suggests the desirability of making an exact phase-shift calculation of $\sigma(\theta)$.

These calculations were actually made in 1951. We feel that in spite of their shortcomings the publication of these results may be of some use at the present time. For, apart from the general desirability of attempting to explain scattering phenomena at these energies in terms of a two body nucleon nucleon interaction satisfying saturation requirements, in certain other types of problems the use of such a potential is not only desirable but necessary. For example, recently Tauber and Wu[4] have attempted a Hartree-Fock-Slater type of self consistent treatment for certain nuclei starting from a given two body nucleon-nucleon interaction. An interaction assumed for such a purpose has to satisfy saturation conditions, and these authors have found it useful to restrict the number of constants in the potential by making use of the condition (4) given below.

We may mention here that with the increasing possibility of obtaining high energy beams of polarized nucleons, an alternative method for determining the nucleon-nucleon interaction may become available in the not too distant future. This consists in first determining the phase shifts from the scattering data at various angles and energies and then determining the potential which would lead to correct phase shifts. Unlike the case of particles of spin zero interacting with central forces, the relevant phase-shifts in n–p and p–p scattering cannot be determined from a knowledge of $\sigma(\theta)$ as a function of θ alone, because of the spin dependent and non-central forces. As has been shown earlier[5–6], however, from a knowledge of angular and azimuthal dependence of σ and of the polarization on scattering of an incident polarised beam of nucleons by unpolarized nucleons, a complete determination of relevant phase-shifts is possible, apart from an ambiguity in signs which can be fixed by other data. A limited number of phase-shifts can also be determined by double and triple scattering of an unpolarized beam by unpolarized targets (see reference 3, chapter XVII).

§ 2. Choice of potential

We write the nucleon-nucleon potential in the form

$$V_{12} = [a_0 + a_\sigma\, \sigma_1 \cdot \sigma_2 + a_\tau\, \tau_1 \cdot \tau_2 + a_{\sigma\tau}\, \sigma_1 \cdot \sigma_2\, \tau_1 \cdot \tau_2] V_0 f_0(r) + [b_\sigma + b_\tau\, \tau_1 \cdot \tau_2]\, S_{12} V_0 f_t(r), \tag{1}$$

where σ and τ are the spin and isotopic spin operators, S_{12} is the usual dipole dipole interaction operator, V_0 is a measure of the strength of the interaction, and $f_0(r)$ and $f_t(r)$ denote the radial dependences of central and tensor interactions. We regard V_0, $f_0(r)$ and $f_t(r)$ to be positive and

$$\lim_{r \to 0} f_t(r)/f_0(r) = 1. \tag{2}$$

The conditions for saturation of nuclear forces in absence of tensor interaction are[7]

$$a_0 \gtrless 0,\ a_0+a_\sigma \gtrless 0,\ a_0+a_\tau \gtrless 0,$$
$$a_0+a_{\sigma\tau} \gtrless 0,\ \text{and}\ a_0+a_\sigma+a_\tau+a_{\sigma\tau} \gtrless 0. \tag{3}$$

Now since the observed n–p scattering cross-section deviates only slightly from symmetry around 90° one must choose the potential to be, as nearly as possible, of the Serber type. Therefore we take different exchange dependences for the singlet and triplet states and assume Serber exchange for the potential in singlet states. This gives us a relation between the a's, viz.,

$$^1V_{odd} = (a_0 - 3a_\sigma - 3a_\tau + 9a_{\sigma\tau}) V_0 f_0(r) = 0. \tag{4}$$

The precise exchange dependence of triplet states is then obtained by assuming the equality sign in the first of the conditions (3), i.e.,

$$a_0 = 0. \tag{5}$$

It will be noticed that with a Serber exchange for triplet states, saturation is not possible irrespective of the exchange dependence assumed for singlet states. Conditions (4) and (5) reduce the number of independent constants in the central potential to two. Further, following Röhrlich and Eisenstein[7], we assume the exchange dependence of the symmetrical theory for the tensor term i.e. $b_{\sigma t}=0$. With the help of (4) and (5), the expressions for the potential in the triplet and singlet even states may then be written as

$$^3V_{even} = -4a_\tau V_0 f_0(r) - 3b_{\tau t} S_{12} V_0 f_t(r), \tag{6a}$$

$$^1V_{even} = -4a_\sigma V_0 f_0(r). \tag{6b}$$

The constants in (6a) and (6b) may be chosen to fit the deuteron and the low energy scattering data, whence all the unknowns in (1) become known. We choose for (6a) and (6b) a set of expressions given by Röhrlich and Eisentein[7] with radial dependences of the Yukawa type. The various unknowns in the potential (1) are then given by†

$$\left.\begin{aligned} &a_0 = b_{\sigma t} = 0,\ a_\sigma = 0.3665,\ a_\tau = 0.250,\\ &a_{\sigma\tau} = 0.2055,\ b_{\tau t} = 0.5663,\ V_0 = 30.817\ \text{Mev};\\ &f_0(r) = \frac{e^{-r/r_0}}{r/r_0},\quad f_t(r) = \frac{e^{-r/r_t}}{r/r_0},\\ &\text{where } r_0 = 1.202\times 10^{-13}\ \text{cm}\\ &\qquad\text{and } r_t = 1.400\times 10^{-13}\ \text{cm}. \end{aligned}\right\} \tag{7}$$

It will be seen that the potential given by (1) and (7) satisfies all the saturation

†) Actually in reference 8 the range for singlet states is 1.183×10^{-13} cm. For simplicity we have taken the ranges for triplet and singlet states for central interaction to be the same, viz., 1.202×10^{-13} cm. The slight difference in the two ranges does not materially affect our results.

requirements in absence of tensor interaction. The necessary conditions for saturation in presence of tensor forces have been worked out by Kanazawa[9]. In addition to (3) one then has two more conditions. If one remembers that (a) the expectation value S of S_{12} has its extreme values $+2$ for a needle shaped ellipsoid, and -1 for a disk shaped ellipsoid of collapsed nuclei, and (b) the tensor interaction is zero for singlet states, while for triplet states $\sigma_1 \cdot \sigma_2 = 1$ the conditions II and IV of Kanazawa may be written as

$$-\tfrac{1}{2}(a_0+a_\sigma) \leq b_{\sigma t} \leq a_0+a_\sigma, \tag{8a}$$

$$-\tfrac{1}{2}(a_0+a_\sigma+a_\tau+a_{\sigma\tau}) \leq b_{\sigma t}+b_{\tau t} \leq (a_0+a_\sigma+a_\tau+a_{\sigma\tau}). \tag{8b}$$

Conditions (8) may be seen to be satisfied by the potential chosen here.

Blatt and Weisskopf[7] have also given, without proof, conditions for saturation in presence of tensor forces. In addition to (8a) and (8b) they give also an inequality

$$-\tfrac{1}{2}(a_0+a_{\sigma\tau}) \leq b_{\tau t} \leq a_0+a_{\sigma\tau}. \tag{9}$$

(9), unfortunately is not satisfied by our potential. This condition presumably arises by considering a nucleus containing equal number of neutrons and protons with all neutron spins pointing in one direction and all proton spins in the opposite direction and may be too stringent from the practical point of view.

Before we give the results of our calculation of n–p scattering cross-sections, we make the following comments. First, the tensor term in our potential is, comparatively speaking, strong. Presence of rather strong non-central interactions is indicated by polarization results in p–p and n–p scattering (see, for example, reference 3, p. 132). Secondly, we have not attempted to include in our potential any spin orbit coupling term or short range repulsive core term. This is because generally the role played by these terms is not yet quite clear and specially the saturation conditions for a potential containing these terms have not been worked out.

§ 3. N-P scattering cross-sections at 100 and 260 Mev

With the help of the potential given by (1) and (7), we have calculated n–p differential scattering cross-section at 100 Mev and 260 Mev in Born's approximation. These results are given as smooth curves in Fig. 1. The experimental points taken from reference 1 are marked by circles and refer to experimental data available prior to 1950. Broken curves (a) and (b) are taken from Bethe and Morrison[3] and represent idealized version of the newer but still rather uncertain data at 100 Mev and 300 Mev. We may mention here that experimental data is available only for angles θ greater than about 35°.

From Fig. 1 the agreement between experimental data and our calculations may be regarded as reasonable at 260 Mev since we have used Born's approximation and neglected relativistic effects both these effects are likely to be corrections of the order of 10 to 20% at this energy. At 100 Mev the experimental and theoretical values agree fairly

well for scattering angles θ ranging from 60° to 180°, and so is the ratio $\sigma(\pi)/\sigma(\pi/2)$. However, for $\theta < 60°$ the agreement is poor indeed. Since at 100 Mev the Born's approximation may be unsatisfactory, we have calculated, by the method given by Jost and

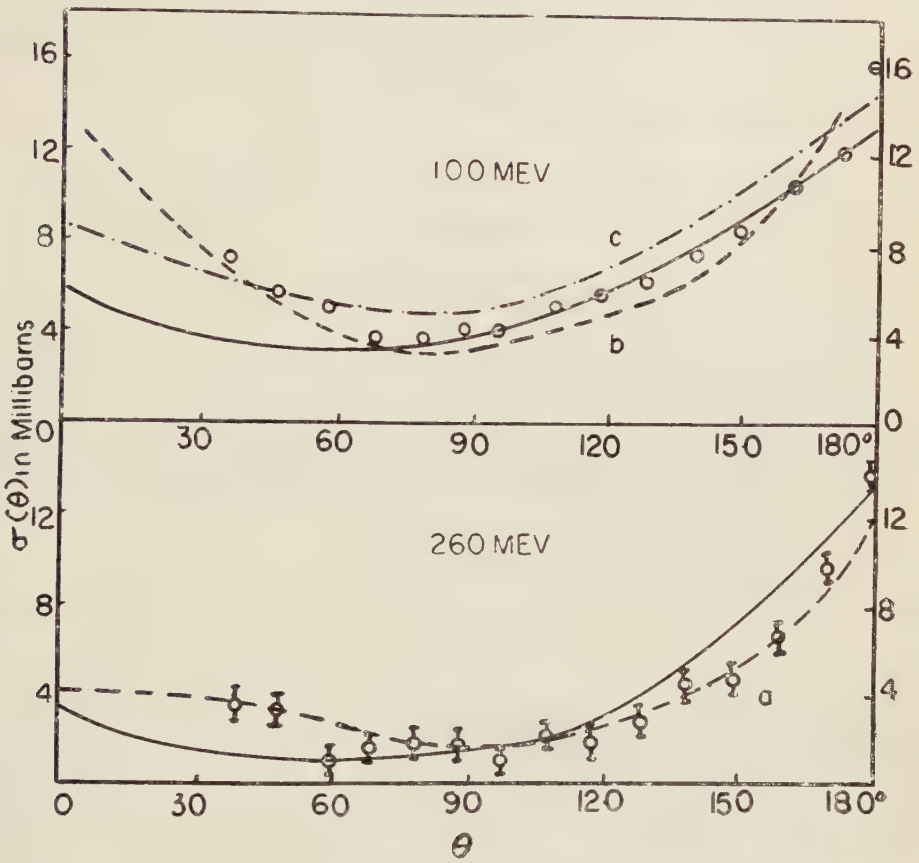

Fig. 1. *n-p* differential scattering cross-section $\sigma(\theta)$ in the centre of mass system at laboratory energies of 100 and 260 Mev.

Calculated $\sigma(\theta)$ in the standard first Born approximation.

Calculated $\sigma(\theta)$ in the first and second Born approximations at 100Mev (see text for details).

○, ⊡ Experimental points (available between about $\theta = 35°$ and $\theta = 180°$).

Idealised version of experimental data at 100 and 300 Mev (reference 3).

Pais[10] corrections $\sigma_c^{(2)}$ at this energy to $\sigma_c^{(1)}$ (cross-section for central potential in the standard Born approximation) in second Born approximation for the central part of our potential. It is found that for triplet states $\sigma_c^{(2)}/\sigma_c^{(1)}$ is about 50% in the forward directions, diminishing with increasing θ and is about 10% in backward directions. For singlet states this correction is small and symmetrical about 90°. The result of adding these corrections to σ in Born's approximation is roughly depicted by the curve *c*; this curve is in better agreement with experiment than the smooth curve. We should mention

that corrections to the contribution to σ from tensor forces has not been calculated in the second Born approximation, as these are complicated.

From the above discussion it would appear that the discrepancy between experimental and calculated values is at least partly due to the use of Born's approximation. It is hoped to publish a phase-shift calculation of cross-sections at a later date. Finally we may mention that the potential such as given here will not explain p–p scattering data at intermediate and high energies without introducing either a repulsive core or spin orbit coupling term or both[+].

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[+] **Note added in proof :** The addition of a repulsive core term in the nucleon-nucleon potential would probably alter the saturation conditions (3), (8) and (9) radically. As such the potential used in this paper is not suggested to the exclusion of other possibilities. The main purpose of this paper is to express the view that is desirable to explain nucleon-nucleon scattering data in terms of a potential satisfying saturation requirements and to examine one such potential. The authors thank the referee for making some very useful comments on their paper.

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The *l*-Forbidden Magnetic Dipole Transitions

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The reduced matrix elements for *l*-forbidden magnetic dipole transitions in medium-weight nuclei are calculated on the basis of configuration mixing. The calculations are based upon simple perturbation theory. The parameters used in this calculation are the same as those adopted for the magnetic and quadrupole moments in previous work. The lifetimes of these radiative processes are well explained by small admixtures of excited configurations to the shell model configurations.

§ 1. Introduction

Much important information about stationary states of the nucleus can be obtained from the study of the interaction of the nucleus with the electromagnetic field. Spins and parities of excited states can best be determined by the type (electric or magnetic) and multipolarity of the radiation. These are experimentally determined by measuring internal conversion coefficients of the radiation without detailed assumptions about nuclear transitions. On the other hand, the gamma-transition probability between two states depends in a sensitive manner on the wave functions of these states, hence knowledge of nuclear wave functions can be obtained by comparing the experimental transition probability with the theoretical value calculated on some specific model. Among the various nuclear models proposed, the shell model seems most suitable to explain the nuclear gamma-transitions, especially the existence of the "island of isomerism".[1] There are, however, some details with which the shell model fails to obtain quantitative agreement.[2] One of these is the *l*-forbidden magnetic dipole ($M1$) transitions which will be discussed in the following, although the shell model does predict their much longer lifetimes in comparison with the usual (or allowed) $M1$ transitions.

The selection rules for $M1$ transitions are that the angular momentum change is not larger than one ($0 \rightarrow 0$ transition is excluded) and there is no parity change. However if the shell model is taken as the basis of theoretical consideration, it gives a more stringent selection rule. Because the magnetic dipole moment operator does not change the orbital angular momentum, the initial and final orbits assigned by the shell model must not differ in orbital angular momentum in the magnetic dipole transition. Mots

of the transitions which are classified as magnetic dipole transitions do not satisfy the latter selection rule and they are called l-forbidden transitions. These l-forbidden transitions have much longer lifetimes than the theoretically expected lifetimes of allowed transitions from the shell model, the average value of the ratio of these quantities being about 10^2, although the values fluctuate considerably.

If the shell model describes the nuclear states very accurately, the observed lifetimes of the l-forbidden transitions give evidence of the inadequacy of the conventional treatment of gamma-transitions. Sachs and Ross[3] have discussed the effect of exchange currents on this type of transition and Ross[4] has concluded that it is impossible to explain the lifetime by mixing excited configurations with the shell model configuration, since, in order to get experimental lifetimes, the extent of the mixing becomes too large. It will be shown in the following that Ross' conclusion is premature and the l-forbidden $M1$ transitions can be explained by configuration mixing. More recently, Spruch and Rotenberg[5] have given extensive results for the accumulated experimental data on the same basis as Ross.

An alternative approach is based on the argument that the shell model is not accurate enough and the departure of the actual wave function from the shell model gives the non-vanishing matrix element for the transition. Volkov[6] has estimated the matrix elements from the standpoint of statistical configuration mixing and has obtained fairly good agreement with experimental values. A second (not statistical) consideration about the deviation of magnetic moment from the Schmidt lines by configuration mixing yields more satisfactory results.[7][8] Therefore, it is expected that the magnetic dipole transitions must be explained by a similar calculation because the relevant operator is the same and the mixing states are similar to those in the case of the static moments. Other evidence of configuration mixing for the nuclear shell model is given by quadrupole moments[9], β-decay matrix elements, and deuteron stripping reactions.[10]

§ 2. Transition probabilities

The matrix elements for magnetic dipole transitions are obtained from the usual magnetic moment operator. The procedure for obtaining the matrix elements for the transitions differs from that for the static moments in that the initial and final states are different. Furthermore, the shell model matrix elements vanish for the l-forbidden transitions, so that nonvanishing elements appear as the effect of configuration mixing, and the dominant contributions come from the matrix elements between shell model configurations and mixing configurations if the mixing is small. (Concerning the more detailed discussion, see the following.) The extent of the admixture was estimated as the order of 1% in the calculation of the magnetic and quadrupole moments.[8][9] Therefore, the ratios of the observed lifetimes of the l-forbidden transitions to theoretical values for allowed transitions by the shell model are expected to be of the order of 10^2, since the lifetimes are inversely proportional to the squares of the matrix elements, which are equal to the amplitudes of admixture multiplied by the allowed transition matrix elements. The compilation of ex-

perimental data by Way et al. gives as the average values of the ratios, 60 for odd-neutron nuclei and 300 for odd-proton nuclei.[2] We shall make more quantitative comparison between theoretical and experimental values of the lifetimes in the next section. Before that, we shall derive the matrix elements for the transitions by the consideration of the configuration mixing.

The radiative transition probability due to a magnetic dipole transition is given by

$$w_\gamma = 1/\tau_\gamma = \frac{mc^2}{\hbar}\left(\frac{E}{mc^2}\right)^3\left(\frac{m}{M}\right)^2 \frac{e^2}{\hbar c}\,\frac{1}{3}\,\mathfrak{m}^2/(2j+1)$$

$$= 0.419\times 10^{13}\, E^3\ \mathfrak{m}^2/(2j+1)\ \text{sec}^{-1}, \tag{1}$$

where τ_γ is a mean lifetime, and $\mathfrak{m}^2$ is given by Eq. (2). Photon energy E is measured in Mev in the second line.

$$\mathfrak{m}^2 = \sum_{qmm'} |(j'm'|\sum \mu_q|jm)|^2 = (j'\|\sum\mu\|j)^2. \tag{2}$$

In (2), $\sum\mu$ means the summation of the magnetic moment operators of each nucleon in the nucleus, and q denotes the component; j is the angular momentum of the initial state and j' is that of the final state, m and m' being their z-components.

Let us denote the initial and final states by $\Psi(jm)$ and $\Psi(j'm')$. We adopt as the zeroth order configurations the shell model wave functions $\Psi_0(jm)$ and $\Psi_0(j'm')$, and let $\Psi_n(jm)$ and $\Psi_{n'}(j'm')$ be typical excited configurations which mix with the zeroth order configurations in respective fashion. If the admixture is small, the initial and final wave functions are

$$\Psi(jm) = \Psi_0(jm) + \sum_n \alpha_n \Psi_n(jm)$$

and (3)

$$\Psi(j'm') = \Psi_0(j'm') + \sum_{n'} \alpha_{n'} \Psi_{n'}(j'm').$$

There α's are given by perturbation theory in terms of non-diagonal elements of the energy matrix:

$$\alpha_n = -(\Psi_0(jm), \sum_{i<k} V_{ik}\, \Psi_n(jm))/\Delta E, \tag{4}$$

where V_{ik} is the interaction between the i-th and k-th nucleons and ΔE is the energy difference between the zeroth order and excited configurations. By using wave functions (3), we get the reduced matrix element in (2). For l-forbidden transitions, the element between the initial and final zeroth order configurations vanishes and the leading terms of the element appear in a linear form involving the coefficients of mixing. We neglect the effects of the second order contribution of these coefficients. This procedure is exactly the same as those for obtaining corrections to magnetic and quadrupole moments in reference 8 and 9. As we shall adopt the same parameters employed there, the magnitude of the coefficient of mixing will be quite similar. Previously they were found to be 0.1 or less, so that it is a plausible assumption that both the reduction of the amplitude of the zeroth order configuration (the reduction of those of the excited configuration is fourth order) and the second order effect on the matrix ele-

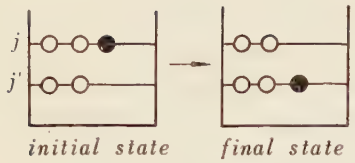

Fig. 1a. Like core transition (L). The number of nucleons in the outermost orbit (initially odd) decreases by one during like-core transition.

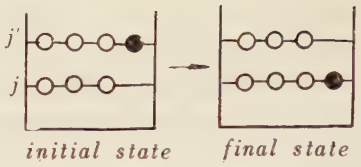

Fig. 1b. Unlike core transition (U). The number increases by one during the unlike-core transition.

ments can be neglected.

It seems convenient to divide the gamma-transitions of odd A nuclei into two groups, one being like-core transitions and the other unlike-core transitions. They have been used for beta-transitions in a similar sense.[1,2] For the gamma-transitions, the number of nucleons in the outermost orbit (initially odd) decreases by one during a like-core transition, and it increases by one during an unlike-core transition. The distinction is easily seen by reference to Figs. 1a and 1b. We may assume the angular momenta of the initial and final configurations to be $j=l-1/2$ and $j'=l'+1/2$, where $l=l'+2$, without loss of generality. (For example $j=d_{3/2}$ and $j'=s_{1/2}$.) For the other case in which $j=l+1/2$, $j'=l'-1/2$ and $l=l'-2$, we need only exchange the roles of the initial and final states of the transition in the following calculation, since the relation

$$(j'\|\sum\mu\|j)=-(j\|\sum\mu\|j')$$

holds, where $|j-j'|=1$.

Like-core (L) transitions

We represent the zeroth order configurations of the initial and final states of the like-core transitions as

$$\Psi_0(jm)=\Psi(j_1^{n_1}(0)\,j_2^{n_2}(0)\,j'^{q}(0)\,j^{p+1}(j);jm) \tag{5}$$

and

$$\Psi_0(j'm')=\Psi(j_1^{n_1}(0)\,j_2^{n_2}(0)\,j'^{q+1}(j')\,j^{p}(0);j'm') \tag{6}$$

where nucleons in $j_1=l_1+1/2$ and $j_2=l_1-1/2$ are like (both proton orbits or both neutron orbits) and n_1 and n_2 are even numbers. Usually $n_2=0$ if $n_1<2j_1+1$ and $n_1=2j_1+1$ if $n_2>0$, and p and q are also even numbers. Then, a sequence of excited states

$$\Psi([j_1^{n_1-1}(j_1)\,j_2^{n_1+1}(j_2)](J)\,j'^{q+1}(j')\,j^{p}(0);jm) \tag{7}$$

can interact with the zeroth order configuration in the initial state (5). The decay by magnetic dipole from these to the zeroth order configuration of the final state (6) would appear to be allowed because configurations (6) and (7) are the same except for one orbit and the different orbits j_1 and j_2 have the same orbital angular momentum. However, only the state with $J=1$ can decay into (6) as is readily seen from the coupling scheme of angular momenta in (6) and (7). Fig. 2 shows schematically the excited configurations (7) and the corresponding ones in the final state. We can get the reduced matrix

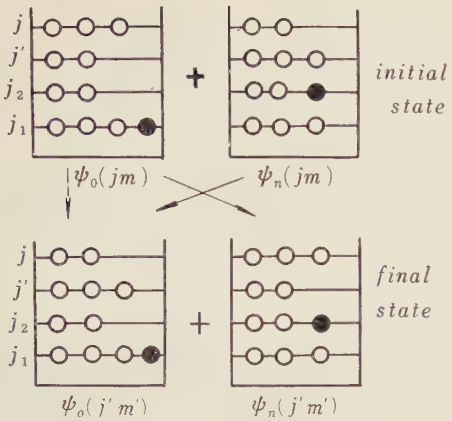

Fig. 2. *LI.* $j=l-\frac{1}{2}$, $j'=l'+\frac{1}{2}$, $l=l'+2$, $j_1=l_1+\frac{1}{2}$, and $j_2=l_1-\frac{1}{2}$. Transitions shown by broken lines are forbidden, while transitions shown by solid lines are allowed. The particle denoted by a solid circle jumps in each case.

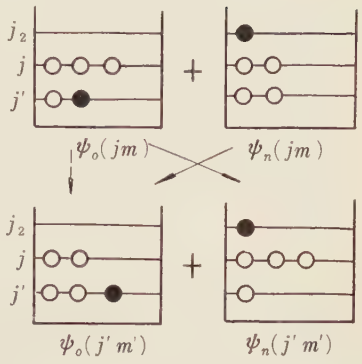

Fig. 3. *LII.* $j=l-\frac{1}{2}$, $j'=l'+\frac{1}{2}$, $j_2=l'-\frac{1}{2}$

element for the magnetic moment operator due to the mixing of the excited configuration (7) in the initial state by the above-mentioned procedure. As the calculation is quite similar to that correction to the static magnetic moment[8], it is not repeated here. The interaction V_{ik} is assumed to have a δ-function radial dependence for the sake of simplicity, and its strengths in triplet and singlet states are denoted by V_t and V_s, respectively. The result is

$$\mathfrak{m}_{LIi}=\lfloor (2j+1-p)\,(2j'+1-q)/(2j+1)\,(2j'+1)\rfloor^{1/2}\times [l(l'+1)/2(l+l'+1)]^{1/2}(g_s-g_l)F_{LIi}, \tag{8}$$

where

$$F_{LIi}=\frac{n_1(2j_2+1-n_2)}{(2j_1+1)\,(2j_2+1)}\,\frac{l_1(l_1+1)}{2l_1+1}\begin{cases} V_s I(j_1\,j_2;\ j\,j')/(-\Delta E), \\ g(V_t-V_s)/2\cdot I(j_1\,j_2;\ j\,j')/(-\Delta E). \end{cases} \tag{9}$$

The upper line in the brace of (9) is valid if the excited nucleon j_1 and the jumping nucleon j are like, and the lower line is valid if they are unlike. g_s-g_l in (8) is 4.585 (nm) for an odd-proton nucleus and -3.826 (nm) for an odd-neutron nucleus, and the value of g in (9) is given as $3.826/4.585=0.834$ for the effect of neutron excitation of odd-proton transition and as $4.585/3.826=1.199$ for the effect of proton excitation on the odd-neutron transition. I is a Slater integral for a delta-function interaction.

$$I(j_1\,j_2;\ j_1'\ j_2')=1/2\cdot\int_0^\infty R(j_1)\,R(j_2)\,R(j_1')\,R(j_2')\,r^2\,dr.$$

The contribution due to similar excitation in the final state:

$$\mathfrak{F}([j_1^{n_1-1}(j_1)\,j_2^{n_2+1}(j_2)](1)\,j'^{q}(0)\,j^{p+1}(j);\ j'm') \tag{10}$$

can be obtained. The results is that F_{LIf} is given by the same formula as F_{LIi} and (8) is valid for changing suffices *LIi* into *LIf*. In some cases, there is more than one pair of j_1 and j_2 which must be taken into account. If this occurs, it is necessary to add each contribution in F_{LIi} and F_{LIf}. Furthermore, the result for all possible excitation

modes in the like-core transition $j^{p+1}(j)\,j'^{q}(0)\rightarrow j^{p}(0)\,j'^{q+1}(j')$ has the same form as (8) and they are different only in respective F. Thus, we can replace $\mathfrak{M}_{LIi}$ by $\mathfrak{M}_L$ and F_{LIi} by F_L and put F_L as the sum of F_{LIi}, F_{LIf}, and other F which appear from the mixing other than *Ii* and *If*.

$$\mathfrak{M}_L=[(2j+1-p)(2j'+1-q)/(2j+1)(2j'+1)]^{1/2}\times \\ \times[l(l'+1)/2(l+l'+1)]^{1/2}(g_s-g_l)F_L, \qquad (8')$$

and

$$F_L=\sum_m F_{Lm}.$$

The form of (8′) is suitable for comparing the results for l-forbidden transitions with allowed transitions. If we consider the allowed transition of like-core type $j^{p+1}(j)\,j'^{q}(0)\rightarrow j^{p}(0)\,\bar{j}'^{q+1}(j')$ where $\bar{j}=\bar{l}+1/2$, $\bar{j}'=\bar{l}-1/2$ and $\bar{l}=(l+l')/2$ (For example, if $j=d_{5/2}$ and $j'=s_{1/2}$, $\bar{j}=p_{3/2}$ and $\bar{j}'=p_{1/2}$.), we get the reduced matrix element for the transition which is just equal to (8′), but with $F=1$. Therefore, F^2 has the meaning of unfavored factor which is a measure of the configuration mixing in our case.

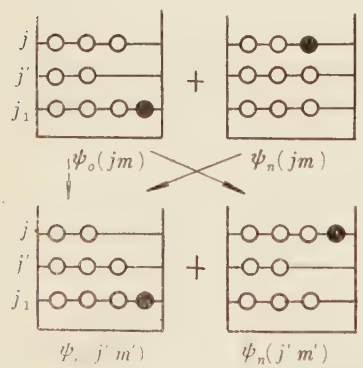

Fig. 4. *LIII.* $j=l-\frac{1}{2}$, $j'=l+j_1=l+\frac{1}{2}$

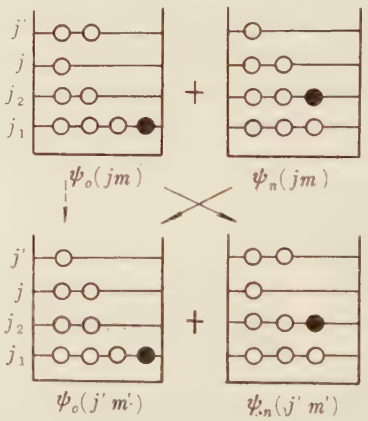

Fig. 5. *UI.* $j=l-\frac{1}{2}$, $j'=l'+\frac{1}{2}$, $l=l'+2$, $j_1=l_1+\frac{1}{2}$, $j_2=l_1-\frac{1}{2}$

When j_1 coincides with j' and $n_2=0$ in (5) and (6), the corresponding mixed configurations in the initial state which give non-vanishing matrix element are

$$\Psi(j'^{q}(0)\;j^{p+1}(j);\,jm)+\sum_J\beta_J\,\Psi(j'^{q}(J)\,j_2\,j^{p}(0);\,jm), \qquad (11)$$

where $j_2=l'-1/2$. Fig. 3 shows this mixing in addition to the mixing in the final state. In (11), β_J are the coefficients of mixing for each J, and $J=2,4,\cdots,2j'-1$, so that $j'^{q}(J)$ represents the state with seniority number two. The derivation of the matrix element is complicated compared with (9), but it becomes easier if we make use of the average value of the reciprocals of the zeroth order energy differences between the first and each of the second configurations in (11). The formula for F_{LIIi} (we designate this excitation mode as *IIi*) is as follows

$$F_{LIIk}= \frac{q}{2j'+1}\,\frac{l'(l'+1)}{2l'+1}\,V_s\,I(j'\,j';\ j\,j_2)\langle -1/\Delta E\rangle. \tag{12}$$

In the same way, the corresponding configuration mixing in the final state

$$\Psi\,(j'^{q+1}(j')\,j^p(0);\ j'm') + \sum\beta'_J\,\Psi\,(j'^{q-1}(j')\,[j_2\,j^{p+1}(j)\,](J);\ j'm'), \tag{13}$$

where $|j_2-j|\leq J\leq j_2+j$, gives the value of F_{LIIf}, which is identical with F_{LIIk} in formulae, but the average value of the energy is different.

Finally, when the mixed configuration is

$$\Psi\,(j_1^{2j_1+1}(0)\,j'^q(0)\,j^{p+1}(j);\ jm) + \sum\gamma_J\,\Psi\,([j_1^{2j_1}(j_1)\,j'^{q+1}(j')\,](J)\,j^{p+1}(j);\ jm), \tag{14}$$

where $j_1=l+1/2$ and $|j_1-j'|\leq J\leq j_1+j'$ (see Fig. 4), the matrix element resulting from this excitation is given by

$$F_{LIIk}=\frac{2j-1-p}{2j-1}\,\frac{l(l+1)}{2l+1}\,V_s I(j_1\,j;\,j'\,j)\langle -1/\Delta E\rangle. \tag{15}$$

Similarly, in the final state, the corresponding excitation is

$$\Psi\,(j_1^{2j_1+1}(0)\,j'^{q+1}(j')\,j^p(0);\ j'm') + \sum\gamma'_J\,\Psi\,(j_1^{2j_1}(j_1)\,j'^q(0)\,j^{p+2}(J);\ j'm'), \tag{16}$$

where $J=2,\ 4,\cdots\cdots,\ 2j-1$. Again, F_{LIIIf} yielded by this modes has the same form as F_{LIIIk}.

Unlike-core (U) transions

There are three modes of excitation in this case too. These are denoted by U_I, U_{II} and U_{III} for both initial and final states, and they are shown in the following. The reduced matrix element $\mathfrak{m}_U$ for the unlike-core transition $j'^q(0)\,j^{p-1}(j)\rightarrow j'^{q-1}(j')\,j^p(0)$ is given by

$$\mathfrak{m}_U=\lfloor pq/(2j+1)\ (2j'+1)\,\rfloor^{1/2}\lfloor l(l'+1)/2\,(l+l'+1)\,\rfloor^{1/2}(g_s-g_l)\,F_U, \tag{17}$$

the meaning of this equation being similar to that for the like-core transitions. The

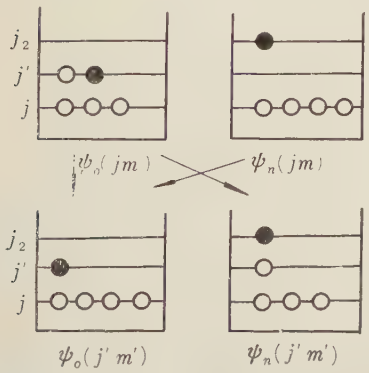

Fig. 6. *UII.* $j=l-\frac{1}{2}$, $j'=l+\frac{1}{2}$, $j_1=l'-\frac{1}{2}$

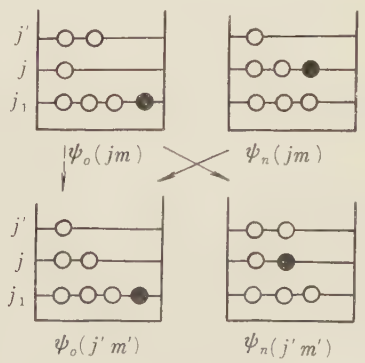

Fig. 7. *UIII.* $j=l-\frac{1}{2}$, $j'=l'+\frac{1}{2}$, $j_1=l+\frac{1}{2}$

$F_{t'}$ are given by the same formula as the corresponding F_t. Therefore, the mixing modes are given in eqs. (18)–(20) and the corresponding figures are Figs. 5–7.

UI. $$\mathcal{F}(j_1^{n_1}(0)\, j_2^{n_2}(0)\, j'^{q}(0)\, j^{p-1}(j);\, jm) + \alpha\mathcal{F}([j_1^{n_1-1}(j_1)\, j_2^{n_2+1}(j_2)](1)\, j'^{q-1}(j')\, j^{p}(0);jm) \tag{18a}$$

$$\mathcal{F}(j_1'^{n_1}(0)\, j_2'^{n_2}(0)\, j'^{q-1}(j')\, j^{p}(0);\, j'm') + \alpha'\mathcal{F}([j_1'^{n_1-1}(j_1)\, j_2'^{n_2+1}(j_2)](1)\, j'^{q}(0)\, j^{p-1}(j);\, j'm') \tag{18b}$$

UII. $$\mathcal{F}(j'^{q}(0)\, j^{p-1}(j);\, jm) + \sum_J \beta_J \mathcal{F}(j'^{q-2}(J)\, j_2\, j^{p}(0);\; jm) \tag{19a}$$

$$\mathcal{F}(j'^{q-1}(j')\, j^{p}(0);\, j'm') + \sum_J \beta'_J \mathcal{F}(j'^{q-1}(j')[j_2\, j^{p-1}(j)](J);\, j'm') \tag{19b}$$

UIII. $$\mathcal{F}(j_1^{2j_1+1}(0)\, j'^{q}(0)\, j^{p-1}(j);\, jm) + \sum_J \gamma_J \mathcal{F}([j'^{q-1}(j')\, j_1^{2j_1}(j_1)](J)\, j^{p+1}(j);\, jm) \tag{20a}$$

$$\mathcal{F}(j_1'^{2j_1+1}(0)\, j'^{q-1}(j')\, j^{p}(0);\, j'm') + \sum_J \gamma'_J \mathcal{F}(j'^{2j_1}(j_1)\, j'^{q}(0)\, j^{p}(J);\, j'm'). \tag{20b}$$

§ 3. Comparison with experimental values

In order to compare the theoretical values for matrix elements $\mathfrak{M}^2$ obtained in the preceding section with the experimental values, knowledge about the quantities which appear in (9), (12), and (15) becomes necessary. All such quantities were already used in the calculation of the correction to magnetic and quadrupole moments by configuration mixing.[5,9] The values estimated there will be adopted for the calculation of $\mathfrak{M}^2$ also and it is not necessary to dwell upon them. Only the values which are used in the present calculation are described very briefly.

The interactions between nucleons are assumed as attractive, with the interaction in the triplet state is stronger than that in the singlet state : $|V_t| = 1.5|V_s|$. The integrals I are inversely proportional to mass number A, since they involve four wave functions. Therefore, the product of the singlet strength and the integral has the form

$$V_s I(j_1\, j_2;\; j_1'\; j_2') = -c_2 F(j_1\, j_2;\; j_1'\; j_2')/2A,$$

where c_2 is a constant having the dimension of energy. And the factor F is a dimensionless constant which does not depend on A but does depend on the shape of the wave functions of the nucleons. We assume that the wave functions are those of a harmonic oscillator. Then, the order of magnitude of c_2 is obtained as 200 to 300 Mev at $A \sim 200$. Actually, the value $c_2 = 250$ Mev is used for the calculation.

In reference 8, we did not take into account the variation of the integral I with respect to the wave functions for nucleons. However, the deviations of the magnetic moments from the Schmidt lines show practically no change (within 0.25 nm) except for nuclei with $3s_{1/2}$ as noted below, even if we make use of the integral in this paper. As for the energy denominator, we neglect the difference between $\langle 1/\Delta E \rangle$ and $1/\Delta E$ in order to avoid complexity, where the latter ΔE means the energy difference between single particle levels in which the shell model configuration and excited configuration differ.

The values adopted are fixed as shown in Table I, irrespective of the mass number A.

Table 1. Doublet splitting assumed in the calculation.

$1f$ 5/2−f 7/2	3 Mev	$1h$ 9/2−h 11/2	2 Mev
$2p$ 1/2−p 3/2	1.5	$2f$ 5/2−f 7/2	1.5
$1g$ 7/2−g 9/2	2.5	$3p$ 1/2−p 3/2	0.5
2d 3/2−d 5/2	1.5	$1i$ 11/2−i 13/2	2

(Table I is identical with Table II of both references 8 and 9.) We can neglect the small energy difference between the initial and final one particle levels (for example $1f_{5/2}-2p_{3/2}$) in the calculation of the matrix elements m. Even if we retain these energy differences for the calculation, their small contribution to the energy denominators in the initial and final mixing are opposite and we need practically only the doublet splitting listed in Table I.

Table II. Experimental life time data for l-forbidden magnetic dipole transitions. $\mathrm{m}^2_{exp}=\dfrac{(2j+1)}{0.419\times10^{13}E^5\tau_T}$

Nucleus	j	j'	E(kev)	τ_T(sec)	m^2_{exp}
${}_{30}Zn_{37}{}^{67}$	$2p_{3/2}$	$1f_{5/2}$	92	9.5×10^{-6}	1.28×10^{-4}
${}_{33}As_{40}{}^{73}$	$1f_{5/2}$	$2p_{3/2}$	66	$<5\times10^{-9}$	$>9.8\times10^{-1}$
${}_{52}Te_{71}{}^{123}$	$2d_{3/2}$	$3s_{1/2}$	159	3.3×10^{-10}	7.16×10^{-1}
${}_{52}Te_{73}{}^{125}$	〃		35.4	4.4×10^{-8}	4.86×10^{-1}
${}_{54}Xe_{77}{}^{131}$	$3s_{1/2}$	$2d_{3/2}$	80	2.1×10^{-9}	4.41×10^{-1}
${}_{56}Ba_{77}{}^{133}$	$2d_{3/2}$	$3s_{1/2}$	11.7	$\sim3.3\times10^{-7}$	~1.80
${}_{55}Cs_{78}{}^{133}$	$2d_{5/2}$	$1g_{7/2}$	81	2.7×10^{-8}	9.9×10^{-2}
${}_{55}Cs_{80}{}^{135}$	〃		248	4.4×10^{-10}	2.12×10^{-1}
${}_{57}La_{82}{}^{139}$	〃		166	2.7×10^{-9}	1.15×10^{-1}
${}_{59}Pr_{82}{}^{141}$	$1g_{7/2}$	$2d_{5/2}$	145	3.9×10^{-9}	1.59×10^{-1}
${}_{61}Pm_{86}{}^{147}$	〃		91	1.1×10^{-8}	2.28×10^{-1}
${}_{73}Ta_{108}{}^{181}$	$2d_{5/2}$	$1g_{7/2}$	480	3×10^{-8}	4.28×10^{-4}
${}_{63}Eu_{9}){}^{153}$	$3s_{1/2}$	$2d_{3/2}$	69.1	1.3×10^{-9}	1.11
				6.0×10^{-8}	2.40×10^{-2}
${}_{80}Hg_{119}{}^{199}$	$3p_{3/2}$	$2f_{5/2}$	50	$<1.6\times10^{-8}$	$>4.76\times10^{-1}$
${}_{81}Tl_{122}{}^{203}$	$2d_{3/2}$	$3s_{1/2}$	280	$\sim1\times10^{-9}$	$\sim4.31\times10^{-2}$

The experimental values for the l-forbidden transitions are listed in Table II. References for each experiment are the same as those quoted in reference 5. The tables for the comparison between the calculated and experimental values are divided according to spins and parities of the initial and final states. The exchange of the initial and final states is possible for the square of the reduced matrix elements as stated in the preceding section. The calculations for the transitions of Ta^{181} and Eu^{153}, were not tried because they are heavy nuclei according to the classification by Way[2] and the single particle model may be not a good approximation to take as a zeroth order wave function.

As the effects of excitation of unlike nucleons are small in comparison with those of like nucleons, we adopt the simplest configuration for unlike nucleons in the following analysis. Except a few examples noted below, the results for m^2 are insensitive to the choice of the configuration of unlike nucleons. In each table in the following, only the final configurations are shown. The initial configurations can be obtained from the final configurations and the type of transitions, L or U.

Table III. l-forbidden magnetic dipole transitions $2p_{3/2} \longleftrightarrow 1f_{5/2}$

nucleus	proton configuration	neutron configuration	type	m^2_{cal}	m^2_{exp}	μ_{cal}	μ_{exp}
odd-neutron nucleus							
$_{30}Zn_{37}^{67}$	$(2p_{3/2})^2(0)$	$(1f_{5/2})^5(5/2)$	U	0.125	1.28×10^{-4}	0.81	0.88
	$(1f_{5/2})^2(0)$	〃		0.069			
odd-proton nucleus							
$_{33}As_{40}^{73}$	$(1f_{5/2})^2(0)\,(2p_{3/2})^3(3/2)$	—	L	0.095			
	〃	—	U	0.294	>0.985		
	$(1g_{9/2})^2(0)\,(2p_{3/2})^3(3/2)$	—	L	0.773			

$2p_{3/2} \longleftrightarrow 1f_{5/2}$ $_{30}Zn_{37}^{67}$ m^2_{cal} is too large in comparison with m^2_{exp}, the ratio being $0.044/(1.28\times10^{-4}) \sim 4\times10^2$. If the effect of configuration mixing is small so as to give a long lifetime for the l-forbidden transition, the deviation of the magnetic moment from the Schmidt line which is 0.49 nm cannot be explained.

$_{33}As_{40}^{73}$ The proton configuration of the nucleus with 23 protons is inferred to be $(2p_{3/2})^3(3/2)\,(1f_{5/2})^2(0)$ by considering that the quadrupole moment of As^{75} is positive. Only one third of the experimental lower limit of m^2 is obtained by this configuration and the U-type transition. Also, the large departure of the magnetic moment of As^{75} which is in the so-called forbidden zone, cannot be explained by the above proton configuration. If we could assume the configuration as $(1g_{9/2})^2(0)\,(2p_{3/2})^3(3/2)$, we get both the large deviation of the magnetic moment As^{75} and the l-forbidden matrix element of As^{73} which is considerably near to the lower limit of the experimental value. However, such configuration is hard to understand from the conventional shell model.[2]

$3s_{1/2} \longleftrightarrow 2d_{3/2}$ The discrepancy between the configurations which give good agreement to the magnetic moment in reference 8 and to the transition matrix element can be removed by considering that the variation of $V_s I$ by the wave function was neglected in the calculation of magnetic moments. The quantity $(-V_s I)$ becomes small in the case of the interaction between $3s$ and $1h$ in comparison with the average $(-V_s I) = 25/A$ Mev.[8] We get consistent results for both magnetic moments μ^* and transition matrix by taking into account the variation of the integral I as stated in the preceding section for the magnetic moments as well. As for the neutron configuration, the $2d_{3/2}$ orbit is occupied before the $1h_{11/2}$ shell is completely closed. This can be seen from the positive quadrupole moments of Ir isotopes. The same situation appears to hold for Xe^{131} and

Table IV. *l*-forbidden magnetic dipole transition $3s_{1/2} \longleftrightarrow 2d_{3/2}$

nucleus	proton configuration	neutron configuration	type	m^2_{cal}	m^2_{exp}	μ_{cal}	μ_{cal}*	μ_{exp}
odd-neutron nuclei								
$_{52}Te_{71}^{123}$	$(1g_{7/2})^2(0)$	$(1h_{11/2})^6(0)\,3s_{1/2}$	L	0.769		−0.17	−0.77	
	〃	$(1h_{11/2})^4(0)\,(2d_{3/2})^2(0)\,3s_{1/2}$	L	0.043	0.716	−0.82	−1.23	−0.74
	〃	〃	U	0.301		〃	〃	
$_{52}Te_{73}^{125}$	$(1g_{7/2})^2(0)$	$(1h_{11/2})^8(0)\,3s_{1/2}$	L	0.925		0.03	−0.68	
	〃	$(1h_{11/2})^6(0)\,(2d_{3/2})^2(0)\,3s_{1/2}$	L	0.074	0.486	−0.60	−1.13	−0.89
	〃	〃	U	0.372				
$_{54}Xe_{77}^{131}$	$(1g_{7/2})^4(0)$	$(1h_{11/2})^{10}(0)(2d_{3/2})^2(0)\,3s_{1/2}$	L	0.140				
	〃	〃	U	0.484				
	$(1g_{7/2})^2(0)\,(2d_{5/2})^2(0)$	〃	U	0.559	0.441			
	〃	$(1h_{11/2})^8(0)\,(2d_{3/2})^4(0)\,3s_{1/2}$	U	0.258				
	$(2d_{5/2})^4(0)$	〃	U	0.339				
$_{56}Ba_{77}^{133}$	$(1g_{7/2})^6(0)$	$(1h_{11/2})^{10}(0)(2d_{3/2})^2(0)\,3s_{1/2}$	L	0.121				
	$(1g_{7/2})^4(0)\,(2d_{5/2})^2(0)$	〃	L	0.160				
	$(1g_{7/2})^2(0)\,(2d_{5/2})^4(0)$	〃	L	0.203				
	$(1g_{7/2})^6(0)$	〃	U	0.442	0.180			
	〃	$(1h_{11/2})^8(0)\,(2d_{3/2})^4(0)\,3s_{1/2}$	U	0.160				
	$(1g_{7/2})^4(0)\,(2d_{5/2})^2(0)$	〃	U	0.223				
odd-proton nucleus								
$_{81}Tl_{122}^{203}$	$(1h_{11/2})^{12}(0)$ $(2d_{3/2})^2(0)\,3s_{1/2}$	$(1i_{13/2})^1(0)\,(3p_{3/2})^4(0)$	L	0.267	0.0431	1.44		1.61

Ba^{133} as the configuration $(1h_{11/2})^{12}(0)\,3s_{1/2}$ gives matrix elements for both elements which are too large, but $(1h_{11/2})^{10}(0)\,(2d_{3/2})^2(0)\,3s_{1/2}\,U$ for Xe^{131} and $(1h_{11/2})^{10}(0)\,(2d_{3/2})^2(0)\,3s_{1/2}L$ or $(1h_{11/2})^8(0)^2(2d_{3/2})^4(0)\,3s_{1/2}L$ for Ba^{133} give good agreement with the experimental values. For Xe^{131} and Ba^{133}, we have considered the effects of the unlike nucleon configurations in more detail, as the variation of the contribution from the like nucleon configuration is comparable with that of the unlike nucleons. In other cases, the latter is considerably smaller than the former.

For Tl^{203}, although the calculated and experimental values are in good agreement for the magnetic moment, the calculated value of the matrix element for the $M1$ transition is larger by factor 5 than the rough experimental value. Because the deviation of the magnetic moment from the Schmidt line is large, the $M1$ transition is expected to be considerably fast.

$2d_{5/2} \longleftrightarrow 1g_{7/2}$ The experimental value of the magnetic moment of Cs^{133} lies between the theoretical values for $(1g_{7/2})^3(7/2)\,(2d_{5/2})^2(0)$ and $1g_{7/2}(2d_{5/2})^4(0)$, and in the case of Cs^{135} the experimental and theoretical value for $1g_{7/2}(2d_{5/2})^4(0)$ of the magnetic moment show good agreement. Similarly, for the *l*-forbidden transitions the experimental value of m^2 for Cs^{133} lies between the calculated values for $(1g_{7/2})^3(7/2)\,(2d_{5/2})^2(0)\,U$ and

Table V. l-forbidden magnetic dipole transitions $2d_{5/2} \longleftrightarrow 1g_{7/2}$.

odd-proton nuclei							
nucleus	proton configuration	neutron configuration	type	m^2_{cal}	m^2_{exp}	μ_{cal}	μ_{exp}
$_{55}Cs_{78}^{133}$	$(1g_{7/2})^3(7/2)\,(2d_{5/2})^2(0)$	$(1h_{11/2})^{12}(0)\,(2d_{5/2})^2(0)$	L	0.224		2.42	
	$1g_{7/2}(2d_{5/2})^4(0)$	〃	L	0.274	0.099	2.75	2.58
	$(1g_{7/2})^3(7/2)\,(2d^5_{/2})^2(0)$	〃	U	0.064		2.42	
	$1g_{7/2}(2d_{5/2})^4(0)$	〃	U	0.123		2.75	
$_{55}Cs_{80}^{135}$	$(1g_{7/2})^3(7/2)\,(2d_{5/2})^2(0)$	$(1h_{11/2})^{12}(0)$	L	0.211		2.37	
	$1g_{7/2}(2d_{5/2})^4(0)$	〃	L	0.260	0.312	2.68	2.72
	$(1g_{7/2})^3(7/2)\,(2d_{5/2})^2(0)$	〃	U	0.060		2.37	
	$1g_{7/2}(2d_{5/2})^4(0)$	〃	U	0.116		2.68	
$_{57}La_{82}^{139}$	$(1g_{7/2})^7(7/2)$	$(1h_{11/2})^{12}(0)$	L	0.012		1.88	
	$(1g_{7/2})^5(7/2)\,(2d_{5/2})^2(0)$	〃	L	0.068		2.19	
	$(1g_{7/2})^3(7/2)\,(2d_{5/2})^4(0)$	〃	U	0.115	0.115	2.50	2.78
	$(1g_{7/2})^5(7/2)\,(2d_{5/2})^2(0)$	〃	U	0.041		2.19	
	$(1g_{7/2})^3(7/2)\,(2d_{5/2})^4(0)$	〃	U	0.133		2.50	
$_{59}Pr_{82}^{141}$	$(1g_{7/2})^6(0)\,(2d_{5/2})^3(5/2)$	$(1h_{11/2})^{12}\,(0)$	L	0.012		3.95	
	$(1g_{7/2})^8(0)\,2d_{5/2}$	〃	U	0.017	0.159	4.53	4.0
	$(1g_{7/2})^6(0)\,(2d_{5/2})^3(5/2)$	〃	U	0.099		3.95	
$_{61}Pm_{86}^{147}$	$(1g_{7/2})^6(0)\,(2d_{5/2})^5(5/2)$	$(1h_{11/2})^{12}(0)\,(2f_{7/2})^4(0)$	L	0.008			
	$(1g_{7/2})^8(0)\,(2d_{5/2})^3(5/2)$	〃	U	0.046	0.228		
	$(1g_{7/2})^6(0)\,(2d_{5/2})^5(5/2)$	〃	U	0.170			

$1g_{7/2}(2d_{7/2})^4(0)\,U$, and the calculated value for $1g_{7/2}(2d_{5/2})^4(0)\,L$ of Cs^{137} is close to the experimental value. There are some other examples for which different isotopes have different ground or excited configurations. (For example, the spins of I and Sb.) For La^{139}, $(1g_{7/2})^5(7/2)$ or $(1g_{7/2})^7(7/2)$, which are necessary to explain the positive quadrupole moment, do not give enough deviation of magnetic moments from the Schmidt line to get agreement with the observed value. The same situation holds for the $M1$ transition. If we can employ $(1f_{7/2})^1(7/2)$, the agreements between experimental and calculated value for both the magnetic moment and the $M1$ transition are satisfactory but the calculated value for the quadrupole moment becomes negative. The quadrupole moment of Pr^{141} is very small (-0.05 barn) and it suggests that the main configuration is $(1g_{7/2})^6(0)\,(2d_{5/2})^3(5/2)$ with slight admixture of $(1g_{7/2})^8(0)\,d_{5/2}$ (see ref. 9). The calculated values of the magnetic moment are 3.95 and 4.53 nm for these configurations respectively, while the experimental value is 4.0 nm. Corresponding to the situation of the magnetic and quadrupole moments, the calculated matrix element for the $M1$ transition $(1g_{7/2})^6(0)\,(2d_{5/2})^5(5/2)$ gives better agreement with the experimental value than by $(1g_{7/2})^8(0)\,2d_{5/2}$. The results for Pr^{141} and Pm^{147} also show that the $1g_{7/2}$ subshell remains unfilled throughout this region of proton number.

$3p_{3/2} \longleftrightarrow 2f_{5/2}$ For Hg^{199} transitions, the calculated value is slightly smaller than the

lower limit of the experimental value.

Table VI. *l*-forbidden magnetic dipole transition $3p_{3/2} \rightarrow 2f_{5/2}$.

odd-neutron nucleus							
nucleus	proto configuration	neutron configuration	type	m^2_{cal}	m^2_{exp}	μ_{cal}	μ_{exp}
$_{80}Hg_{119}^{199}$	$(1h_{11/2})^{12}(0)$	$(1i_{13/2})^{12}(0)(2f_{5/2})^{6}(0)3p_{3/2}$	U	0.281	>0.476	1.44	1.61

§ 4. Conclusion

There are two major objections to the explanation of the *l*-forbidden $M1$ transitions by the consideration of the configuration mixing. One is that given by Ross[4] and the other is that given by De Waard and Gerholm[12]. The former objection is that we need unreasonably large admixture of the excited configuration in order to explain the lifetime of Xe^{131}, so that we must take into account the contribution by the exchange current. However, the configuration mixing which he employed is only one of several modes of excitation and, furthermore, the only contribution was underestimated. We have shown that the correct lifetime can be obtained by the mixing of configurations which have the probability of about 1%. Therefore, the lifetimes of the *l*-forbidden $M1$ transition do not necessarily give the evidence for the exchange current. On the other hand, the objection by De Waard and Gerholm is that the extent of admixture must increase with mass number from Cs^{133} to Pm^{147} although there is no such tendency in experimental values. But there is also no tendency for the effect of configuration mixing to become larger for heavier nuclei in the calculated values in the preceding section.

The calculated values by Spruch and Rotenberg[5] due to the exchange current give fairly good agreement with the experiments. However, it must be observed that their calculated values are normalized to the experimental values as the effects of the exchangecurrent cannot be determined definitely. In their results, $\tau_\gamma(theor)/\tau_\gamma(exp) \approx 2 \sim 3$ for odd-neutron nuclei and $\tau_\gamma(theor)/\tau_\gamma(exp) \approx 1/3 \sim 1/5$ for odd-proton nuclei. This contradicts the experimental result that the $\tau_\gamma(exp)$ of the odd-neutron nuclei are small than those of the odd-proton nuclei[2]. In our results in the preceding section, $\tau_\gamma(theor)/\tau_\gamma(exp) \approx 2 \sim 1/2$ holds for both odd-neutron and odd-proton nuclei. Furthermore, it must be stressed that the parameters adopted in this paper are the same as those used in the consideration of the magnetic and quadrupole moments of odd A nuclei,[8,9] and it is interesting that there is some correlation between the analysis of the magnetic and quadrupole moments and that of the *l*-forbidden $M1$ transitions, as was shown in §3.

There is another explanation of the *l*-forbidden $M1$ transitions which employs the spin-orbit interaction.[12,13] However, this effect vanishes for odd-neutron nuclei, and moreover, even if the effective charge of neutron is considered, the lifetimes of the odd-neutron nuclei are found to be considerably larger than those of the odd-proton nuclei. This contradicts the experimental results quoted above.

We should like to thank Professor T. Yamanouchi and Professor M. Kobayasi for their kind interest and valuable advices. Appreciation is also due to Professor S. Nakamura.

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Errata : Configuration Mixing and Magnetic Moments of Odd Nuclei, A. Arima and H. Horie, Prog. Theor. Phys. **12** (1954), 623.

		Wrong	Correct
Table III	$_{81}Tl^{205}$	1.45	1.43
	$_{50}Sn^{117}$	−1.26	−1.18
	$_{52}Te^{123}$	$(d^5_{\frac{3}{2}})^2(h^{11}_{\frac{1}{2}})^1s^1_{\frac{1}{2}}$	$(d^5_{\frac{3}{2}})^2(h^{11}_{\frac{1}{2}})^1s^1_{\frac{1}{2}}$
		−0.04	−0.17
		−1.39	−1.46
	$_{52}Te^{125}$	−0.03	+0.03
Table V	$_{33}As^{75}$	1.62	1.54
	$_{35}Br^{79}$	2.06	1.91
	$_{37}Rb^{87}$	$(g^9_{\frac{9}{2}})^9$	$(g^9_{\frac{9}{2}})^{10}$
Table VI	$_{77}Ir^{193}$	$(i^{13}_{\frac{3}{2}})^8$	$(i^{13}_{\frac{3}{2}})^{10}$
	$_{79}Au^{197}$	$(h^{11}_{\frac{1}{2}})^{12}(d^3_{\frac{3}{2}})^3$	$(h^{11}_{\frac{1}{2}})^{10}(d^3_{\frac{3}{2}})^3$
		0.43	0.45
		0.28	0.29
	$_{56}Ba^{137}$	0.77	0.75
Table X	$_{51}Sb^{123}$	$(d^{11}_{\frac{1}{2}})^6$	$(h^{11}_{\frac{1}{2}})^8$
		2.46	2.49
	$_{53}I^{129}$	2.84	2.67
		2.69	2.59
Table XII	$_{83}Bi^{209}$	3.43	3.30

P. 633 3 line from the bottom

Wrong

Adopting I(2*d*, 2*d*) as the standard of magnitude and inserting these values of ratios, we obtain μ^* cal in the above table.

Correct

Adopting I(2*d*, 2*d*) as the standard of magnitude and inserting these values of ratio, briefly

I(1*d*, 1*d*) : I(2*d*, 2*d*) : I(2*d*, 1*g*) : I(2*d*, 1*h*) ≈ 3 : 2 : 1 : 1

we obtain μ^* cal in the above table.

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Pion-Nucleon Scattering and Structure of Fourth-Order Matrix Elements in Ps(Ps) Meson Theory*

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The fourth-order S-matrix elements (ME) for elastic pion-nucleon scattering at low energies are analytically evaluated by expanding in powers of mass ratio μ/M, and are analyzed in detail so as to compare relative advantages of the existing theories. The scattering phase shifts are then evaluated using the variation principle for K-matrices. It is found that the main s-wave parts of the ME contain $\ln(M/\mu)$ term which has no bearing on the recoil effect and has the different sign for crossed and ladder graphs. In connection with this, the reason why the 5-box TD approximation gave essentially isotopic spin independent s-wave scattering is given. It is also suggested that the main recoil p-wave term $\ln(M/\mu)$ appearing only in the crossed graphs can be a possible reason why our $I=J=3/2$, p-wave phase shift does not show resonant behavior, contrary to that in the 5-box TD method as well as the Chew and Low theory.

§ 1. Introduction

In order to understand the pion-nucleon scattering experiments[1] in energy region below about 300 Mev, many theoretical approaches were done using symmetric ps-meson theory with ps- or pv-coupling:

$$H_{ps}(x) = ig\bar{\psi}(x)\,\gamma_5\tau_\rho\psi(x)\,\phi_\rho(x)\,, \tag{1}$$

$$H_{pv}(x) = i(f/\mu)\,\bar{\psi}(x)\,\gamma_5\gamma_\nu\tau_\rho\psi(x)\,(\partial\phi_\rho/\partial x_\nu)\,. \tag{2}$$

Here and in below the standard notations and natural rationalized units are used.

The pv-coupling theory or fixed-source theory was discussed in detail by Chew et al.[2] and was successful in explaining the p-wave scattering data. The approximation method adopted there was similar to the Tamm-Dancoff (TD) one which takes into account the Lévy graphs with two or fewer pions.[2a] Meanwhile the excellent approach for the fixed-source model was put forward by Low and Chew.[3] Their theory was very successful in explaining the p-wave scattering data even in the one-meson approximation.** Nevertheless, it must be noted that no appreciable s-wave scattering can be expected in the pv-coupling theory.

The ps-coupling or γ_5 theory is in marked contrast to the above theory. Some of

* Preliminarily reported at 1955 Annual Meeting of Physical Society of Japan, Tokyo.

** The result of this approximation seems to agree almost with the result expected from the calculation which is much similar to ours (§ 4) but applied to the pv-coupling theory. Cf. Blair and Chew, Phys. Rev. **90** (1953), 1065. For validity of one-pion approximation, see ref. 4).

the qualitative features of the former can be best disclosed by the Tani-Foldy transformation applied to (1). This gives the following effective interaction up to order M^{-2}.[7]

$$H_{\text{eff}}=H_c+H_{pv}'+H_\pi\,,$$

$$H_c=(g^2/2M)\int dv\,\psi^*\psi\boldsymbol{\phi}^2,\quad H_\pi=(g/2M)^2\int dv\,\psi^*(\boldsymbol{\tau}[\boldsymbol{\phi}\times\boldsymbol{\pi}])\psi\,, \tag{3}$$

$$H_{pv}'=(f'/\mu)\int dv\,\psi^*\boldsymbol{\sigma}\cdot\boldsymbol{\Gamma}(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\cdot\psi\,,\quad f'=g\mu/2M\,,$$

where H_c, H_{pv}' are the strongly repulsive core term and the equivalent pv-coupling respectively; the relatively small H_π is the isotopic-spin dependent s-wave interaction. The theory based on (3) will be referred as H_{eff} theory in below. Of course, cutting off pion energies at $\sim M$ must be understood. In calculating the s-wave scattering, Fukuda et al.[6] used the TD method considering H_c+H_π as an effective pion-nucleon potential.* They showed that the s-wave scattering amplitude is suppressed largely. But, in order to get enough isotopic-spin dependence, it was essential to take into account the s-wave damping due to nucleon self-energy effects.[7,8] The $I=1/2$, s-wave phase shift thus obtained was much sensitive for cut-off energies. In addition, Goebel[5] recently showed that H_π and also H_{pv}' are strongly suppressed by virtual p-waves; this effect was not considered by Fukuda et al.[6]

On the other hand, the TD method applied to the original γ_5 theory was also presented by Dyson et al. and Chiba et al.,[9] in the 5-box TD approximation which includes up to one-pair or/and two-pion states. The results obtained by the latter authors showed no appreciable isotopic-spin dependence in the s-wave scattering. Recently, Kalos and Dalitz[10] reported the results of a similar and more accurate calculation, confining to the $I=3/2$, s-state and $I=J=3/2$, p-state. Their results showed the resonant p-wave scattering which was in good agreement with the experiment for the reasonable value of $f^2/4\pi$. They pointed out that strong interaction in the region of high virtual momenta is important for sufficiently sharp resonance, and this suggests that an uncautious cutting off in γ_5 theory should be avoided.

Thus, we can expect no appreciable s-wave scattering in the pv-coupling theory. On the other hand, in H_{eff} theory and the 5-box TD method we have the too large scattering with incorrect isotopic-spin dependence. As for p-wave scattering, the former theory seems to be less promising than the latter. But the 5-box TD method violated the symmetry theorem of Gell-Mann and Goldberger[11] from the outset. In connection with this, why the resonance in the p-wave scattering was reproduced and also why the essentially isotopic-spic independence in the s-wave scattering was obtained will be asked.

In order to get some insight on these points and to compare the various existing theories reviewed above, we examine the covariant fourth-order matrix elements in γ_5 theory. We evaluate the matrix element for each Feynman graph, expanding in powers

* Note added after completion of MS. F. L. Lomon, Nuovo Cimento **IV** (1955), 106, solved this problem exactly.

of μ/M and retaining the first few terms for *s*- and *p*-waves (§ 2). Similar detailed studies were already done by Wyld,[11] in zero pion-momentum limit. We furthur try to see the energy dependences in higher momentum region for the purposes above stated. We put stress on looking into structure of γ_5 theory and interpreting each term in the above expansion in connection with H_{eff} theory (§ 3). Finally, we estimate the scattering phase shifts by the variation principle for *K*-matrix. By this method which was applied by Ito and Tanaka and also by Sartori and Wataghin,[12] iterated graphs with two-pair or three-pion which were omitted in the 5-box TD method are also taken into account. Our results are compared with those of the above mentioned various approaches and with experimental data (§ 4).

§ 2. S-matrix elements in fourth order

The interaction Hamiltonian in γ_5 theory is given by (1), supplementing the mass counter terms of nucleon and pion. The Feynman graphs of second and fourth orders for elastic pion-nucleon scattering are shown in Fig. 1, where the graphs of crossed and ladder type are denoted by *I* and *II* respectively.

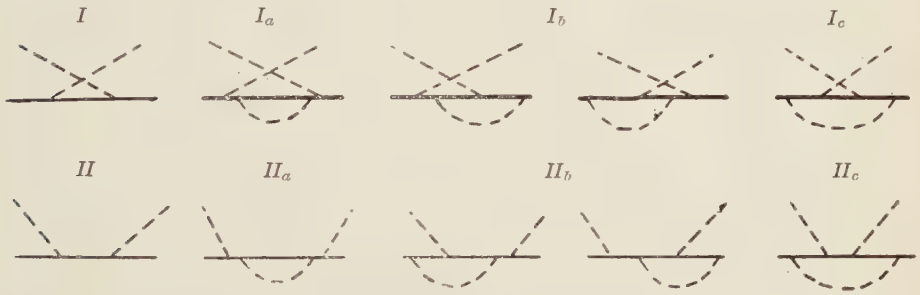

Fig. 1. Feynman graphs for scattering in second and fourth orders, pion and nucleon lines being denoted by full and dotted lines, respectively. All graphs labelled with *I* are of crossed type, and ones with *II* are of ladder type.

The elements of covariant *S*-matrix are readily written down for each graph. After conventional renormalization,* the matrix elements of fourth order take the following form :

$$(g/4\pi)^4 \bar{\psi}(p_2) I_\tau [\pm i k_{2,1} A_{I,II} + M B_{I,II}] \psi(p_1), \tag{4}$$

where p_1, p_2 (k_1, k_2) are four-momenta of the initial and final nucleon (pion) respectively. *A*, *B* are multiple integrals over Feynman parameters. Almost all of these integrals are reducible to single parameter integrals, thus allowing elementary but tedious approximate evaluation in the form of power series in μ/M at lower energies.

In the *NR* approximation for nucleon, the required matrix element can be written in the following form :

* A modification for vertex renormalization, different from ref. 14), is implied ; see Appendix.

$$(g/4\pi)^4 I_\tau \left\{cM+c'\varepsilon\ln\frac{M}{\mu}+\varepsilon\cdot s+\frac{\boldsymbol{k}_1\boldsymbol{k}_2}{M}\left[r\ln\frac{M}{\mu}+p\right]+\frac{(\sigma\cdot\boldsymbol{k}_2)\,(\sigma\cdot\boldsymbol{k}_1)}{M}\cdot p_\sigma\right\}, \quad (5)$$

omitting the common factor i/M^2 and NR nucleon spinors. $\varepsilon=(k^2+\mu^2)^{1/2}$, $k=|\boldsymbol{k}_1|=|\boldsymbol{k}_2|$ are pion energy and momentum in the barycentric system. The isotopic spin factor I_τ and the coefficients c, c' and r are given in Table I. The functions s, p and p_σ, which are at most of order M^0 and are estimated up to this order,* are given by Eq. (6).

Table I. The coefficients in core and recoil terms, c, c' and r, and the isotopic spin factor I_τ. $\frac{1}{2}I$, $\frac{1}{2}II$ are contributions from each of the two graphs of vertex type.

	c	c'	r	I_τ
I_a	1/2	−1	−1	$3\tau_1\cdot\tau_2$
II_a	1/2	+1	0	$3\tau_2\cdot\tau_1$
$\frac{1}{2}I_b$	1	−1	−1	$-\tau_1\cdot\tau_2$
$\frac{1}{2}II_b$	1	+1	0	$-\tau_2\cdot\tau_1$
I_c	1	−1	−1	$4\delta\rho_1\rho_2-\tau_1\cdot\tau_2$
II_c	1	+1	0	$4\delta\rho_1\rho_2-\tau_2\cdot\tau_1$

$$s\approx\frac{1}{2}+x^2,\quad p\approx 3+\frac{1}{3}x^2,\quad p_\sigma\approx-\frac{3}{8}. \quad (6\,\mathrm{Ia})$$

$$s\approx-\frac{1}{2}-x^2+i\pi x,\quad p_\sigma\approx\frac{1}{8}. \quad (6\,\mathrm{IIa})$$

$$s\approx 0.43+1.28x^2,\quad p\approx\frac{7}{2}+\frac{1}{2}x^2,\quad p_\sigma\approx-\frac{1}{2}. \quad (6\,\mathrm{Ib})$$

$$s\approx 2.23-3.40x^2+i\pi x,\quad p_\sigma\approx 1.15-1.34x^2. \quad (6\,\mathrm{IIb})$$

$$s\approx-1.70+2.32x^2-\frac{1}{4}x^2(1+x^2)\ln(2/x^2),$$

$$p\approx 5.07-2.95x^2+\frac{1}{4}x^2\left(1+\frac{7}{4}x^2\right)\ln(2/x^2), \quad (6\,\mathrm{Ic})$$

$$p_\sigma\approx 0.43_5+0.52x^2-\frac{1}{8}x^2(1+x^2)\ln(2/x^2).$$

$$s\approx 1.63-0.90x^2+\frac{1}{32}x^4(1-2x^2+3x^4)\ln(2/x^2)+i\pi x-i\frac{\pi}{4}x^2\left(1+\frac{3}{2}x^2\right),$$

* Although corrections of order M^{-1} and $M^{-1}\ln(M/\mu)$ are not so large as is inferred from the exact zero-momentum values given by Wyld,[11] our expansion approximation may be by no means so much good for the actual ratio $\mu/M=0.15$.

$$p \approx -\frac{1}{6} + i\frac{2}{3}\pi x\,,$$

$$p_\sigma \approx 0.90 - 0.31x^2 + \frac{1}{64}x^4(1-2x^2+3x^4)\ln(2/x^2) - i\frac{\pi}{4}x - i\frac{\pi}{8}x^2\left(1+\frac{3}{2}x^2\right), \tag{6 IIc}$$

where* $x \equiv k/\mathcal{E}$.

The matrix elements of second order are written in a similar form as (5) :

$$I \approx (g/4\pi)^2 \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left\{-\frac{M}{2} - \frac{\mathcal{E}}{4} + \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{M}\left[\frac{M}{2\mathcal{E}} + \frac{1}{2} - \frac{x^2}{2}\right] + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{k}_1)}{M}\left[-\frac{M}{4\mathcal{E}} - \frac{1}{4} + \frac{x^2}{8}\right]\right\}, \tag{7 I}$$

$$II \approx (g/4\pi)^2 \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_1\left\{-\frac{M}{2} + \frac{\mathcal{E}}{4} + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{k}_1)}{M}\left[-\frac{M}{4\mathcal{E}} + \frac{x^2}{8}\right]\right\}. \tag{7 II}$$

Partial wave analysis of Eqs. (5) and (7) is easy, recalling the well-known relation :

$$\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 = -T_1 + 2T_2\,, \quad \boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_1 = 3T_1\,,$$
$$\boldsymbol{k}_1\cdot\boldsymbol{k}_2/k^2 = P_1 + P_2\,, \quad (\boldsymbol{\sigma}\cdot\boldsymbol{k}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{k}_1)/k^2 = 3P_1, \tag{8}$$

where T_{2I}, P_{2J} are the projection operator.

§ 3. Structure of fourth-order matrix elements

The matrix elements (ME) obtained in § 2 compose of terms of different kinds. For example, the *s*-wave parts of the second-order ME (7) contain one term of order M and one of order μ. The former is as a whole isotopic spin independent, and is just the lowest order ME of H_c in (3) as was shown also by Wyld.[11] On the other hand, the minor terms $\mp\mathcal{E}/4$ of order μ are readily seen to be equivalent to the isotopic spin dependent H_π in H_{eff} theory.

Regarding the *p*-wave parts of the second-order ME, the leading terms of order M is ascribed to H'_{pv} in (3). The recoil effects contribute to the minor terms of order μ.

We now consider the fourth-order ME. The main *s*-wave parts : $cM + c'\mathcal{E}\ln(M/\mu)$ turn out

$$4\,\delta_{p_1p_2}[M - \mathcal{E}\ln(M/\mu)] - \tfrac{3}{2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 M\,, \tag{9'}$$
$$4\,\delta_{p_1p_2}[M + \mathcal{E}\ln(M/\mu)] - \tfrac{3}{2}\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_1 M \tag{9''}$$

* All functions of k or ε appearing after the evaluations were expanded with respect to x and the relatively small terms were omitted. The only singular function $\ln(\varepsilon/\mu)$ in the limit $\mu \to 0$ was also expanded.

for crossed or ladder type respectively. The first terms are well explained as being due to iteration of the core potential, $H_c a^{-1} H_c$, where a^{-1} is the propagator of the system. For example, the corresponding graph of ladder type which is collection of II_a', II_b', II_c' in Fig. 2 and the inverted graph of II_b' has the ME proportional to

$$\delta_{p_1 p_2} (g^2/2M^2) \int d\boldsymbol{k}'/\mathcal{E}' (W - W' + i\eta) . \tag{10}$$

W, W' are energies of the initial and intermediate states respectively and $\mathcal{E}'$, $\boldsymbol{k}'$ refer to the virtual pion. If we neglect in (10) nucleon recoil energy and cut off $\mathcal{E}'$ at M, the integral in (10) turns out to be $-M - \mathcal{E} \ln (M/\mu) - i\pi k$. Similarly in the case of crossed type, we get $-M + \mathcal{E} \ln (M/\mu)$. We can also confirm the above conclusion by evaluation of contributions from the convergent graphs II_a', etc., neglecting recoil effects and introducing cut-off.

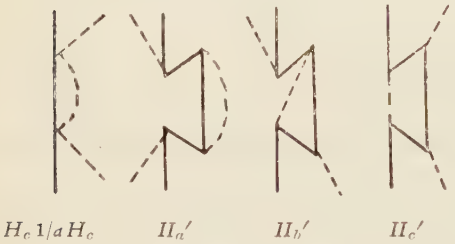

Fig. 2. Lévy graphs for iteration of core potential and equivalent ones belonging to fourth-order graphs of ladder type.

Fig. 3. Lévy graphs from fourth-order graphs of crossed type, one of which is responsible for the leading recoil effect in the p-wave scattering.

Thus the $\ln (M/\mu)$ term is entirely indifferent to the nucleon recoil effect and is due to having one intermediate state with no pair and one virtual pion.* We note that at higher energies this term is comparable to the first one in the bracket in (9).

The s term contains the minor contributions (including the recoil effects) from the graphs above considered, and also the contributions from Lévy graphs such as I_c'' and I_c''' in Fig. 3. Its magnitude is, however, by no means small compared with the c and c' terms for the actual ratio μ/M.

Nextly, we consider the p-wave part. The remarkable point is that the most predominant $\ln (M/\mu)$ term appears only for the graphs of crossed type. As was noted firstly by Wyld,[11] this is due to the main recoil effect associated with pair production in the graph I_c' in Fig. 3 and the similar graphs I_a', I_b'. These graphs involve three-pion state and therefore are omitted in the 5-box TD approximation.

* In fact, the graphs lacking this state give rise to no c'-term. For example, the four Lévy graphs belonging to II_c which have a two-pair (and one-meson) intermediate state contribute only to the c-term and each contribution is about 8 times smaller than that from II_c'. This explains the second term in (9″). It is noted that these four graphs are omitted in the 5-box TD approximation.

The above recoil term is of course absent in the fixed-source theory. It lacks also in H_{eff} theory. For, firstly the virtual pion in I_c' responsible for our recoil term is emitted and reabsorbed as an s-wave pion. Secondly the graph in H_{eff} theory which corresponds to collection of I_c'' and similar graphs I_a'', I_b'' is due to product action of H_c and iteration of H_{pv}' and is easily seen to contribute only to the p_σ term in (5).*

Thus the leading recoil term $\ln(M/\mu)$ is characteristic of γ_5 theory. It is evidently spin independent and moreover the whole contribution from all the fourth-order Feynman graphs, $-4\,\delta_{p_1p_2}\ln(M/\mu)$, is independent of isotopic spin.

Finally, we remark on the symmetry theorem for s-wave scattering. It is observed from Table I and (8) that the isotopic spin independence is already valid for I_a+II_a, I_b+II_b and I_c+II_c separately. Further, the total s-wave part turns out to be $5M\delta_{p_1p_2}$ because we observe from (9) cancellation of the $\ln(M/\mu)$ terms. On the other hand, we expect in the 5-box TD approximation nearly the first term in (9''), recalling the remark in the footenote, p. 586. This is the reason why there occurs no essential difference between the s-wave phase shifts obtained by the 5-box TD treatment and our treatment, as will be seen in § 4.

§ 4. Scattering phase shifts and comparison with experiments

The K-matrices in second and fourth order for each eigenstate of given I, J are easily obtained from Table. I, (5), (6) and (7); putting them

$$\begin{aligned} K_{2I}^{(n)} &= -(g^2/4\pi)\,(g/4\pi)^{n-2}(\varepsilon/2M)\,D_{2I}^{(n)}, \\ K_{2I,2J}^{(n)} &= -(g^2/4\pi)\,(g/4\pi)^{n-2}(2k^2/3\,\varepsilon^2M^2)\,D_{2I,2J}^{(n)}, \end{aligned} \tag{11}$$

where $n=2$ or 4, we have $(x=k/\varepsilon)$

$$\begin{aligned} &D_1^{(2)}=-1+\varepsilon/M, \quad D_3^{(2)}=-1-\varepsilon/2M, \\ &D_{11}^{(2)}=-2+(\varepsilon/4M)\,(1+4x^2), \\ &D_{13}^{(2)}=D_{31}^{(2)}=-\tfrac{1}{2}D_{33}^{(2)}=-\tfrac{1}{2}-(\varepsilon/2M)\,(1-\tfrac{1}{2}x^2), \end{aligned} \tag{12}$$

while the leading terms in $D_{2I}^{(4)}$ and $D_{2I,2J}^{(4)}$ are 5 and $(\varepsilon/M)\,|-4\ln(M/\mu)|$ respectively.

The variation principle for K-matrix leads to $K\approx K^{(2)}[1-K^{(4)}/K^{(2)}]^{-1}$ in the first approximation, from which we obtain the following phase shifts**:

$$\begin{aligned} \tan\delta_1 &\approx -(k/\mu)\,a(0.7-0.15x^2)\cdot[1+b(1.8+1.4x^2)]^{-1}, \\ \tan\delta_3 &\approx -(k/\mu)\,a\cdot 0.93[1+b(5.3+0.4x^2)]^{-1}, \\ \tan\delta_{11} &\approx -(k^3/\mu^2\varepsilon)\,a'\cdot\tfrac{8}{3}(0.85-0.15x^2)[1+b'\cdot\tfrac{1}{2}(8.0+20x^2+12x^4)]^{-1}, \\ \tan\delta_{13} &\approx -(k^3/\mu^2\varepsilon)\,a'\cdot\tfrac{2}{3}(1-0.09x^2)[1+b'\cdot 2(13.6-6.8x^2)]^{-1}, \end{aligned} \tag{13}$$

* The graph I_c''' gives an s-wave contribution of order $(g^2/2M)^2(\mu/M)\ln(M/\mu)$, which may be partly responsible for s-wave suppression by virtual p-wave pion as was pointed out by Goebel.[5]

** The unimportant $\ln(2/x^2)$ terms in (6) are here omitted.

$$\tan\delta_{31}\approx-(k^3/\mu^2\varepsilon)\,a'\cdot\tfrac{2}{3}(1-0.09x^2)\left[1+b'\cdot 2(15.2+1.9x^2+2.9x^4)\right]^{-1},$$
$$\tan\delta_{33}\approx+(k^3/\mu^2\varepsilon)\,a'\cdot\tfrac{4}{3}(1-0.07x^2)\left[1-b'(5.2-2.6x^2)\right]^{-1}, \qquad (14)$$

where

$$a=(g^2/4\pi)\,(\mu/M),\quad b=(g/4\pi)^2,$$
$$a'=f'^2/4\pi,\quad b'=(f'^2/4\pi)\,(M/\pi\mu).$$

We compare these results with the experimental data. As regards the s-wave phase shifts in low energy region, Orear[17] pointed out that his low energy fit,

$$\tan\delta_1=+0.16\,(k/\mu),\quad \tan\delta_3=-0.11\,(k/\mu), \qquad (15)$$

is consistent with rather higher energy (up to about 300 Mev) data.

Comparing the Orear's value (15) with ours in low energy limit, we get the ps-coupling constant $g^2/4\pi=1.2$ for $\mu/M=0.15$. This is consistent with the value obtained by Kalos and Dalitz[9] but is about 10 times smaller than the reasonable value. This may be due to insufficient suppression of the core potential H_c. With the above value, we obtain $\tan\delta_1\approx-0.09\,(k/\mu)$ at low energies in complete disagreement with the experimental data (15). This is in marked contrast to the result of Sartori and Wataghin[11] who obtained a peculiarly large isotopic spin dependence of s-wave phase shifts. Their approach was similar to ours, but cutting off was adopted in evaluation of the ME and the renormalization introduced by Deser, Thirring and Goldberger[10] was applied. In connection with this, Ishida[17] pointed out that their result would be reproduced if one used therein an unreasonable value -2.7 as the conventional renormalized coupling constant.

It is remarked that Lévy's covariant theory[13a] gave the large isotopic spin dependence, but violated the symmetry theorem since only the graphs I and II in Fig. 1 and their iterated graphs were taken into account. Thus, the problem of s-wave scattering has not yet been explained.** The possible pion-pion interaction which was recalled for high energy pion phenomena* also does not seem to be available.[18] In this situation, the effects of strange particles are desirable to be studied in view of uncloseness of meson theory.

Next, we concern ourselves in the p-wave phase shifts. Experimentally, the phase shifts except for the resonant $I=J=3/2$ state are small, and only δ_{33} is known in reliable accuracy; this is

* These phenomena as well as the asymptotic behavior of renormalized vertex function seem to suggest that the s-wave interaction must be rather very weak in essence; cf. Lehmann, Symanzik and Zimmermann, Nuovo Cimento **II** (1955), 425. We are indebted to Ishida for calling attention to this point.

** After completion of MS., the interesting paper of A. Martin, Nuovo Cimento **IV** (1956), 369, appeared. He modified Lévy's theory so as to satisfy the symmetry theorem, by including all the graphs which are obtained from the above noted iterated graphs by interchanging the initial and final pion lines. (We note that the graph I_c in Fig. 1 is this time a member of these graphs.) For $g^2/4\pi\sim 15$, the s-wave phase shifts thus obtained turned out to be correct in sign and relative ratio, although their magnitudes were adout 10 times larger than the data (51). It is very interesting to investigate whether this theory can reprobuce the resonant p-wave scattering.

$$\delta_{33}=0.235\,(k/\mu)^3. \tag{16}$$

at low energies. The analysis of the data up to energies fairly above the resonance energy ≈ 190 Mev (in the laboratory system) was done by Grandley and Clark[20] according to the effective range theory of Brueckner. We refer Orear's best fit[17]:

$$(k/\mu)^3 \cot\delta_{33}=(\mathcal{E}^*/\mu)\left[8.05-3.8\,(\mathcal{E}^*/\mu)\right], \quad \mathcal{E}^*=\mathcal{E}+k^2/2M. \tag{17}$$

With this the result from Chew-Low theory for the fixed-source model in one-pion approximation is in excellent agreement, if one takes* $f^2/4\pi=0.08$, $\mathcal{E}_{\max}\sim 6\mu$.

For comparison with (17), our result is conveniently rewritten as

$$(k/\mu)^3 \cot\delta_{33}\approx \tfrac{3}{4}\,(f'^2/2\pi)^{-1}(\mathcal{E}/\mu)\left[1-(M/\pi\mu)\,(f'^2/4\pi)\,(5.2-2.6x^2)\right]. \tag{18}$$

The low energy fit requires $f'^2/4\pi=0.09$ or $g^2/4\pi=15$, which is quite a reasonable value. There are, however, discrepancies: the second term in the bracket of our (18) is about 2 times large and has wrong energy dependence. Hence, although this term is quite sensitive to the neglected corrections of M^{-1} and $M^{-1}\ln(M/\mu)$ our (18) can predict no resonance in the p-wave scattering. This is an essential difference from the result of Chew's theory[2] which was obtained by the similar approximation method as ours. This difference is mostly attributed to the existence of the graphs with 3 pions (e.g., I_c') in our γ_5 theory which gave rise to the characteristic recoil term $\ln(M/\mu)$, as was emphasized in § 3. We recall also that the above graphs are absent in the 5-box TD method applied especially by Kalos and Dalitz[9] and also in the variational method by Ito and Tanaka.[10] It is very likely that the recoil effect in question is anomalously large in fourth order because the virtual pion in I_c' propagates as an s-wave and interacts with nucleon-antinucleon pairs, and it will be important for any possible suppression effect that higher-order corrections are to be considered.

As for the other p-wave phase shifts, the b' terms are all by far large and positive, so that these phase shifts remain relatively small (and always positive), favorably from the existing experimental evidences.

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Appendix

The procedure of removing divergence from the vertex function occurring in I_b, II_b is slightly modified, rather than that adopted in the case of anomalous magnetic moment,[14] with the following changes in Eq. (13), ref. 14):

$$N=(i\gamma p+M)\,\gamma_5\,(i\gamma q+M)\,uv+\left[(i\gamma p+M)\gamma_5+\gamma_5(i\gamma q+M)\right]M(1-uv)-\gamma_5 G\Lambda_0^{-1}\mu^2 uv,$$

* Note added after completion of MS. Recently Cini, Fubini and Stanghellini, Nuovo Cimento ***III*** (1956), 1380; ***IV*** (1956), 168, obtained the value $f^2/4\pi=0.11$ from the experimental δ_{33} and δ_{31}, using the exact Low equations directly.

$$G=(p^2+M^2)\,u(1-u)\,v+(q^2+M^2)\,u^2v(1-v)+|(p-q)^2+\mu^2|\,u(1-u)\,(1-v),$$

$$\Lambda_0=M^2(1-uv)^2-\mu^2u(1-2v-u(1-v)),$$

$$A=(1/i\pi^2)\int_0^1 du\,dv\,u\int d^4k\,[k^2-M^2(1-uv)^2+\mu^2u(1-u)\,(1-v)]\,(k^2+\Lambda_0)^{-3}.$$

With this, it proves that evaluation of the radiative correction in question to pion-nucleon scattering can be performed quite simpler than otherwise, and the final result obtained takes a form having much systematic similarity as ones from the other fourth order graphs. The parameter integrals A_{Ib}, B_{Ib} in (4) are given by

$$A_{Ib}=\int_0^1 du\,dv\,\lambda u^4v^2(1-v)\,\Phi_1^{-1}\Phi^{-1}+2\int_0^1 du\,dv\,dz\,u^3v(1-v)\,\Phi_{1,z}^{-1},$$

$$B_{Ib}=\int_0^1 du\,dv\,u(1-uv)\,\Phi_1^{-1},$$

where

$$\Phi=(1-uv)^2-\lambda u(1-2v-u(1-v)),$$

$$\Phi_1=\Phi+x_1u^2v(1-v),\quad \Phi_{1,z}=\Phi+x_1u^2v(1-v)z,$$

$$\lambda=(\mu/M)^2,\quad x_1=-\lambda+2\,p_1k_2/M^2,\quad x_2=-\lambda+2\,p_1k_1/M^2,$$

with understanding that these functions have a small imaginary part $-i\eta$ if there occur singularities such as pole or branch point in the course of parameter integrations. For A_{IIb}, B_{IIb}, we have only to put x_2 in place of x_1; note that Φ_2 thus obtained clearly has a pole from the outset owing to negative x_2. The above integrals are all reducible to single parameter ones; we encounter quite similar ones also in one part of I_c, II_c, while evaluation of the other part is of course most complicated and tedious.

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Mathematical Formulation of the Gell-Mann-Nishijima Scheme for New Particles

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It is shown that the theory of d'Espagnat and Prentki (henceforth referred to as DP) on a mathematical formulation of the Gell-Mann-Nishijima scheme can be generalized so as to accomodate particles of higher isofermion number. Modifications made for this purpose are: (i) the isospinor formalism is adopted for all kinds of particles instead of DP's isospinor-isotensor mixed formalism; and (ii) the invariance under inversion of isobaric spin space, imposed by DP as a postulate on strong and electromagnetic interactions, is replaced by the invariance under a continuous phase transformation group G_U generated by the operator U representing isofermion number. These modifications lead not only to extension of the framework but also to refinement of the theory as follows. In our formalism, (a) the transformation properties of field variables with respect to G_U are unambiguously determined without resort to the assumption, made by DP, that only the Yukawa interactions appear in the interaction Lagrangian; (b) the isofermion number U proves to be the number of suffices of the first kind minus that of the second kind, with respect to inversion, of the isospinor describing a particle; and (c) DP's identity $Q=I_3+\frac{1}{2}U$ among the electric charge Q, the third component of isobaric spin I_3, and U as well as Racah's identity $P=\exp(i\frac{1}{2}\pi U)$ between U and the isobaric parity P follows directly from an identical relation among the transformation groups concerned.

§ 1. Introduction

The Gell-Mann-Nishijima (henceforth referred to as GN) scheme[1] for new particles seems to have been firmly established by experimental data accumulated through recent years. Several attempts[2] have been made to formulate mathematically the GN scheme or its variations and to interpret the strangeness quantum number S from a group-theoretical view-point. Of these the theory of d'Espagnat and Prentki[3] (henceforth referred to as DP) seems to be most successful. The advantages of DP's theory over the others are that it involves no difficulty of predicting the existence of unwanted isobaric particles, and that it restricts the values of electric charge (in units of the electric quantum) allowed for elementary particles to integers without *ad hoc* rejection of half odd integers.

DP's theory may be outlined as follows. The electric charge properties of elementary particles are assumed to be described by assigning to them either an isospinor or an isotensor field in the isospace* according to whether their isospin* is 1/2 or an integer.

* The terms "isospin", "isospace", and "isoparity" or "isobaric parity" will be used throughout this paper for the abbreviation of "isobaric spin", "isobaric spin space", and "parity with respect to the inversion of isobaric spin space through the origin", respectively.

Then, an equivalent scheme to GN's can be formulated mathematically by postulating, besides the conservation law of baryon number N, that (I) the strong interaction Lagrangian L_s should be invariant under the inversion P and the full rotation group G_I in isospace; (II) the electromagnetic interaction Lagrangian L_e should be invariant under the inversion P and the rotation group G_{I_3} around the third axis of isospace; and (III) the weak interaction Lagrangian L_w should be invariant only under the gauge transformation group G_Q. The link to the GN scheme is given by setting the identity

$$S=U-N\,, \tag{1·1}$$

where U is the isofermion number determined from the isoparity P* by Racah's identity[4]

$$P=\exp(i\tfrac{1}{2}\pi U)\,. \tag{1·2}$$

It is a merit of DP's theory that they have succeeded in deriving the identity

$$Q=I_3+\tfrac{1}{2}U\,, \tag{1·3}$$

which holds among the electric charge Q, the third component of isospin I_3, and the isofermion number U for operators as well as for eigenvalues. The Hermitian operator U has been given its explicit expression by constructing the generator of a phase transformation group, to which we shall refer as G_U, operating on isospinors. It is observed from (1·2) that the (minimal) U value is unambiguously determined as zero for isotensors of even isoparity, whereas it is indeterminate for isotensors of odd isoparity between $+2$ and -2.**

Faced with this ambiguity, DP were forced to choose either of the following two attitudes: (i) isotensors of odd isoparity must be classified into two kinds, of which one is labelled by $U=+2$ and the other by $U=-2$; or (ii) particles described by isotensors of odd isoparity should be rejected. We wish to remark here that, adopting the attitude (i), one has to impose on L_s and L_e the G_U invariance as a supplement of the P invariance in order to secure the conservation of U. On the other hand, if one adopts the attitude (ii) which implies the restriction $|U|\leq 1$, the postulate of P invariance suffices for the additive conservation of U. This attitude is of interest in the sense that it provides us with such an economical catalogue of elementary particles as records without omission all those particles of which the existence has firmly been proved by experiment, and yet leaves scarcely any columns blank. However, as DP themselves have admitted, one would have to choose the more extensive attitude (i), if the Eisenberg particle Y_E^- turned out to be real and to have $U=-2$.[5]

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* No confusion will be introduced by using the same notation P for both the operator of inversion and its eigenvalue, i. e., isoparity. Similarly, we shall use later the same notations I, I_3, Q, U and N to denote operators as well as eigenvalues.

** Strictly speaking, DP have groped for U values to be assigned to isotensors by writing down explicit expressions for L_s satisfying the G_U invariance under the assumption that only Yukawa interactions appear in L_s. It is Racah who first pointed out that U can be defined from P on the basis of (1·2).

Since the values for the isofermion number are restricted within the range $|U| \leq 2$ even for the wider attitude (i), the framework of DP's theory is narrower than that of the GN scheme which originally involves no such restriction. Hence it seems preferable to widen the framework of the theory, so as to prepare room for those particles of higher U value which might be observed in future experiments, although we do not necessarily expect their existence.

It is the main purpose of this paper to show that DP's theory can easily be generalized so as to accomodate particles of higher isofermion number (and higher isospin) by introducing modifications in the following two respects: (i) we adopt the isospinor formalism for all kinds of particles throughout instead of DP's isospinor-isotensor mixed formalism; and (ii) we replace the inversion invariance by the G_U invariance in DP's postulates (I) and (II) imposed on the strong and the electromagnetic interactions. These modifications will lead not only to extension of the framework but also to refinement of the theory as follows.

In our formalism, (a) the transformation properties of field variables with respect to G_U are unambiguously determined without resort to the assumption of the Yukawa interactions;* (b) the isofermion number U proves to be the number of suffices of the first kind minus that of the second kind, referring to inversion, of the isospinor describing a particle; and (c) DP's identity (1·2) and Racah's identity (1·3) can be derived from identical relations among the groups of gauge transformations, rotations around the third axis of isospace, G_U transformations and the inversion of isospace.

§ 2. Transformation groups and constants of motion

Let us assume that the electric charge properties of elementary particles are completely described by assigning to them isospinors, ξ_m, $\xi^{\dot{m}}$, ξ_{mn}, $\xi^{\dot{m}}{}_{\dot{n}}$, $\xi^{\dot{m}\dot{n}}$, $\cdots$ ($m, n = 1, 2$), where ξ_m and $\xi^{\dot{m}}$ undergo the same linear transformations of 2×2 irreducible representation with respect to the full rotation group G_I operating on the three-dimensional isospace, while they are subject to transformations of the first and the second kind, respectively, with respect to an inversion P of isospace through the origin as

$$\begin{aligned} \xi_m &\to \xi_m{}' = P\xi_m = i\xi_m\,, \\ \xi^{\dot{m}} &\to \xi'^{\dot{m}} = P\xi^{\dot{m}} = -i\xi^{\dot{m}}\,. \end{aligned} \tag{2·1}$$

An isospinor $\xi^{\dot{m}\dot{n}\cdots}{}_{kl\cdots}$ ($m, n, \cdots, k, l, \cdots = 1, 2$) is assumed, of course, to transform like a direct product $\xi^{\dot{m}}\xi^{\dot{n}}\cdots\xi_k\xi_l\cdots$. We shall write ψ or φ for ξ, if necessary, according as it refers to a fermion or a boson. The undotted suffices, $k, l, \cdots$, and the dotted suffices $\dot{m}, \dot{n}, \cdots$, will be called suffices of the first and the second kind, respectively. Suffices will be suppressed so long as arguments hold irrespective of them.

We shall later treat various groups, G_U's, of phase transformations of isospinors. Since, as is well known, the Lagrangian L_0 for a system of free fields is generally given in a

* See footnote **.

bilinear form in $\tilde{\xi}^{*T}$ and ξ or their first derivatives with respect to space-time coordinates, L_0 is always invariant under G_C's.* Hence, the invariance of the Lagrangian $L=L_0+L_1$ for the total system will be guaranteed by postulating the invariance of the interaction Lagrangian L_1 under G_C's.

Let $G_C(\beta)$ denote an element of a phase transformation group G_C specified by a real and continuous phase parameter β. Then, consider a Hermitian operator defined by

$$C=\int c_\mu d\sigma_\mu \tag{2·2}$$

with

$$c_\mu=-\sum_\xi\left[\frac{\partial\tilde{\xi}'^{*T}}{\partial\beta}\frac{\partial L}{\partial(\partial\tilde{\xi}^{*T}/\partial x_\mu)}+\frac{\partial L}{\partial(\partial\xi/\partial x_\mu)}\frac{\partial\xi'}{\partial\beta}\right]_{\beta=0}, \tag{2·3}$$

where ξ' is an isospinor transformed from ξ by $G_C(\beta)$ as

$$\xi\to\xi'=G_C(\beta)\xi\,, \tag{2·4}$$

and $d\sigma_\mu$ ($\mu=1, 2, 3, 4$) is a four-vector representing a three-dimensional hypersurface element of an arbitrary space-like hypersurface, in the Lorentz space-time, over which the integration is to be carried out.[6] In (2·3) $\sum_\xi$ stands for the summation over all ξ fields. With the use of the Lagrangian equations of motion for field variables and the condition of invariance $\partial L'/\partial\beta=0$, it can be elementarily verified that c_μ satisfies the equation of continuity and, consequently, C is a constant of motion.

Of the phase transformation groups G_C's, those we are interested in are the rotation group G_{I_3} around the third axis of isospace, the gauge transformation group G_Q, and the G_U group first introduced by DP.

Under a rotation $G_{I_3}(\gamma)$ through angle γ around the third axis of isospace, ξ_m and $\xi^{\dot{m}}$ undergo the same phase transformation :[7]

$$\begin{aligned}\xi_m\to\xi_m'&=G_{I_3}(\gamma)\xi_m=[\exp(-i\tfrac{1}{2}\gamma\tau_3)]_{mn}\xi_n\,,\\ \xi^{\dot{m}}\to\xi'^{\dot{m}}&=G_{I_3}(\gamma)\xi^{\dot{m}}=[\exp(-i\tfrac{1}{2}\gamma\tau_3)]^{\dot{m}\dot{n}}\xi^{\dot{n}}\,,\end{aligned} \tag{2·5}$$

where τ_3 is the Heisenberg matrix given by (A·12) in Appendix. Substituting $G_{I_3}(\gamma)$ for $G_C(\beta)$ in (2·2) to (2·4), we obtain for C the operator I_3 representing the third component of isospin of the total system, which is conserved in any interactions invariant under G_{I_3}.

As is well known, the conservation of electric charge Q of the total system results from the gauge invariance of the total Lagrangian L. The transformation equations for isospinors of first rank with respect to a gauge transformation $G_Q(\lambda)$ are given by

$$\begin{aligned}\xi_m\to\xi_m'&=G_Q(\lambda)\xi_m=[\exp(i\tfrac{1}{2}\lambda(1-\tau_3))]_{mn}\xi_n,\\ \xi^{\dot{m}}\to\xi'^{\dot{m}}&=G_Q(\lambda)\xi^{\dot{m}}=[\exp(i\tfrac{1}{2}\lambda(-1-\tau_3))]^{\dot{m}\dot{n}}\xi^{\dot{n}}\,.\end{aligned} \tag{2·6}$$

* The asterisk * denotes complex conjugate, the T transposition of isospinor components, and the tilde $\sim$ transposition of spinor components if ξ refers to a fermion.

The charge operator Q (in units of the electric quantum) is defined by substituting $G_Q(\lambda)$ for $G_Q(\beta)$ in equations (2·2) to (2·4).

It is DP who first introduced the following phase transformations

$$\begin{aligned}\hat{\xi}_m &\to \hat{\xi}_m' = G_U(\alpha)\,\hat{\xi}_m = \exp(i\alpha)\,\hat{\xi}_m\,,\\ \hat{\xi}^{\dot{m}} &\to \hat{\xi}'^{\dot{m}} = G_U(\alpha)\,\hat{\xi}^{\dot{m}} = \exp(-i\alpha)\,\hat{\xi}^{\dot{m}}\,.\end{aligned} \tag{2·7}$$

The operator U defined by substituting $G_U(\alpha)$ for $G_Q(\beta)$ into equations (2·2) to (2·4) is that which has been interpreted by DP as representing the total "isofermion number", which is conserved in any interactions invariant under G_U. It is important to notice that an identical relation

$$G_Q(\lambda) = G_{1^3}(\lambda)\,G_U(\tfrac{1}{2}\lambda) \tag{2·8}$$

holds among the three phase transformation groups owing to the definitions given by (2·5), (2·6) and (2·7). Furthermore, by comparing (2·1) with (2·7), one may easily find that the inversion P is included in the G_U group as an element, i. e.,

$$P = G_U(\tfrac{1}{2}\pi)\,. \tag{2·9}$$

It is to be noted here that the upper equations of (2·5), (2·6) and (2·7) should be regarded as the definitions of the transformations, while the lower equations can be derived from the upper equations from the considerations of the unitary and unimodular properties of the rotation group and the convention that $(\hat{\xi}^1, \hat{\xi}^{\dot{1}})$ transforms like $(\hat{\xi}_2{}^*, -\hat{\xi}_1{}^*)$ with respect to a rotation and an inversion of isospace. The equation (2·6) implies that $\hat{\xi}^1 = \hat{\xi}_2$ and $\hat{\xi}^2 = -\hat{\xi}_1$ describe a positively charged and a neutral particle, while $\hat{\xi}^{\dot{1}} = -\hat{\xi}^{\dot{2}}$ and $\hat{\xi}^{\dot{2}} = \hat{\xi}^{\dot{1}}$ a neutral and a negatively charged particle, respectively. This is consistent with the fact that an antiparticle of the $\hat{\xi}_m$ field is described by its charge conjugate isospinor, which may be made equivalent to the complex conjugate isospinor $\hat{\xi}_m{}^*$ (by choosing a suitable representation for the Dirac matrices in the case where $\hat{\xi}$ refers to a fermion).[6]

Since an isospinor of higher rank is assumed to transform like a direct product of isospinors of first rank with respect to G_{1^3}, G_Q, G_I and P, the transformation equations for isospinors of arbitrary rank are determined unambiguously by (2·5), (2·6), (2·7) and (2·1) in our formalism without resort to the assumption of the Yukawa interactions. This is the reason why one may say that the identities, (2·8) and (2·9), hold not only for the 2×2 representations but also for the transformation groups, G_Q, G_{1^3}, G_I and P, themselves irrespective of special representations. In particular, the transformation equation for an isospinor of higher rank with respect to G_I is derived from (2·7) as

$$\hat{\xi}^{\dot{m}\dot{n}\cdots}{}_{kl\cdots} \to \hat{\xi}'^{\dot{m}\dot{n}\cdots}{}_{kl\cdots} = G_U(\alpha)\,\hat{\xi}^{\dot{m}\dot{n}\cdots}{}_{kl\cdots} = \exp(i(U^+ - U^-)\alpha)\,\hat{\xi}^{\dot{m}\dot{n}\cdots}{}_{kl\cdots}\,, \tag{2·10}$$

where U^+ and U^- denote the number of suffices of the first kind, k, l, $\cdots$, and that of the second kind, $\dot{m}$, $\dot{n}$, $\cdots$, respectively.

In the above arguments the field variables have been treated as unquantized. The

transformation equations for quantized field variables with respect to $G_C(\beta)$ are given by

$$\hat{\xi} \to \hat{\xi}' = G_C^{-1}(\beta)\,\hat{\xi}\,G_C(\beta) \tag{2·11}$$

with

$$G_C(\beta) = \exp(iC\beta), \tag{2·12}$$

where $G_C(\beta)$ stands for $G_{I_3}(\gamma)$, $G_Q(\lambda)$, or $G_U(\alpha)$, C being the Hermitian operator defined by equations (2·2) to (2·4). It need scarcely be said that the proof follows from the commutation relations between quantized field varibles. By inserting into (2·8) the expressions for $G_Q(\lambda)$, $G_{I_3}(\lambda)$ and $G_U(\frac{1}{2}\lambda)$ obtained with the use of (2·12), we have DP's identity cited in (1·3). Similarly, Racah's identity (1·2) can be derived at once by inserting into (2·9) the expression for $G_U(\frac{1}{2}\pi)$ obtained with the use of (2·12).

It is not difficult to prove, with the commutation relations in mind, that the Hermitian operator U takes an integral eigenvalue

$$U = U^+ - U^- \tag{2·13}$$

in a state in which there is present a single particle described by an isospinor $\hat{\xi}^{\dot{m}\dot{n}\cdots}_{kl\cdots}$ with number U^+ undotted suffices and number U^- dotted suffices. Thus, it turns out that the "isofermion number" in DP's nomenclature is nothing but the number of suffices of the first kind minus that of the second kind of the isospinor describing a particle, or, in short, the "algebraic rank" of the isospinor.

Finally, there remains another phase transformation group to be discussed. It is the "heavy particle gauge transformation" group, G_N, in the nomenclature of Lee and Yang.[5] The transformation equations for unquantized field variables with respect to this group are given by

$$\begin{aligned} &\psi' \to \psi'' = G_N(\nu)\,\psi' = \exp(i\nu)\,\psi', \\ &\varphi \to \varphi' = G_N(\nu)\,\varphi = \varphi\,, \end{aligned} \tag{2·14}$$

where ν is a real phase parameter labelling a group element $G_N(\nu)$, and ψ and φ field variables describing a baryon and a boson, respectively. The procedure, given by (2·2) to (2·4), for constructing a conservative Hermitian operator applies also to $G_N(\nu)$. It is, however, to be noted that isospinors of higher rank transform not like direct products but like isospinors of first rank under the G_N group. Further, ψ'_m and $\psi'^{\dot{m}}$ undergo the same phase change, $+\nu$, under $G_N(\nu)$, in contrast to the opposite phase changes under the other phase transformations, $G_{I_3}(\gamma)$, $G_Q(\lambda)$ and $G_U(\alpha)$. This is because G_N operates on a degree of freedom which has no relation to the isospace. The transformation properties with respect to G_N distinguish between fermion and antifermion. For instance, ψ^m and $\psi^{\dot{m}}$ describe baryons, while $\psi_m{}^*$ and $\psi^{\dot{m}*}$ antibaryons. The conservation of baryon number N is, of course, guaranteed by postulating the G_N invariance on the interaction Lagrangian L_1.

§ 3. Mathematical formulation of the GN scheme

The GN scheme can be mathematically reformulated by imposing the following postulates on the interaction Lagrangians:

(I′) the invariance of L_s under G_N, G_U and G_I;

(II′) the invariance of L_e under G_N, G_U and G_{I_3}; and

(III′) the invariance of L_w under G_N and G_Q;

in our formalism in which the electric charge properties of elementary particles are assumed to be completely described by assigning to them isospinors of suitable rank, which transform under the phase transformation groups in accordance with (2·5), (2·6), (2·7) and (2·14). The P as well as the G_Q invariance is automatically satisfied by obeying the postulates (I′) and (II′) owing to the identities (1·2) and (1·3). The G_I invariance of L_s leads, of course, to the charge independence of the strong interactions.

Table 1. Classification of baryons ($N=1$)

Isospinors	ψ_{mn}				ψ_m		$\dot{\psi}^m{}_n$						$\dot{\psi}^m$		$\dot{\psi}^{mn}$					
U	2				1		0						−1		−2					
S	1				0		−1						−2		−3					
P	−1				i		1						$-i$		−1					
DP's symbols	$\psi_{a'}$	$\psi_{b'}$			ψ_A		ψ_a	ψ_b					ψ_B		$\psi_{a''}$	$\psi_{b''}$				
I	0	1			$\frac{1}{2}$		0	1					$\frac{1}{2}$		0	1				
I_3	0	1	0	−1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	−1			$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	−1		
Q	+1	+2	+1	0	+1	0	0	+1	0	−1			0	−1	−1	0	−1	−2		
Assignment					p	n	Λ^0	Σ^+	Σ^0	Σ^-			Ξ^0	Ξ^-	Y_E^-					

Table 2. Classification of bosons ($N=0$)

Isospinors	φ_{mn}				φ_m		$\dot{\varphi}^m{}_n$						$\dot{\varphi}^m$		$\dot{\varphi}^{mn}$					
U	2				1		0						−1		−2					
S	2				1		0						−1		−2					
P	−1				i		1						$-i$		−1					
DP's symbols	$\varphi_{\Upsilon'}$	$\varphi_{\delta'}$			φ_Δ		$\varphi_{\bar{\Upsilon}}$	φ_δ					$\varphi_{\Delta'}$		$\varphi_{\Upsilon''}$	$\varphi_{\delta''}$				
I	0	1			$\frac{1}{2}$		0	1					$\frac{1}{2}$		0	1				
I_3	0	1	0	−1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	−1			$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	−1		
Q	+1	+2	+1	0	+1	0	0	+1	0	−1			0	−1	−1	0	−1	−2		
Assignment					θ^+	θ^0	γ	π^+	π^0	π^-										

The assignment of isospinors to the elementary particles of which the existence has been well established or at least suggested is shown in Tables 1 and 2. The correspondence of our isospinor formalism to DP's isospinor-isotensor mixed formalism is given in Appendix. In our formalism, a fermion of $U=-2$ ($S=-3$) may be described by an isospinor $\psi^{\dot{m}\dot{n}}$, which can be reduced, on the basis of the irreducible representations of the full rotation group G_I, to an isopseudoscalar of the second kind $\psi_{a''}$ and an isovector of the second kind $\boldsymbol{\psi}_{b''}$ (see Appendix). If the Eisenberg particle Y_K^- turned out to be real and to have $U=-2$, then it would be described by $\boldsymbol{\psi}_{b''}$ or $\psi_{a''}$ according to whether there exists also a doubly charged isobar Y_K^{--} or not. The latter alternative has been chosen in Table 1.

We shall here briefly argue about isospinors of higher rank. Consider, for example, $\psi^{\dot{m}\dot{n}\dot{k}}{}_{\dot{l}}$ which describes a fermion of $U=-2$. Classified according to the irreducible representations of G_I, it is reduced to $\psi_{a'''}=\varepsilon^{\dot{m}\dot{n}}\varepsilon^{\dot{k}}{}_{\dot{l}}\psi^{\dot{m}\dot{n}\dot{k}}{}_{\dot{l}}$, $\boldsymbol{\psi}_{b'''}=\varepsilon^{\dot{m}\dot{p}}\varepsilon^{\dot{k}}{}_{\dot{l}}\tau^{\dot{p}\dot{m}}\psi^{\dot{m}\dot{n}\dot{k}}{}_{\dot{l}}$, and ψ_c belonging to $I=0$, 1, and 2, respectively. Since $\psi_{a'''}$ and $\boldsymbol{\psi}_{b'''}$ transform like $\psi_{a'}$ and $\boldsymbol{\psi}_{b''}$, respectively, under G_N, G_U and G_I, the former isotensors degenerate to the latter. The remaining ψ_c describes a mass quintet of isobaric particles with $Q=1$, 0, -1, -2, and -3. From a similar argument it follows that ψ_a and φ_τ may be treated from the outset as an invariant isoscalar, i. e., an isospinor of zeroth rank, so far as the phase transformation groups, G_N, G_U and G_I, are concerned. As an example of a boson described by φ_τ, Bethe's neutral meson[9] may be cited besides photon. If there should exist particles of $|U|\geq 3$ in nature, they would be described by isospinors of algebraic rank higher than second.

§ 4. Discussions

First we shall discuss the extent of the frameworks of DP's theory and ours in comparison with the GN scheme. As a matter of course, DP's theory of attitude (ii), which restricts U within the range $|U|\leq 1$, is the narrowest. Even DP's theory of wider attitude (ii) is narrower than the GN scheme, as DP themselves have acknowledged, because this scheme allows the coexistence of both kinds of particles of which one is labelled by $U=+2$ and the other by $U=-2$, whereas it excludes either one of the two kinds.* On the contrary, involving no restriction concerning the range of U, our theory is equivalent to the GN scheme in the extent of the framework.** The difference in the extent of the two theories is due to the difference between DP's postulates, (I) and (II) (§ 1), and ours, (I′) and (II′) (§ 3), imposed on L_s and L_e. In the latter

* For this subject refer to § 6 of the second paper cited in reference 3). The article given in § 5 is somewhat obscure as to whether the two kinds of particles should exclude each other or not.

** Very recently Gotô has proposed an attempt to classify the weak interactions, responsible for those decay processes in which at least one light fermion is produced, by assigning $U=+2$, 0, and -2 to μ^+, ν and e^-, respectively, and $I=0$ to all these fermions. [S. Gotô, to be published.] To this model DP's theory is inapplicable, whereas our theory is applicable. The author is indebted to Mr. Suteo Gotô for informing him of the content of his work before publication.

postulates the G_U invariance takes the place of the P invariance in the former. Since the inversion P together with the unit operator 1 constitutes a cyclic Abelian group of fourth order, $\{1, P, P^2, P^3\}$, P takes on only four eigenvalues: $P=\exp(i\pi U/2)$ $(U=1, 0, -1, -2 \text{ modulo } 4)$.[10] Hence the minimal values for U are necessarily restricted to either $(1, 0, -1, -2)$ or alternatively $(2, 1, 0, -1)$. On the contrary, an element $G_U(\alpha)$ belonging to the continuous phase transformation group G_U takes on eigenvalues $G_U(\alpha)=\exp(iU\alpha)$ $(U=0, \pm 1, \pm 2, \cdots, \pm\infty)$. This accounts for the difference in the range of U between the two theories.

Next we shall try a geometrical interpretation of the G_U group. Consider a set of two isospinors, ξ_m and $\xi^{\dot{m}}$:

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi^{\dot{m}} \\ \xi_m \end{pmatrix}. \tag{4·1}$$

Then (2·7) may be rewritten as

$$\eta_r \to \eta_r' = G_U(\alpha)\eta_r = [\exp(-iu_z\alpha)]_{rs}\eta_s, \quad (r, s=1, 2) \tag{4·2}$$

with

$$\| u_{zrs} \| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4·3}$$

These equations indicate that $G_U(\alpha)$ may be interpreted, by way of analogy to (2·5), as a 2×2 representation of a rotation through angle 2α around the origin in an abstract plane, S_U, spanned by two mutually perpendicular axes, B_1 and B_2. DP's identity (1·3) suggests that the "charge space" R_Q to describe the electric charge properties of elementary particles is constructed out of two totally orthogonal subspaces, one S_U and the other the familiar isospace R_I spanned by three orthogonal axes, A_1, A_2 and A_3.* Interpreted as an operator acting on the five-dimensional charge space R_Q, $G_I(\gamma)$ represents a rotation through angle γ around the three-dimensional subspace spanned by A_3, B_1 and B_2, while $G_U(\alpha)$ a rotation through angle 2α around the subspace R_I. Then, the identity (2·8) leads to an interpretation that $G_Q(\lambda)$ represents a rotation through angle λ around the A_3 axis shared by the two invariant subspaces under $G_I(\lambda)$ and $G_U(\frac{1}{2}\lambda)$, respectively.

A more intuitive picture can be obtained by adopting a new interpretation that G_I operates on a point P_I in R_I while G_U on another point P_U in the same three-dimensional isospace R_I, instead of the older interpretation that both G_I and G_U operate on a single point P_Q in the five-dimensional charge space R_Q. The interrelation between these two pictures is analogous to that between the γ(gas)-space and the μ(molecule)-space pictures in the statistical mechanics.[11] In the three-dimensional picture, a gauge transformation $G_Q(\lambda)$ may be reinterpreted as a simultaneous rotation of the two points, P_I and

* We shall from now on call the three axes of R_I A_1, A_2 and A_3 in order to avoid confusion with the axes, B_1 and B_2.

P_U, through angle λ around the A_3 axis of R_I. Further, the identity (2·9) indicates that the inversion of the point P_I through the origin may be superseded by the 180° rotation of the point P_U around the A_3 axis,* although these two operations are not equivalent to each other in this picture. After having eliminated the inversion that is a discontinuous operation, we are allowed to constrain the points P_I and P_U to move continuously on a (unit) spherical surface and along its equator, respectively.

Finally, we shall discuss the full rotation group G_U which includes $G_{U3}\equiv G_U$ as a subgroup. The other subgroups, G_{U1} and G_{U2}, can be constructed out of the generators of infinitesimal rotations

$$\|u_{1rs}\|=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad \|u_{2rs}\|=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \tag{4·4}$$

around the B_1 and the B_2 axes, respectively.** The Hermitian operators, U_1 and U_2, can be defined on the basis of equations (2·2) to (2·4). I_i and $\frac{1}{2}U_i$ $(i=1, 2, 3)$ correspond to τ_i and μ_i in the theory of Salam and Polkinghorne.[2] The operator $U^2=U_1^2+U_2^2+U_3^2$ thus defined will not play any role in cataloging elementary particles, unless one imposes the postulate of invariance on L_s under the full rotation group G_U. Imposing this postulate, one obtains a theory equivalent to Feldman's variation of the theory of Salam and Polkinghorne.[12]

The author wishes to thank Professor T. Miyazima for his kind interest in this work and for valuable advice in preparing the manuscript.

Appendix

The correspondence of our isospinor formalism to DP's isospinor-isotensor mixed formalism is given as follows:

isospinor of the 1st kind $(U=1)$:

$$\psi_A=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \varphi_\Delta=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{A·1}$$

isospinor of the second kind $(U=-1)$:

$$\psi_B=\begin{pmatrix}\psi^{\dot 1}\\ \psi^{\dot 2}\end{pmatrix}, \qquad \psi_{\Delta'}=\begin{pmatrix}\varphi^{\dot 1}\\ \varphi^{\dot 2}\end{pmatrix}, \tag{A·2}$$

isoscalar $(U=0)$:

$$\psi_\alpha=\varepsilon^{\dot m}{}_n\psi^{\dot m}{}_n, \qquad \varphi_\tau=\varepsilon^{\dot m}{}_n\varphi^{\dot m}{}_n, \tag{A·3}$$

isopseudovector $(U=0)$:

* It has already been pointed out by Caianiello that an inversion of a three-dimensional space may be replaced by a 180° rotation in a five-dimensional space. [E. R. Caianiello, Phys. Rev. **86** (1952), 564.] The author is indebted to Dr. T. Okabayashi for calling his attention to this reference.

** If we wish to adopt a higher dimensional formalism analogous to the γ-space picture, we have to introduce a three-dimensional abstract space R_U spanned by three orthogonal axes, B_1, B_2 and B_3. Then the charge space $R_Q=R_I\times R_U$ becomes six-dimensional.

$$\psi_b = \varepsilon^{\dot{m}}{}_{\dot{k}}\,\tau_{kn}\psi^{\dot{n}}{}_{\dot{n}}\,, \qquad \varphi_\delta = \varepsilon^{\dot{m}}{}_{\dot{k}}\,\tau_{kn}\varphi^{\dot{n}}{}_{\dot{n}}\,, \tag{A·4}$$

is opseudoscalar of the 1st kind ($U=2$) :

$$\psi_{a'} = \varepsilon_{mn}\psi_{mn}\,, \qquad \varphi_{\tau'} = \varepsilon_{mn}\varphi_{mn}\,, \tag{A·5}$$

isovector of the 1st kind ($U=2$) :

$$\psi_{b'} = \varepsilon_{nk}\tau_{km}\psi_{mn}\,, \qquad \varphi_{\delta'} = \varepsilon_{nk}\tau_{km}\varphi_{mn}\,, \tag{A·6}$$

isopseudoscalar of the 2nd kind ($U=-2$) :

$$\psi_{a''} = \varepsilon^{\dot{m}\dot{n}}\psi^{\dot{m}\dot{n}}\,, \qquad \varphi_{\tau''} = \varepsilon^{\dot{m}\dot{n}}\varphi^{\dot{m}\dot{n}}\,, \tag{A·7}$$

isovector of the 2nd kind ($U=-2$) :

$$\psi_{b''} = \varepsilon^{\dot{n}\dot{k}}\tau^{\dot{k}\dot{m}}\psi^{\dot{m}\dot{n}}\,, \qquad \varphi_{\delta''} = \varepsilon^{\dot{n}\dot{k}}\tau^{\dot{k}\dot{m}}\varphi^{\dot{m}\dot{n}}\,, \tag{A·8}$$

with

$$\|\varepsilon_{mn}\| = \|\varepsilon^{\dot{m}}{}_{\dot{n}}\| = \|\varepsilon^{\dot{m}\dot{n}}\| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A·9}$$

$$\|\tau_{1mn}\| = \|\tau_1{}^{\dot{m}}{}_{\dot{n}}\| = \|\tau_1{}^{\dot{m}\dot{n}}\| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A·10}$$

$$\|\tau_{2mn}\| = \|\tau_2{}^{\dot{m}}{}_{\dot{n}}\| = \|\tau_2{}^{\dot{m}\dot{n}}\| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{A·11}$$

$$\|\tau_{3mn}\| = \|\tau_3{}^{\dot{m}}{}_{\dot{n}}\| = \|\tau_3{}^{\dot{m}\dot{n}}\| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A·12}$$

where τ denotes (τ_1, τ_2, τ_3).

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Letters to the Editor

On the Air Shower Core

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The structure of the air shower core is of great importance not only for interpretation of the structure of air shower but for the study of the angular distribution of secondary mesons produced by high energy nucleon-nucleus collisions. Recently, J. Nishimura has pointed out that for the analysis of the experimental data of the lateral distribution of soft components at sea level we should take $N-K$[1] function for shower age $s=1.2$.[2] This conclusion is deduced from the analysis of the altitude variation of shower particles assuming that π^0-mesons are produced only along the line of shower axis. Here, the effect of the angular spread of π^0-mesons, or, strictly, the angular distribution of energy flow, produced by high energy jet showers is considered, which will make the lateral distribution of the soft components near the core less steep than that predicted only by the single cascade theory. Assuming the shape of the angular distribution of π^0-mesons as well as the altitude at which they are produced, the theoretical lateral distribution of soft component at the place of the observation is calculated. Then, from the comparison of this result with experimental data, we will get the information about the angular spread of π^0-mesons produced by high energy jet showers. In this calculation, we consider the spatial distribution of π^0-mesons only, assuming the one-dimensional development of subsequent cascade showers. Put the spatial distribution of π^0-meson as $F(\boldsymbol{r})d\boldsymbol{r}$, for which we use the normalized gaussian distribution $(\alpha/\pi)\exp(-\alpha r^2)$ or $(1/\pi r_0^2)\exp(-(r/r_0)^2)$ as trial function, the observed structure function is

$$G(\boldsymbol{r})=\int_0^\infty F(\boldsymbol{r}-\boldsymbol{r}')f(\boldsymbol{r}',s)d\boldsymbol{r}', \qquad (1)$$

where $f(\boldsymbol{r},s)d\boldsymbol{r}$[3] is the normalized $N-K$ function with $s=1.2$.

Setting the Hankel transforms of $G(\boldsymbol{r})$, $F(\boldsymbol{r})$ and $f(\boldsymbol{r})$ as $J_G(\zeta)$, $J_F(\zeta)$ and $J_f(\zeta)$ respectively, we get from eq. (1)[4]

$$J_G(\zeta)=2\pi J_F(\zeta)J_f(\zeta) \qquad (2)$$

$J_F(\zeta)$ and $J_f(\zeta)$ can be easily obtained, and $J_G(\zeta)$ is

$$J_G(\zeta)=-i/4\pi^2\int_{-i\infty}^{i\infty}dp(K^2\zeta^2/4\mathcal{E}^2)^p \times\exp(-\rho^2/4\alpha)\,\Gamma(-p)\,\Gamma(2p+s) \times\frac{\mathfrak{M}(p,-2p-s,s,t)}{\{\Gamma(s)\,\mathfrak{M}(o,-s,s,t)\}} \qquad (3)$$

From the inverse Hankel transform of $J_G(\zeta)$, we have

$$G(\boldsymbol{r})=-i/4\pi^2\int_{-i\infty}^{i\infty}\left(\frac{K^2\alpha}{\mathcal{E}^2}\right)^p\Gamma(-p)\Gamma(2p+s) \times\frac{\mathfrak{M}(p,-2p-s,t)}{\{\Gamma(s)\,\mathfrak{M}(o,-s,s,t)\}}$$

$$\sum_{m=0}^{\infty}(-1)^m(\alpha r^2)^m \times \Gamma'(p+m+1)/\{\Gamma'(m+1)\}^2. \qquad (4)$$

The numerical calculation of this integral can be carried out by the saddle point method, but we can get the characteristic features of the structure function near and far from the core simply by series expansion using the pole of Γ'-function

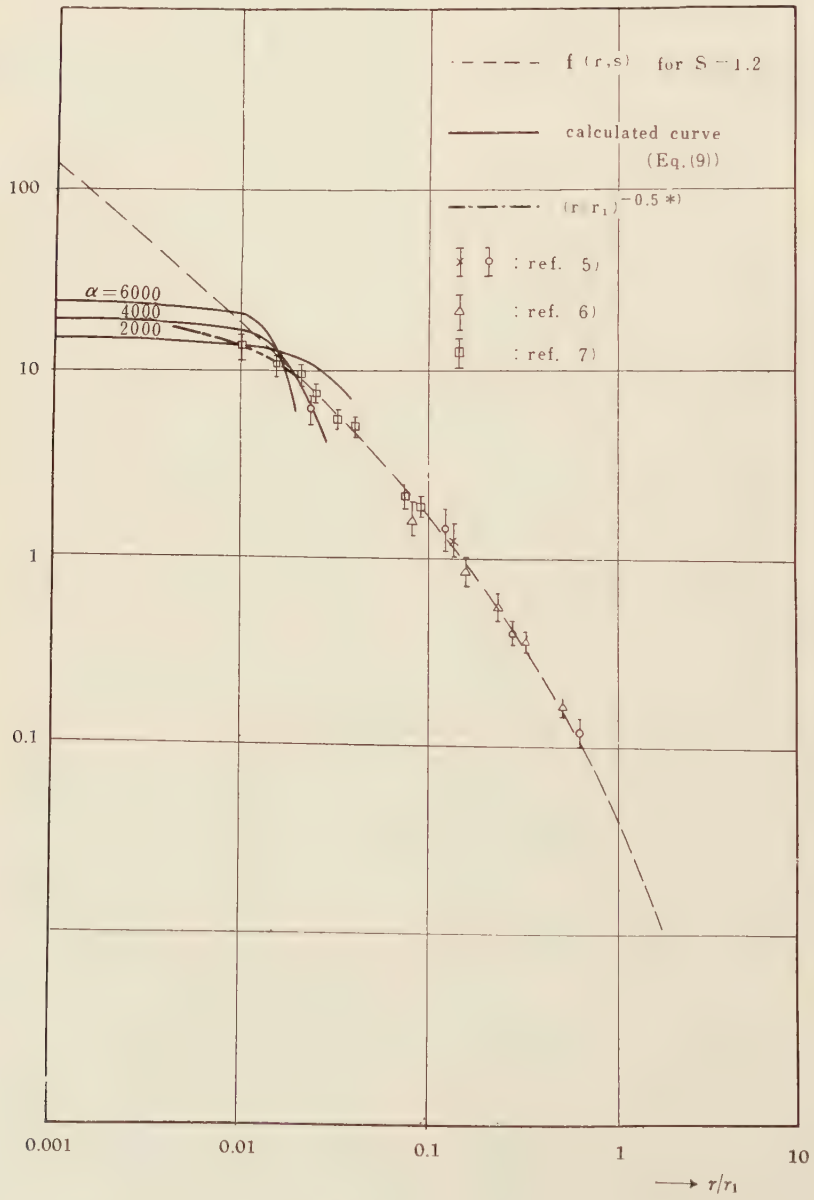

Fig. 1. The experimental data and calculated curves.

*) See reference 7)

appearing in $G(\boldsymbol{r})$.

I) Near the core; $r \lesssim r_1 = K/\mathcal{E}$. Using the pole of $\Gamma(2p+s)$

$$G(\boldsymbol{r}) = \alpha \cdot \Gamma(s/2)\,(K^2\alpha/\mathcal{E}^2)^{-s/2} \times \frac{\mathfrak{M}(p, -2p-s, s, t)}{\{2\pi\Gamma(s)\,\mathfrak{M}(o, -s, s, t)\}} \sum_{m=0}^{\infty} (-1)^m (\alpha r^2)^m \times \Gamma(-s/2+m+1)/\{\Gamma(m+1)\}^2. \quad (5)$$

II) Far from the core; $r \gtrsim r_1$. In this case we have, after some mathematical processes,

$$G(\boldsymbol{r}) = f(r, s) + \sum_{m=1}^{\infty} \left\{ \frac{1}{4\alpha} \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} f(r, s) \right\}^m. \quad (6)$$

$G(\boldsymbol{r})$ is almost usual single $N-K$ function with $s=1.2$, because the numerical value of α is considered to be very large as in Fig. 1, and the only first term is dominant. In Fig. 1, we may choose $\alpha \approx 4000$, i.e, $r_0 \approx 1.1$ m and this means the energy flow is concentrated in about 10^{-4} radian if we take the effective altitude of π^0-meson production about 10 km above sea level. It will be discussed later on this physical significance and the energy spectrum of the air shower core relevant to this angular spread of π^0-mesons.

In conclusion, the author would like to express his sincere thanks to Prof. J. Nishimura for his continuous encouragement.

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We call the distribution function of cascade shower derived in this reference as $N-K$ function.

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A Kinematical Test for the Relation between the Coupling Constants in Meson Theory

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Eqs. (1) and (8) in the preceding letter[1] have been obtained with the use of the renormalizability conditions for the vertex part and the propagators only in the g_0^2 order. From this fact a question arises: When eq. (I·1) or (I·8) is expanded in a power series of g_0^2, is this expression equal to the largest part of the result obtained by perturbation calculations in every orders of g_0^2? Though in the g_0^2 order both expressions coincide, we can say nothing of what may happen until we shall calculate the higher order divergent integrals. Unfortunately it is very hard to calculate uniquely the diverging parts of renormalization constants up to higher order than g_0^2. For this reason we shall present in this letter, in $S(S)$ and $PS(PS)$ meson theories, a kinematical method to

compare eq. (I·8) with the result to be obtained by perturbation calculations, using the properties of the isotopic spin operator. With this method it is not necessary to calculate directly the diverging parts of renormalization constants.

If the interaction Hamiltonian is given by eqs. (I·6) and (I·7), the vertex part and the proper self energy parts of nucleon and meson are expanded as

$$\left.\begin{aligned} &\Gamma_j'(p,p)=\gamma_j[A_j+(\gamma p-im)A_j'(p)],\\ &\Sigma'(p)=-(\gamma p-im)B_j\\ &\qquad+(\gamma p-im)^2B_j'(p),\\ &\Pi_j(p^2)=-i(p^2+\mu^2)C_j\\ &\qquad+(p^2+\mu^2)^2C_j'(p^2),\\ &\qquad(j=1 \text{ or } 2,\ 3 \text{ and } 4) \end{aligned}\right\} \tag{1}$$

then it can be shown by a similar method as that given by Dyson[4] that

$$\left.\begin{aligned} &Z_{1,j}=|1+A_j|^{-1},\quad Z_{2,j}=|1+B_j|^{-1}\\ &\text{and}\quad Z_{3,j}=|1+C_j|^{-1},\\ &\qquad(j=1 \text{ or } 2,\ 3 \text{ and } 4) \end{aligned}\right\} \tag{2}$$

where $\sum'$, B_j, B' and $Z_{2,j}$ are independent of suffix j, and A_j, B_j and C_j are the functions of the unrenormalized coupling constant, the renormalized masses and a_i^2 factors.

First, let us consider the simplest case where the interaction Hamiltonian is given by eq. (I·3). If we define A, B and C as

$$\left.\begin{aligned} &A=A_1\big|_{\alpha_1=\alpha_3=0},\quad B=B_{j\ \alpha_1=\alpha_3=0}\\ &\text{and}\quad C=C_1\big|_{\alpha_1=\alpha_3=0}, \end{aligned}\right\} \tag{3}$$

and denote the contribution to A (B or C) from each Feynman's diagram in a g_0^{2n} order, of which the vertex part (the proper self energy part of nucleon or meson) consists, by A_i^n (B_i^n or C_i^n), then A (B or C) can be expressed as

$$\left.\begin{aligned} &A=\sum_{n=1}^{\infty}{}'\sum_i g_0^{2n}A_i^n\quad (B=\sum_{n=1}^{\infty}{}'\sum_i g_0^{2n}B_i^n\\ &\text{or}\quad C=\sum_{n=1}^{\infty}{}'\sum_i g_0^{2n}C_i^n), \end{aligned}\right\} \tag{4}$$

where suffix i runs from 1 to the total number of the graphs in a g_0^{2n} order. In the next place, we shall return to the general case. In this case the vertex part and the proper self energy parts differ from the corresponding ones in the pure neutral case only by the isotopic spin parts, and the isotopic spin space is orthogonal to the Lorentz space. The following equations, therefore, are obtained instead of eq. (4)

$$\left.\begin{aligned} &A_j=\sum_{n=1}^{\infty}{}'\sum_i g_0^{2n}D_{i,j}^nA_i^n,\quad B_j=\sum_{n=1}^{\infty}{}'\sum_i g_0^{2n}E_i^nB_i^n\\ &\text{and}\quad C_j=\sum_{n=1}^{\infty}{}'\sum_i g_0^{2n}F_{i,j}^nC_i^n, \end{aligned}\right\} \tag{5}$$

where $D_{i,j}^n$, E_i^n and $F_{i,j}^n$ depend only on a_i^2 factors.

Inserting this expression for A_j into eq. (2) and expanding $Z_{1,j}$ given by both eqs. (2) and (I·8) in the power of g_0^2, the comparison between the two expressions for $Z_{1,j}$ yields

$$\left.\begin{aligned} &D_{1,j}^1A_1^1=-b_{1,j}\alpha_j\log\lambda^2/m^2,\\ &(D_{1,j}^1A_1^1)^2-D_{1,j}^2A_1^2-D_{2,j}^2A_2^2\cdots\\ &\quad=\{b_{1,j}(b_{1,j}-1)/2\}\ (\alpha_j\log\lambda^2/m^2)^2\\ &\qquad\times[1+\text{correction terms}],\\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\} \tag{6}$$

when $D_{i,j}^n$, $b_{1,j}$ and α_j are some functions of a_i^2 factors.

When eq. (I·8) gives us a correct information about the relation between $Z_{1,j}$ and g_0^2, the magnitude of the correction terms in eq. (6) is $0\left(\frac{1}{\log \Lambda^2/m^2}\right)$ and this terms may be neglected. In this case we get, from eq. (6), $A_i^{n'}$s which are independent of a_i^2 factors. In the other case where both expressions for $Z_{1,j}$ are inconsistent with each other, the correction terms in eq. (6) cannot be neglected. If we neglect these terms in this case and solve eq. (6) for $A_i^{n'}$s, then some A_i^n depend explicitly on a_i^2 factors. The same arguments may be done for $Z_{2,j}$ and $Z_{3,j}$ as for $Z_{1,j}$.

Using this method we may conclude, in the $S(S)$ and $PS(PS)$ meson theories, that the relation between the renormalized and unrenormalized coupling constants obtained by the method of Landau and his co-workers, cannot coincide with that to be obtained by perturbation calculations. Details will be published elsewhere.

The author would like to thank Prof. K. Sakuma for his continued encouragement.

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Electron-Neutron Interaction

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Recent experiment on electron-neutron interaction at Brookhaven[1] gives -3860 ± 370 eV for the potential of well depth V_0, which is defined

$$-eV_0 = e/2r_0 \int \rho r^2 dv,$$

where r_0 is the classical electron radius, and ρ the charge density of the nucleon. Physically the electron-neutron interaction is mainly due to two effects; the electrostatic interaction due to the separation of neutron into proton and pion, and the Foldy term due to the spin-orbit coupling between the magnetic moment of the neutron and the electron current. Our object is to interpret the electrostatic interaction. Such interpretations have been given by several authors.[2)3)4)5)6)] Salzman[5] and others[6] calculated the interaction by means of static p-wave pion theory, but their results are too large, though plausible qualitatively. Our theory is based on the S-wave pion theory.[7] We know little about the S-wave interactions, but its effect may be important for nuclear forces. Within the framework of the static model, implying the momenta of nucleons can be neglected, no difficulty exists. In this case dispersion relations for the S-wave

pion-nucleon scattering were derived by Oehme,[8] and are in good agreement with experiment.

One can draw a Feynman diagram for the electron-neutron interaction as in Fig. 1.

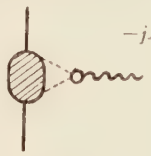

Fig. 1

The shaded area we call a scattering part,[6] and we write the scattering matrix element as

$$\langle j, q \mid S_s \mid i, k\rangle = 2\pi i \delta(k_0 - q_0) \times [D(k_0)\tau_i\tau_j + E(k_0)\tau_j\tau_i]$$

where

$$D(k_0) = \frac{2\pi}{3}(a_1 + 2a_3) - \frac{2\pi k_0}{3\mu}(a_1 - a_3) + \frac{k_0^2 - \mu^2}{2\pi}\int_0^\infty \frac{dp}{\omega_p}\left[\frac{\sigma_3(p)}{\omega_p - k_0 - i\varepsilon} + \frac{\sigma_3 + 2\sigma_1}{3(\omega_p + k_0 - i\varepsilon)}\right],$$

$$E(k_0) = \frac{2\pi}{3}(a_1 + 2a_3) + \frac{2\pi k_0}{3\mu}(a_1 - a_3) + \frac{k_0^2 - \mu^2}{2\pi}\int_0^\infty \frac{dp}{\omega_p}\left[\frac{\sigma_3 + 2\sigma_1}{3(\omega_p - k_0 - i\varepsilon)} + \frac{\sigma_3}{\omega_p + k_0 - i\varepsilon}\right].$$

The terms containing cross section are smaller than others. Then we can write approximately as

$$D(k_0) = \frac{2\pi}{3}(a_1 + 2a_3) - \frac{2\pi k_0}{3\mu}(a_1 - a_3),$$

$$E(k_0) = \frac{2\pi}{3}(a_1 + 2a_3) + \frac{2\pi k_0}{3\mu}(a_1 - a_3).$$

These are in good agreement with experiment.

Pion charge density is

$$\rho_\pi = -ie[\pi\varphi - \pi^*\varphi^*],$$

and an S-matrix element corresponding to this is

$$\langle j, q \mid \int \rho\, dx^4 \mid i, k\rangle = (2\pi)^4 e \times \delta^4(k - q)\,(k_0 + q_0)\,(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}).$$

We are to calculate following matrix elements

$$S = \frac{1}{2}\sum\iint \langle j, -q \mid \rho \mid i, -k\rangle\langle j, q \mid S_s \mid i, k\rangle \times \frac{-1}{(2\pi)^8}\frac{d^4k}{k_0^2 - \omega_k^2}\frac{d^4q}{q_0^2 - \omega_q^2}.$$

The calculations are carried out straightforwardly, and we obtain $-0.48e$ for total charge due to S-wave pions. Cut off procedures are made at $k = 6\mu$ in this case.

Next we are to calculate the effective potential due to electron-neutron interaction. The following matrix element plays an important role in this case;

$$\langle j, q \mid \int \rho r^2 d^4x \mid i, k\rangle = -e(2\pi)^4(k_0 + q_0) \times \Gamma_k \cdot \Gamma_k \delta^4(k - q)\,(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}),$$

and we calculate the following;

$$S = \frac{1}{2}\, 2\pi i \iint \delta(k_0 - q_0)\,[D(k_0)\tau_i\tau_j + E(k_0)\tau_j\tau_i] \frac{-1}{(2\pi)^8}\frac{d^4k}{k_0^2 - \omega_k^2}\frac{d^4q}{q_0^2 - \omega_q^2} \times e(2\pi)^4(k_0 + q_0) \times \Gamma_k \cdot \Gamma_k \delta^4(k - q)\,(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}).$$

The result is

$$-2\pi i e\,\delta(0)\,\tau_3\left\{\frac{4}{3\pi\mu}(a_1 - a_3)\right.$$

$$\times \int \left(\frac{3}{8} \frac{k^2}{\omega_k^3} - \frac{1}{4} \frac{k^4}{\omega_k^5} \right) dk \Big\} .$$

The bracketed part is the expectation value of $\int \rho r^2 dv$ and we can easily find thd effective potential from this. If we cut off at $k = 6\mu$ for this case also, the effective potential is about -1.8 KeV and attractive. Of course, a cut-off dependent term $F(k)$ must be added but it is unimportant for our calculations.

S-wave effects are smaller in comparison with p-wave effects,[5] which are about $-7 \sim -8$ kev, and attractive. On the other hand, the Foldy term alone accounts for -4.1 KeV of the value of $-ev_0$. Since the experimental value is -3.9 ± 0.4 KeV, it follows that our result should be $-0.2 \sim +0.6$ KeV together with p-wave effects. The result is larger. Our calculations are based on the static model and we assume only the pion current contributes dominantly. Then our result suggests the following alternatives ;

1) the nucleon core or nucleon current effect may be important,

2) a heavy meson current may cancel the pion current.[9]

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Viscosity and Heat Conductivity of Liquid He^3

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In recent years, several authors[1),2),3),4)] have discussed the viscosity and heat conductivity of liquid He^3 on the basis of an ideal F. D. gas model. Although the measured rise in viscosity of liquid He^3 with decreasing temperature, as reported by Weinstock et al.,[5] is not inconsistent with the requirements of an ideal F. D. gas model, the experiments certainly do not support this model, since the condition of $T \ll T_0 (\approx 5\,^\circ\mathrm{K}$, the degeneracy temparature of an ideal F. D. gas having mass and density of liquid He^3) is not fulfilled in the reported temperature range. Moreover, the recent measurements of surface tension and of thermodynamic and magnetic properties of liquid He^3 are found to be in complete disagreement with the predictions of this simple limiting model (see in this connection references (6), (7), (8)). One may therefore conclude that the reported temperature dependence of the viscosity of liquid He^3 resembles that of an ordinary liquid. In the present note we have attempted to bring out this fact more clearly by an empirical approach which leads us to the evaluation of viscosity and heat conductivity and an estimation of the zero-point energy of liquid He^3.

Treating liquid He^3 as an ordinary liquid, the viscosity η is related to the absolute temperature T by the equation[9]

$$\ln \eta = A + (B/T), \qquad (1)$$

where A and B are constants. Using the experimental results of Weinstock et al.,[5] we have $A=2.899$, and $B=0.536$, when η is expressed in μ-poise. The values of η thus calculated, as a function of temperature, by using equation (1) are compared in Fig. 1 with the correspondingly calculated values on the basis of an ideal F. D. gas model.[4] It is evident from Fig. 1 that the calculated values are different in both the cases.

Using the empirical relation[1] for the specific heat, C_v, the heat conductivity K

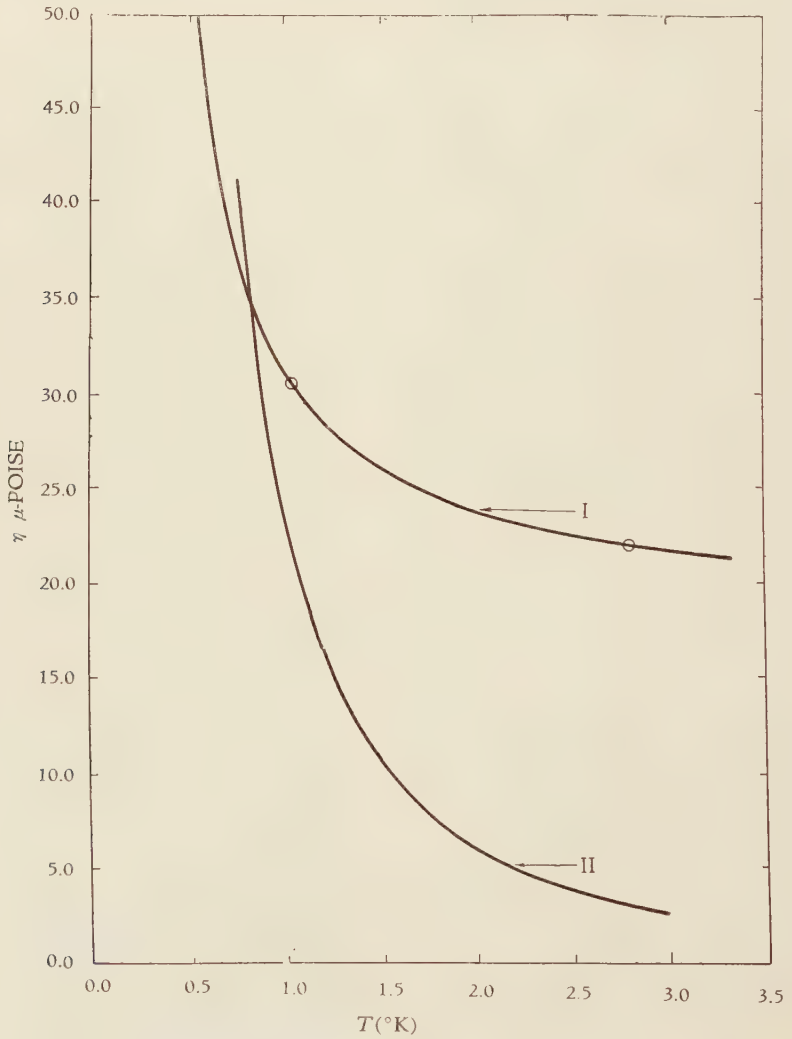

Fig. 1. Viscosity of Liquid He^3 versus T(°K).
Curve I: Present Model, eq. (1).
Curve II: Ideal F. D. Gas Model.[4]
(•): Experimental results of Weinstock et al.[5]

of liquie He^3 is expressed by the relation[11),12)]

$$K=f\cdot\eta C_v=f\cdot\eta(\alpha+\beta T+\gamma T^3), \quad (2)$$

where α, β and γ are constants while f is a numerical factor. Unfortunately, no measurements have been reported so far about the heat conductivity of liquid He^3 and the data of the viscosity as reported by Weinstock et al.,[5)] is insufficient to allow us to make any decisive conclusion about the model of the liquid He^3. In view of these facts more data about the viscosity and the heat conductivity of liquid He^3 is highly desirable. Further, the vapour pressure p, for ordinary liquids, is represented roughly by the integrated Clausius-Clapeyron's equation

$$\ln p=A'-(L/RT), \quad (3)$$

where A' is a constant. Here L is the latent heat of evaporation of the liquid while R is the gas constant ($=1.987$ cal. mole.$^{-1}$ deg^{-1}). Eliminating T between (1) and (3) we get

$$\ln p+n\ln\eta=C, \quad (4)$$

where n and C are constants. This equation has been found to hold good for various non-associated liquids[13),14)] with the value of $n\approx 4$. Thus for any two temperatures T_1 and T_2, equation (4) may be expressed as

$$\ln(p_1/p_2)=4\ln(\eta_2/\eta_1). \quad (5)$$

Combining equations (3) and (5) and using equation (1) the Trouton's ratio can thus be expressed as

$$L/T_b=4RB/T_b, \quad (6)$$

where T_b is the boiling point of the liquid. For liquid He^3, equation (6) gives us a value of 1.3 for (L/T_b). This is much too low than is expected on the basis of Trouton's rule, the reason being that for atoms of low atomic weight the zero-point energy cannot be ignored. Following the procedure outlined in reference 15, it leads us to an estimation of the zero-point energy of liquid He^3, the value of which is found to be 63 cal. mole^{-1}. It is of the same order of magnitude as that obtained by other methods.[15)]

My grateful thanks are due to Professor D. S. Kothari for taking interest in the above investigation.

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On the Annihilation Cross Section of the Antinucleon and the Nucleon

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We have recently obtained further information[1] about the electro-magnetic structure of nucleons, from the high energy electron scattering experiments of the Stanford group (electron-proton and electron-deuteron scattering experiments) and the electron-neutron interaction. According to these results, the proton has the electro-magnetic radius,

$$r_{e,p} \simeq r_{m,p} = (0.77 \pm 0.10) \times 10^{-13} \text{ cm}, \tag{1}$$

while the neutron seems to have the charge radius

$$r_{e,n} \simeq 0. \tag{2}$$

The latter requires that, although the charge distribution inside the neutron should extend intrinsically over a range of the order of (1), it is essentially neutral at every local points.

Let us here ask what is expected from such an electromagnetic size of the nucleon as to the annihilation cross section[2),3)] of the anti-nucleon and the nucleon. The most intuitive picture for the anti-proton-proton annihilation process may be the neutralization of the charges. An argument based on the charge conjugation proves that the electromagnetic structure of the antiproton is identical with that of the proton except for the sign of charge. One can expect that the annihilation process takes place at the instant that the proton and the antiproton with the finite size come into contact. Thus the anti-proton with high energies (where the classical trajectory of motion can be pictured) should have the annihilation cross section σ_a,

$$\sigma_a \simeq \pi (2r_{e,p})^2 \simeq (74 \pm 20) \text{ mb}, \tag{3}$$

and the total cross section σ_t, including the diffraction scattering σ_d ($\simeq \sigma_a$, optical model)

$$\sigma_t \simeq 2\sigma_a \simeq (150 \pm 40) \text{ mb}. \tag{4}$$

Experimentally it was found that[3)]

$\sigma_t \simeq 100$ mb for high energy ($\gtrsim 350$ Mev) antiproton on hydrogen.

It is not clear to the author whether this value of the total cross section is free from an ambiguity of the diffraction scattering. Considering the present stages of both experimental and theoretical aspects of this process, one may conclude that our crude model is quite promising to predict the annihilation cross section.

For a low energy antiproton, the classical approximation adopted above will break down, and an increase of the cross section due to the wave mechanical effect is expected. We shall estimate critical energy E_c (in lab. sys.), above which the classical picture should be allowed. Suppose that an antiproton annihilation process involves partial waves up to the order l in the incident channel. Then E_c may be found from the following equation;

$$\sigma_a \sim \pi \lambda\!\!\!^{-}_c{}^2 \sum_{l'=0}^{l} (2l'+1), \quad l \lambda\!\!\!^{-}_c \sim r_{e,p}; \tag{5}$$

where

$$\lambda\!\!\!^{-}_c \simeq 9 \times 10^{-13} / \sqrt{E_c} \quad (E_c \text{ in Mev}).$$

Using eq. (3), we find

$$E_c \simeq 34 \text{ Mev} \quad \text{for } l=0,$$

and

$$E_c \simeq 136 \text{ Mev} \quad \text{for } l=1.$$

If the antiproton is captured by the proton from the p-states beside the s-states, the wave mechanical increase of the cross section will be expected for the antiproton with a kinetic energy smaller than $E_c \simeq 136$ Mev. Such tendency was clearly seen in the experimental results.

Finally, let us add a speculation about the structure of the "physical" nucleon. Recently, Sakata[4] has proposed a new theory of elementary particles in which the pion is considered as a composite system of the nucleon and the antinucleon.[5] According to this model, a physical nucleon is described in terms of (a bare nucleon) + (nucleon pairs). Thus, the difference of (1) and (2) can easily be understood[6] and it is natural to suppose that the annihilation of the antinucleon and the nucleon should take place just after the contact of the rims of the extended nucleons.

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$$(r_{e,p})^2 = \int dr\, r^2 [\rho_c(r) + \rho_\pi(r)],$$

$$(r_{e,n})^2 = \int dr\, r^2 [\rho_c(r) - \rho_\pi(r)].$$

From the experimental results, eqs (1) and (2), we can learn that the nucleon-core extends over the radius $\simeq r_{e,p}$ ($\gg \hbar/Mc$, the nucleon Compton wave length), which is difficult to understand from the meson theories, see ref. 1).

Interaction with the Sound Quanta in a System consisting of Two Kinds of Charged Particles

The Method of Supplementary Variables

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In some cases one considers the solid either as a system of particles interacting by means of central forces, or as a system of electrons interacting with the sound quanta.

In this paper it is shown that with the use of the method of supplementary variables[1),2),3)] we can transform the Hamiltonian of system composed of two kinds of particles which interact by means of central forces into a form in which there appear terms describing interaction of electrons with sound quanta.

The Hamiltonian of electrons and ions which interact by means of central forces has the form;

$$H=\sum_{j/1}^{N_1}\frac{p_j^2}{2m}+\frac{1}{2}\sum_{j_1\neq j_2}\sum_{\boldsymbol{k}}\frac{4\pi e^2}{k^2}\times e^{i\boldsymbol{k}(\boldsymbol{r}_{j_1}-\boldsymbol{r}_{j_2})}+\sum_{l/1}^{N_2}\frac{\pi_l^2}{2M}+$$

$$+\frac{1}{2}\sum_{l_1\neq l_2}\sum_{\boldsymbol{k}}\mathcal{Z}^2e^2w(k)\,e^{i\boldsymbol{k}(\boldsymbol{R}_{l_1}-\boldsymbol{R}_{l_2})}$$

$$-\sum_{j,l}\sum_{\boldsymbol{k}}\mathcal{Z}e^2u(k)\,e^{i\boldsymbol{k}(\boldsymbol{r}_j-\boldsymbol{R}_l)}. \qquad (1)$$

Here $\boldsymbol{r}_j$, $\boldsymbol{p}_j$, $-e$ and m refers to the j-th electron, $\boldsymbol{R}_l$, π_l, $\mathcal{Z}_e$ and M to the l-th ion, N_1 is the density of the electrons, N_2 that of the ions. $4\pi e^2/k^2$, $w(k)\mathcal{Z}^2e^2$, $-u(k)\mathcal{Z}e^2$ are respectively the Fourier coefficients of electron-electron, ion-ion, electron-ion interactions.

We split all these interactions into "long-range" ($|\boldsymbol{k}|=k\leq k_0$) and "short-range" ($k>k_0$) interactions. The value k_0 is the quantum mechanical analogue of the value $k_D\cong 1/\lambda_D$ appearing in the classical plasma theory, where λ_D is the Debye length. The "long-range" interactions we express as the functions of Fourier components of the density-of-mass operators:

$$\rho_k^{(1)}=m\sum_{j/1}^{N_1}e^{-i\boldsymbol{k}\boldsymbol{r}_j},\quad \rho_k^{(2)}=M\sum_{l/1}^{N_2}e^{-i\boldsymbol{k}\boldsymbol{R}_l}. \qquad (2)$$

In analogy with the method of the papers quoted above, supplementary variables $Q_k^{(i)}$ ($i=1,\,2$, $k\leq k_0$) are introduced and subsidiary conditions are imposed on the wave function φ:

$$Q_k^{(i)}\varphi(\boldsymbol{r}_1,\cdots\boldsymbol{r}_{N_1},\boldsymbol{R}_1,\cdots\boldsymbol{R}_{N^2}/\cdots Q_k^{(1)}\cdots Q_k^{(2)}\cdots)=0\quad\text{for}\quad k\leq k_0. \qquad (3)$$

We transform (1) and (3) with the unitary operator:

$$U=\exp\{(-i/\hbar)\cdot\sum_{|k|\leq k_0}(P_k^{(1)}\rho_k^{(1)}+P_k^{(2)}\rho_k^{(2)})\} \qquad (4)$$

where $P_k^{(i)}$ are canonical conjugate to $Q_k^{(i)}$ and $\rho_k^{(i)}$ are given by (2).

Using the transformed subsidiary conditions and introducing the new canonical variables $(q_k,\,p_k)$ we can represent the transformed Hamiltonian as the sum of three terms:

$$\mathcal{H}=\mathcal{H}_{\text{part}}+\mathcal{H}_{\text{osc}}+\mathcal{H}_{\text{int}}. \qquad (5)$$

1. $\mathcal{H}_{\text{part}}$ is the Hamiltonian of particles interacting by means of potentials which are "cut off" in comparison with the potentials occuring in the Hamiltonian (1). Especially when we put $\boldsymbol{R}_l=\boldsymbol{n}_l+\Delta\boldsymbol{R}_l$ ($\boldsymbol{n}_l$ is the radius vector of the l-th lattice point, $\Delta\boldsymbol{R}_l$ the small deviation of the ion) the "cut off" potentials can be averaged to give a periodical potential.

2. $\mathcal{H}_{\text{osc}}$ is the Hamiltonian of the coupled oscillators:

$$\mathcal{H}_{\text{osc}}=(1/2)\cdot\sum_{|\boldsymbol{k}|\leq k_0}\{p_k^{(1)}p_{-k}^{(1)}+\omega_1^2q_k^{(1)}q_{-k}^{(1)}+p_k^{(2)}p_{-k}^{(2)}+\omega_2^2q_k^{(2)}q_{-k}^{(2)}-\gamma_k(q_k^{(1)}q_{-k}^{(2)}+q_{-k}^{(1)}q_k^{(2)})\}, \qquad (6)$$

where $\omega_1=\sqrt{4\pi e^2N_1/m}$ is the so-called plasma frequency, $\omega_2=k\sqrt{\mathcal{Z}^2e^2N_2w(k)/M}$ the frequency of the ion oscillations. $\gamma_k=(u(k)k/\sqrt{4\pi w(k)})\omega_1,\omega_2$ are the coupling constants of both systems of oscillators.

3. $\mathcal{H}_{\text{int}}$ gives the interaction of electrons with the quanta $\hbar\omega_1$ and the ions with the quanta $\hbar\omega_2$.

Performing the normal transformation of the Hamiltonian (5) we get a new Hamiltonian $\widetilde{\mathcal{H}}$ which contains only noninteracting oscillators vibrating with the frequency:

$$\Omega_{1,2}^2=1/2\{\omega_1^2+\omega_2^2\pm\sqrt{(\omega_1^2+\omega_2^2)^2+4(\gamma_k^2-\omega_1^2\omega_2^2)}\}. \qquad (7)$$

When $(\omega_1^2+\omega_2^2)^2 \gg 4(\gamma_k^2-\omega_1^2\omega_2^2)$, and $u(k)=4\pi/(k^2+a^2)$, $w(k)=4\pi/(k^2+b^2)$ (screend Coulomb potentials, with the screenning constants a and b) we get:

$$\Omega_1 \simeq \sqrt{4\pi e^2 N_1/m},$$

$$\Omega_2 \simeq k\sqrt{(Z^2e^2N_2/M)\,(4\pi/k^2+a^2)}. \quad (8)$$

It is seen from (8) that the frequency Ω_1 (connected with the plasma frequency) gives the "optical branch" and the frequency Ω_2 the "acoustical branch" of frequencies.

$\tilde{\mathcal{H}}_{\text{int}}$ gives now the interactions of electrons and ions as well as with the two kinds of bosons. The coupling constants of the interaction of electrons with the sound (acoustical) quanta depends on the ion mass.

The methods used here are similiar to those used in the paper of Bardeen and Pines.[4] It seems that the differences consist in the introduction of two kinds of supplementary variables in order to describe the collective motion of electrons as well as ions. Moreover we do not assume the ion oscillations; they appear rather as a result of central forces between the particles. Bardeen and Pines assume elastic departure of the ions from equilibrium positions. This leads to a periodical potential which differs from the potential obtained in this paper. The exact formulae for frequencies of vibrations are obtained in the same way as in the papers of Wentzel[5] or Bardeen.[6] Bardeen and Pines obtain the approximate values for the frequencies.

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The Catalysis of Nuclear Fusion Reactions by μ^--Mesons

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Recently, Alvarez et al.[1] found the catalysis of the nuclear fusion reactions by μ^--mesons in the liquid hydrogen bubble chamber. They observed such events that a μ^--meson comes to rest in the hydrogen to form μ^--mesic atom from which the μ^--meson is ejected again with kinetic energy of 5.4 Mev, which in turn decays by emitting an electron. In this note we shall try to explain these events and show that our calculated results agree, but semi-quantitatively, with their experimental data.

In the liquid hydrogen, a μ^--meson from $\pi-\mu$ decay loses its energy, knocks out the electron from the hydrogen atom and forms a μ^--mesic hydrogen atom. This μ^--mesic atom, generally excited,

makes radiative transitions to the ground state. The time from 4.1 Mev to the ground state is an order of 10^{-10} sec.,[2] which is very short compared with the μ-meson lifetime. The μ^--mesic hydrogen atom, then, reacts with other hydrogen nuclei in various modes as follows,

(a) μ^--mesons in the $(\mu^- H)$ or $(\mu^- D)$ mesic atom decays into electron and neutrinos (the decay probability is $w_d = 5\times 10^5$ sec^{-1}).

(b) Rearrangement process in which a deuteron robs the μ^--meson from proton, because of the 5% larger reduced mass:

$$(\mu^- H) + D \rightarrow (\mu^- D) + H + 135 \text{ eV}.$$

(c) Elastic Scattering: $(\mu^- D) + H \rightarrow (\mu^- D) + H$, etc.

(d) μ^--mesic hydrogen atom forms a molecular ion with hydrogen nucleus: $(\mu^- H) + H \rightarrow (\mu^- HH)$, $(\mu^- D) + H \rightarrow (\mu^- DH)$, etc.

To estimate the cross sections of processes (b), (c) and (d) referred to as σ_e, σ_s and σ_f, respectively, we use the analogy of nuclear reactions. Using one-level dispersion formula,[3] we have

$$\sigma_f = 8\zeta^2 \lambda\!\!\!^{-2}\, \Gamma'_\gamma / KD \sim 2\zeta^2 \times 10^{-25} \text{ cm}^2 \quad \text{at } 20^\circ\text{K}, \tag{1}$$

where $\pi\lambda\!\!\!^{-2} = 5\times 10^{-16}$ cm^2 at 20°K, D is level spacing ($\simeq \hbar\omega$), K is wave number in $(\mu^- HD)$-ion ($\sim 3\times 10^{12}$) and $\zeta = 1/2\,(D/E_{rot})^{1/2} > 1$. ω is the zero-point oscillation frequency of proton for the adiabatic potential between $(\mu^- D)$-atom and proton with the following form: $V(r) \simeq e^2/r$ for $r \ll a_\mu$, $\simeq D[e^{2(r_0 - r)/a} - 2e^{(r_0 - r)/a}]$ for $r > a_\mu$, where $D = 0.56$ Kev, $a_\mu = \hbar^2/m_\mu e^2 \sim 2.6\times 10^{-11}$ cm (m_μ being μ-meson mass) and $r_0 = 2a_\mu$. The radiative width Γ'_γ is given by a dipole one $\sim \alpha^3\omega$. Also we have

$$\sigma_e \simeq 8\zeta'^2 \lambda\!\!\!^{-2}\, \Gamma'_H / KD \simeq \zeta'^2 \times 10^{-18} \text{ cm}^2 \quad \text{at } 20^\circ\text{K}, \tag{2}$$

where $\Gamma'_H (\simeq 2D/\pi\lambda\!\!\!^{-2}_H K)$ is a proton emission width at 135 eV, and

$$\sigma_s \simeq 8\zeta'^{\prime 2} \lambda\!\!\!^{-2}_H \Gamma'^{\prime}_H / KD \simeq 5\zeta'^{\prime 2} \times 10^{-21} \text{ cm}^2 > 100\,\sigma_f. \tag{3}$$

Thus, the ratios of the transition probabilities of processes (b) and (d), w_e and w_f, to that of the $\mu - e$ decay, are given by

$$e \equiv w_e/w_d = n_H \sigma_e v/w_d \simeq 6\zeta'^2 \times 10^3,$$
$$f \equiv w_f/w_d = n_H \sigma_f v/w_d \simeq \zeta^2 \times 10^{-3}. \tag{4}$$

The $(\mu^- D)$ mesic atom, ejected in the process (b) with kinetic energy of 45 eV, is slowed down to thermal energy by elastic scatterings and then forms the molecular ion $(\mu^- DH)$ by capturing a proton. It should be noticed that the ion-exchange process as $(\mu^- HH) + D \rightarrow (\mu^- HD) + H$, etc., can be neglected, because of the Coulomb barrier effect.

Next, let us consider the nuclear reaction occurring in the molecular ion $(\mu^- DH)$. The barrier penetrating factor P can be estimated as e^{-2G}, with $G \simeq (2\mu/\hbar^2)^{1/2} \int_R^{a_\mu} \sqrt{V(r) - E}\, dr$ where R is the nuclear radius and $\mu = (2/3)\, m_H$ (m_H being hydrogen mass). We find $P \sim 10^{-5}$.

The cross section of $D(p, \gamma)$ He^3 is expressed as $\sigma_\gamma = (0.1 \text{ Mev}/E)\cdot P \times 10^{-30}$ cm^2 (E in Kev),[4] from which the probability $w_\gamma\,(\equiv \gamma w_d)$ of the reaction in the molecular state is given by

$$w_\gamma = n\sigma_\gamma v \simeq (2a_\mu)^{-3}(3\times 10^{-33}) \times (3\times 10^7) \text{ sec}^{-1} \simeq 6\times 10^5 \text{ sec}^{-1} \simeq w_d. \tag{5}$$

In some fraction of these nuclear reactions, the reaction energy is taken up

by the μ^--meson, which appears in the bubble chamber with a kinetic energy of 5.4 Mev. This process is referred to as $\mu\mu$-ending while other process as μe-ending in reference 1). This process is an analogue of K-electron ejection caused by $0^+ - 0^+$ transition in atomic nuclei and we can obtain the nuclear transition width of μ^--emission, Γ'_μ, as 0.04 eV.[3] Comparing this value with $\Gamma'_\gamma \simeq 0.24$ eV of the nuclear reactions, $H+D \rightarrow He^3+\gamma$,[4] we estimate the reaction probability of the above process w_μ ($\equiv \mu w_d$) is $\sim 0.2\, w_d$.

Thus, the ratio of the frequency of μe-ending to that of $\mu\mu$-ending is given by

$$y = 1+f+xe\left\{\frac{1}{1+f}+\frac{f}{1+f}\,\frac{1+\gamma}{1+\gamma+\mu}\right\}\Big/ xe\,\frac{f}{1+f}\,\frac{\mu}{1+\gamma+\mu}, \qquad (6)$$

where x is the concentration of deuterium in the liquid hydrogen, while one finds an empirical relation $y = a/x + b$ ($a = 1/43$, $b = 33$) from the experimental results by Alvarez et al. Taking $\gamma \simeq 1$ and $\mu/\gamma \sim 1/6$ from the above values of a, b, and (6), we obtain $f \sim 1/10$ and $e \quad 1.4 \times 10^3$. These results are not inconsistent with our estimations[4] and show that the explanation outlined above is correct.

Further, the 1 mm length of gaps at the end of some of the stopping μ-mesons observed in the experiment can be explained by using the cross section (3) for the elastic scattering of $(\mu^- D)$-mesic atom which is slowed down to thermal energy ($\sim 1/600$ eV) from 45 eV.

Finally, we wish to remark that other negative particles than μ^--mesons, i. e., π^--and K^--mesons, antiprotons, and Σ^-- and Ξ^--particles cannot be useful as catalysers for nuclear fusion reactions, because of either their strong interactions with nucleons or their shorter decay lives.

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ERRATA

Helium Capturing Reactions in Stars

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Prog. Theor. Phys. **16** (1956), 507

Page	Equation or Figure	Original	Corrected
509	Eq. (2·5)	$\cdots\sqrt{(B-V)}/\cdots$	$\cdots\sqrt{(B-E)}/\cdots$
510	Eq. (2·10)	$\cdots/(A_1+A_2)]^{1/6}T^{5/6}$	$\cdots/(A_1+A_2)]^{1/3}T^{2/3}$
512	Eq. (2·16)	$\cdots T^{-3/2}10^{-50.401E_0/kT}$	$\cdots T^{-3/2}10^{-50.401E_0/T}\cdot n_1 n_2$
512	Eq. (2·16′)	$\cdots T^{-3/2}10^{-50.401E_0/kT}$	$\cdots T^{-3/2}10^{-50.401E_0/T}\cdot n_1 n_2$
521	On the horizontal axis in Fig. 7	0	1
522	Eq. (5·3)	$\gamma=\alpha\xi$	$r=\alpha\xi$

On the Spins and Decay Interactions of $K_{\mu 3}$ and K_{e3}

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The expression for $P^{V(V)}$ in equation (5) is not correct, and hence the statement that $P^{V(V)}$ coincides with $P^{V(P)}$ except constant factor is erroneous. It should be corrected as

$$P^{V(V)}(E)=(F^{V(V)2}/F^{V(P)2})\,2m_k{}^2\,P^{V(P)}(E)+(F^{V(V)2}/24\pi^3)\,m_k{}^4\{(m_k{}^2-2m_kE)\,I_4-2(m_k-E)\,I_3+I_2\}(W-E)^3$$
$$=(F^{V(V)2}/48\pi^3)\,2m_k{}^3[m_k{}^3\,I_5(W-E)^4-m_k{}^2\{W-E+m_k\}\,I_4(W-E)^3+$$
$$+m_k\{E^2-m_\mu{}^2+2m_k(W-E)\}\,I_3(W-E)^2-\{E^2-m_\mu{}^2+m_k(W-E)\}\,I_2(W-E)^2]$$

and the following expressions for the interference terms should be added in equation (7).

$$P^{V(V,A)}=(Re[F^{V(V)}F^{V(A)*}]/Re[F^{V(P)}F^{V(T)*}])\,2P^{V(P,T)}$$
$$P^{V(V,T)}=(Re\{iF^{V(V)}F^{V(T)*}]/Re[iF^{V(P)}F^{V(A)*}])\,2m_k{}^2\,P^{V(P,A)}$$
$$P^{V(V,T)'}=(Re[iF^{V(V)}F^{V(T)'*}]/Re[F^{V(V)}F^{V(A)*}])\,2m_\mu\,P^{V(V,A)}.$$

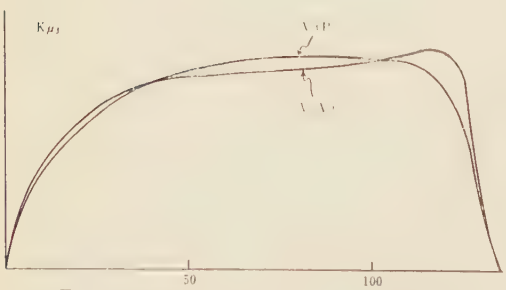

Fig. 1. Kinetic energy of the muon (Mev)

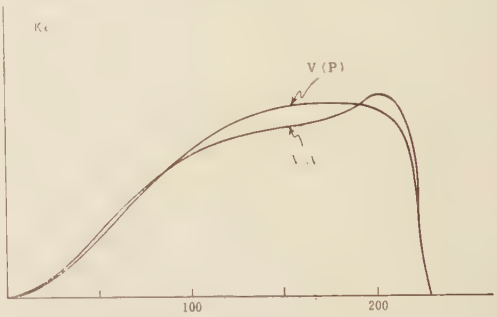

Fig. 2. Kinetic energy of the electron (Mev)

The corrected curves for $P^{V(V)}$ is indicated in the Figs. together with $P^{V(P)}$ curve. Fortunately they do not affect the discussions there and in our succeeding note, Nuovo Cimento Vol. 5. No. 1.

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Quantum-Electrodynamical Corrections to Energy-Levels of Helium

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Quantum-electrodynamical corrections to energy levels of helium are calculated with an accuracy of $\alpha^5 mc^2$ on the basis of the Bethe-Salpeter equation according to the perturbation method of Fulton and Karplus. The interaction kernel of the equation is derived according to the method of Martin and Schwinger. The corrections are carefully calculated by the method which is free from any doubt. The result is expressed in terms of non-relativistic eigenfunctions. It is found for $\alpha^5 mc^2$ corrections that correction factors for spin-orbit and spin-spin couplings are in complete agreement with those derived by G. Araki on the basis of a simple consideration. Since $\alpha^4 mc^2$ corrections often gave rise to confusions and discussions among many authors these corrections are carefully examined. It is found on the firm basis that the $\alpha^4 mc^2$ terms involve a contact interaction but not the Breit surplus term. Further it is found that the energy correction for the excited bound state includes an imaginary part. Of course the ground-state energy is real. In fact the absolute value of the imaginary part represents half the width of the level in question. A discussion for this fact is given.

Introduction

It is well known that quantum electrodynamics of a system including one fermion gives correct energy levels of a hydrogen-like atom by shifting eigenvalues of the Dirac electron in agreement with experiment. Salpeter and Bethe[1] introduced a relativistic two-body equation and Salpeter[2] was first able to calculate recoil effects relativistically according to this equation. Karplus and Klein[3] calculated the singlet-triplet splitting in the ground state of positronium and Arnowitt[4] applied their method to the hyperfine structure of hydrogen. Their method was deficient in that all intermediate states were free-particle states. The method was improved by Fulton and Karplus,[5] and further by Martin and Schwinger[6] so as to take into account the binding in intermediate states. Energy levels of positronium were calculated by the improved method.[5),7] The next problem may be to apply the method to the two-electron system in helium. This is the purpose of the present paper.

The fine-structure intervals of energy levels of helium was calculated by G. Araki[8] on the basis of the Breit interaction. As is well known the intervals amount to magnitudes of order $\alpha^4 mc^2$ where α is the Sommerfeld fine structure constant and m is the electron mass. Recently G. Araki[9] calculated corrections of order $\alpha^5 mc^2$ for the triplet intervals. His calculation was based on corrected spin-orbit and spin-spin interactions. The correction factors were derived by a simple consideration on quantum-electrodynamical calculations of the Lamb shift for hydrogen and an intrinsic magnetic moment for an electron. The calculated corrections amounted to observable magnitude in the case of

microwave experiment,[10] though they were within experimental errors in the case of optical measurement.[11] His method cannot be applied to a calculation of the Lamb shift for singlets. The Lamb shift in the ground state of neutral helium was experimentally studied by Herzberg.[12] In view of this situation it is very interesting to derive the corrections in the form which is applicable to both singlets and triplets and to examine whether the correction factors derived by G. Araki[9] are justified by the quantum-electrodynamical method.

In the present paper energy eigenvalues of helium-like atom will be calculated with an accuracy of $\alpha^5 mc^2$, on the basis of the Bethe-Salpeter equation.[1] The interaction kernel of the equation will be derived according to the method of Martin and Schwinger[6] so as to include terms up to the fourth order in the interaction, taking the binding into account. The corrections to the relativistic eigenvalues which include only the Coulomb energy will be calculated according to the perturbation method of Fulton and Karplus.[7] In order to take into account the binding effect on the Lamb shift in the two-body system owing to an introduction of the nuclear Coulomb field, some devices will be needed. The final result will be represented by the integral which readily gives us numerical values of the corrections up to terms of order $\alpha^5 mc^2$ if we have correct non-relativistic wave functions of helium. In § 1 a general account for the fundamental equation will be given. In § 2 the general principle of calculation and the method of approximation will be accounted for together with the result of calculation. It will be verified that the correction factors derived by G. Araki[9] on the basis of a simple consideration is still correct in the detailed consideration. The outline of the present calculations will be explained in succeeding five sections. Since $\alpha^4 mc^2$ corrections often gave rise to confusions and discussions among many authors, these corrections will be carefully examined. It will be confirmed on the firm basis that the corrections contain a contact interaction term but not the Breit surplus term. We shall see that the energy correction for the excited bound state contains an imaginary part, although the ground-state energy is real. We shall find that the absolute value of the imaginary part is equal to a half of the level width and that this fact is deeply connected with the essential nature of the Bethe-Salpeter equation. A discussion on this fact will be given in the final section.

§ 1. Fundamental equations

We consider a system of two electrons which are bound in the nuclear Coulomb field due to the charge Ze. The Bethe-Salpeter equation[1] of the system is given in the usual form, as follows:

$$[\mathrm{F}_1(11')\mathrm{F}_2(22') - \mathrm{I}(12, 1'2')]\Psi(1'2') = 0, \qquad (1\cdot1)$$

where F_1 is an operator of the first electron, F_2 is that of the second electron, and I represents the interaction between two electrons. F_1 is defined by

$$\mathrm{F}_1 = \gamma^{(1)}p_1 + m - \gamma_0^{(1)}A_0^{(1)}, \quad A_0^{(1)} = Ze^2/r_1 + A_0^{(1)}{}_{\text{vac pol}}, \qquad (1\cdot2)$$

where $\gamma^{(1)}$ and p_1 are the gamma operator and the four-momentum of the first electron respectively, r_1 is the distance of the first electron from the nucleus, and $A_0^{(1)}{}_{\text{vac pol}}$ is the vacuum-polarization correction to the potential energy of the first electron due to the external Coulomb field. We employ the unit system in which $\hbar=c=1$. The scalar product of gamma operators and momentum is denoted by $\gamma p=\beta\boldsymbol{\alpha}\boldsymbol{p}-\beta p_0$.

We derive the interaction kernel, I, according to the method of Martin and Schwinger.[6] If we take into account the terms up to the fourth order in the interaction between electrons, photons and the Coulomb field, we have the interaction kernel as follows:

$$\mathrm{I}(12,\ 1'2')=\mathrm{I}(\xi_1\xi_2,\ \xi_1'\xi_2')=\mathrm{I}_C+\mathrm{I}_{CC}+\mathrm{I}_t+\mathrm{I}_{tt}+\mathrm{I}_{eL}+\mathrm{I}_{nL}+\mathrm{I}_v, \tag{1·3}$$

$$\mathrm{I}_C=4\pi ie^2\gamma_0^{(1)}\gamma_0^{(2)}D_C(\xi_1\xi_2)\,\delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2')$$
$$=(ie^2/r_{12})\gamma_0^{(1)}\gamma_0^{(2)}\delta(t_1-t_2)\,\delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2'), \tag{1·4a}$$

$$\mathrm{I}_{CC}=(4\pi ie^2)^2\gamma_0^{(1)}(\xi_1)\gamma_0^{(2)}(\xi_2)\mathrm{G}_1{}^C\mathrm{G}_2{}^C\gamma_0^{(1)}(\xi_1')\gamma_0^{(2)}(\xi_2')D_C(\xi_1\xi_2')D_C(\xi_1'\xi_2), \tag{1·4b}$$

$$\mathrm{I}_t=-4\pi ie^2[\gamma_t^{(1)}(\xi_1)\mathrm{F}_2\mathrm{G}_{12}^C\mathrm{F}_1\gamma_t^{(2)}(\xi_2')-\gamma_t^{(1)}(\xi_1)\gamma_t^{(2)}(\xi_2')\,\delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2')/2]D_t(\xi_1\xi_2')$$
$$+(1\rightleftarrows 2), \tag{1·4c}$$

$$\mathrm{I}_{tt}=(4\pi ie^2)^2\gamma_t^{(1)}(\xi_1)\gamma_t^{(2)}(\xi_2)\mathrm{G}_1{}^C\mathrm{G}_2{}^C\gamma_t^{(1)}(\xi_1')\gamma_t^{(2)}(\xi_2')D_t(\xi_1\xi_2')D_t(\xi_1'\xi_2), \tag{1·4d}$$

$$\mathrm{I}_{eL}=-4\pi ie^2\gamma^{(1)}(\xi_1)\mathrm{F}_2(\mathrm{G}_{12}^C-\mathrm{G}_1{}^C\mathrm{G}_2{}^C)\mathrm{F}_2\gamma^{(1)}(\xi_1')D(\xi_1\xi_1')$$
$$+(4\pi ie^2)^2\gamma^{(1)}(\xi_1)\mathrm{G}_1{}^C\gamma_t^{(1)}(\xi_1'')\mathrm{G}_1{}^C\gamma^{(1)}(\xi_1')\gamma_t^{(2)}(\xi_2)\,\delta(\xi_2-\xi_2')D(\xi_1\xi_1')D_t(\xi_1''\xi_2)$$
$$-2ie^4B_1\gamma^{(1)}(\xi_1)\gamma^{(2)}(\xi_2)D(\xi_1\xi_2)\,\delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2')+(1\rightleftarrows 2), \tag{1·4e}$$

$$\mathrm{I}_{nL}=-4\pi ie^2\mathrm{F}_2\gamma^{(1)}(\xi_1)\,(\mathrm{G}_1{}^C-\mathrm{G}_1{}^O)\gamma^{(1)}(\xi_1')D(\xi_1\xi_1')+i(e^2/2\pi)B_1\mathrm{F}_2\gamma_0^{(1)}A_0^{(1)}$$
$$+(1\rightleftarrows 2), \tag{1·4f}$$

$$\mathrm{I}_v=-(4\pi ie^2)^2\gamma^{(1)}(\xi_1)\gamma^{(2)}(\xi_2)[D(\xi_1\xi)\,\mathrm{tr}\{\gamma(\xi)\mathrm{G}^O\gamma(\xi')\mathrm{G}^O\}D(\xi'\xi_2)$$
$$+(i/8\pi^2)B'D(\xi_1\xi_2)]\,\delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2'), \tag{1·4g}$$

where in the second line of (1·4e) and the first term in the bracket of (1·4g) integrations with respect to ξ_1'', and ξ and ξ' respectively should be understood. $(1\rightleftarrows 2)$ means the terms which are obtained by exchanging operators and coordinates of the first and second electrons in the preceding terms, G and D are the Green functions of electrons and photons respectively, B_1 and B_2 are constants coming from renormalization-corrections of wave functions to the single-quantum exchange and nuclear Coulomb field, and B' is that coming from the charge renormalization already performed. In order to define the Green functions we introduce the Hamiltonian of the first electron as follows:

$$\mathrm{H}_1=\mathrm{H}_1{}^0(\boldsymbol{p}_1)-Ze^2/r_1,\quad \mathrm{H}_1{}^0(\boldsymbol{p})=\alpha^{(1)}\boldsymbol{p}+\beta^{(1)}m. \tag{1·5}$$

The Hamiltonian of the second electron is defined in the same way. The Green functions are now given by the following equations:

$$G_1^{C}(\hat{\xi}\hat{\xi}') = \int (H_1 - p_0)^{-1} \gamma_0^{(1)} \exp[-ip_0(t-t')]\, dp_0/(2\pi), \tag{1·6a}$$

$$G_1^{0}(\hat{\xi}\hat{\xi}') = \int [\gamma^{(1)} p + m]^{-1} \exp[ip(\hat{\xi}-\hat{\xi}')]\, d^4p/(2\pi)^4, \tag{1·6b}$$

$$G_{12}^{C} = |F_1 F_2 - I_C|^{-1}, \tag{1·6c}$$

$$D(\hat{\xi}\hat{\xi}') = (2\pi)^{-4} \int e^{ik(\xi-\xi')}\, d^4k/(k^2 - k_0^2), \tag{1·7a}$$

$$D_{jl}^{t}(\hat{\xi}\hat{\xi}') = (2\pi)^{-4} \int e^{ik(\xi-\xi')} (\delta_{jl} - k_j k_l/k^2)\, d^4k/(k^2 - k_0^2), \tag{1·7b}$$

$$D_C(\hat{\xi}\hat{\xi}') = (2\pi)^{-4} \int e^{ik(\xi-\xi')}\, d^4k/k^2. \tag{1·7c}$$

G_2^{C} and G_2^{0} are defined in the same way. G^{0} is obtained from G_1^{0} by replacing $\gamma^{(1)}$ with γ. The interaction kernel contains F_1 and F_2. We can omit $A_{0\,\mathrm{vac\ pol}}$ in them because the omitted terms give no significant contribution to the final result in the required accuracy, while such terms are significant in F_1 and F_2 of (1·1).

A scalar product of two four-vectors, $p = (p_\nu) = (\boldsymbol{p}, p_0)$ and $q = (q_\nu) = (\boldsymbol{q}, q_0)$, is denoted by $pq = \sum_{\nu=1}^{4}{}' p_\nu q_\nu = \boldsymbol{p}\cdot\boldsymbol{q} - p_0 q_0$. Space-time coordinates of an electron is denoted by $\hat{\xi} = (\boldsymbol{x}, t)$. Further the following abbreviations are employed:

$$\gamma(\hat{\xi}) \cdots \gamma(\hat{\xi}') \cdots D(\hat{\xi}\hat{\xi}') = \sum_{\nu=1}^{4} \gamma_\nu(\hat{\xi}) \cdots \gamma_\nu(\hat{\xi}') \cdots D(\hat{\xi}\hat{\xi}'), \tag{1·8a}$$

$$\gamma_l(\hat{\xi}) \cdots \gamma_l(\hat{\xi}') \cdots D_l(\hat{\xi}\hat{\xi}') = \sum_{j,l=1}^{3} \gamma_j(\hat{\xi}) \cdots \gamma_l(\hat{\xi}') \cdots D_{jl}^{t}(\hat{\xi}\hat{\xi}'). \tag{1·8b}$$

The argument of γ means arguments of operators which embrace it; for example, the trace in (1·4g) means

$$\mathrm{tr}[\gamma(\hat{\xi})\, G^0 \gamma(\hat{\xi}')\, G^0] = \mathrm{tr}[\gamma(\hat{\xi})\, G^0(\hat{\xi}\hat{\xi}')\, \gamma(\hat{\xi}')\, G^0(\hat{\xi}'\hat{\xi})].$$

Among the various terms in (1·4), I_C corresponds to the Coulomb interaction between two electrons, $I_{CC'}$ to the crossed Coulomb interaction, I_t to the interaction including one photon, I_{tt} to the interaction including two photons, $I_{t'}$ to the vertex correction to the single-quantum exchange between two electrons (e), which gives rise to a part of the Lamb shift (L), $I_{t'e}$ to the vertex correction to the external (nuclear) Coulomb field, and I_v to the vacuum-polarization correction to the single-quantum exchange. The following relations can be shown to hold between the Green functions defined by (1·6) and (1·7):

$$G_{12}^{C} = G_1^{C} G_2^{C} + G_1^{C} G_2^{C} I_C G_{12}^{C} = G_1^{C} G_2^{C} + G_{12}^{C} I_C G_1^{C} G_2^{C}, \tag{1·9a}$$

$$G_1^{C} = G_1^{0} + G_1^{0} A_t^{(1)} \gamma_0^{(1)} G_1^{C}, \quad G_2^{C} = G_2^{0} + G_2^{0} A_t^{(2)} \gamma^{(2)} G_2^{C}, \tag{1·9b}$$

$$\gamma_l \gamma_l D_t - \gamma_0 \gamma_0 D_t = \gamma\gamma D \quad \text{if} \quad \gamma k = 0. \tag{1·10}$$

§ 2. Principle and result of calculation

The purpose of the present calculation is to obtain the eigenvalue of the Bethe-Salpeter equation, (1·1), with an accuracy of $e^{10}m\,(=\alpha^5 mc^2)$, according to the perturbation method of Fulton and Karplus.[5] In this section an outline of the calculation will be accounted for. For this purpose we employ an abbreviation as follows:

$$O_E(t,\,t') = \int_{-\infty}^{\infty} O(t_1 t_2,\; t_1' t_2')\, e^{iE(T-T')}\, dT', \tag{2·1}$$

where $T=(t_1+t_2)/2$, $t=t_1-t_2$, and $O(t_1t_2,\,t_1't_2')$ is an operator having a property that O_E is independent of T. We introduce the following operators:

$$\Lambda^C(\tilde{\xi}_1\tilde{\xi}_2,\;\tilde{\xi}_1'\tilde{\xi}_2') = G_1{}^C(\tilde{\xi}_1\tilde{\xi}_1')\gamma_0^{(1)}\delta(\tilde{\xi}_2-\tilde{\xi}_2') + G_2{}^C(\tilde{\xi}_2\tilde{\xi}_2')\gamma_0^{(2)}\delta(\tilde{\xi}_1-\tilde{\xi}_1'), \tag{2·2}$$

$$\Lambda(\boldsymbol{x}_1\boldsymbol{x}_2,\;\boldsymbol{x}_1'\boldsymbol{x}_2') = -i\Lambda_E^C(\boldsymbol{x}_1\boldsymbol{x}_2 t,\;\boldsymbol{x}_1'\boldsymbol{x}_2' t) = \Lambda_1^{C+}\Lambda_2^{C+} - \Lambda_1^{C-}\Lambda_2^{C-}, \tag{2·3}$$

where $\Lambda_1^{C\pm} = [\,|H_1| \pm H_1\,]/[\,2|H_1|\,]$, for example. It can be shown for the operator defined by (2·2) that

$$\gamma_0^{(1)}\gamma_0^{(2)}(F_1F_2)_E\Lambda_E^C = \Lambda_E^C\gamma_0^{(1)}\gamma_0^{(2)}(F_1F_2) = [H_1+H_2-E]\,\delta(t-t'). \tag{2·4}$$

For the purpose of solving the Bethe-Salpeter equation we first consider the following eigenvalue problem:

$$[H_1+H_2+(e^2/r_{12})\Lambda - W]\chi(\boldsymbol{x}_1,\,\boldsymbol{x}_2) = 0, \quad (\chi,\,\Lambda\chi) = 1. \tag{2·5}$$

We represent the eigenfunction of the Bethe-Salpeter equation in terms of the solution of this equation and calculate the difference of the required eigenvalue from W by making use of relations which are satisfied by these eigenfunctions. Since the difference between W and the relativistic eigenvalue including only the Coulomb interaction can easily be calculated, most task is to calculate the above-mentioned difference. Although the operator in the equation (2·5) is not Hermitian, its eigenvalue is real for the reason as follows. If we consider a Hermitian product of $\Lambda\chi$ and the left side of the first equation of (2·5) we have

$$W = (\Lambda\chi,\,[H_1+H_2+(e^2/r_{12})\Lambda]\chi). \tag{2·6}$$

This equation shows that W is real because Λ is Hermitian and commutes with H_1+H_2. By means of (2·4) it can be shown that

$$\chi^*\delta(t)\Lambda_E^C\gamma_0^{(1)}\gamma_0^{(2)}[F_1F_2-I_C]_E = -\chi^*(E-W)\,\delta(t'), \tag{2·7}$$

$$[F_1F_2-I_C]_E\Lambda_E^C\delta(t')\chi = (W-E)\,\gamma_0^{(1)}\gamma_0^{(2)}\delta(t)\chi. \tag{2·8}$$

The last equation provides us means of writing the eigenfunction of the Bethe-Salpeter equation in terms of χ as follows:

$$\Psi(12) = \Psi_E(\boldsymbol{x}_1,\,\boldsymbol{x}_2,\,t)\,e^{-iET}, \tag{2·9a}$$

$$\Psi_E = \Lambda_E^C\delta(t')\chi + G_E I_E'\Lambda_E^C\delta(t')\chi + G_E\gamma_0^{(1)}\gamma_0^{(2)}(E-W)\,\delta(t')\chi, \tag{2·9b}$$

where E is the exact energy level which we will calculate and

$$G=[F_1F_2-I]^{-1}, \quad I'=I-I_C=I_{CC}+I_t+I_{tt}+I_{eL}+I_{nL}+I_v. \tag{2·10}$$

The relations (2·7) and (2·9) together with the normalization condition for χ give the following equation for the required energy difference with an accuracy of $e^{10}m$:

$$E-W=i\int\chi^*(x_1, x_2)[\Lambda^C\gamma_0^{(1)}\gamma_0^{(2)}(I'+I'GI')\Lambda^C]_{E0}\chi(x_1', x_2')\,d(x_1, x_2, x_1', x_2')+(\Delta E)_m, \tag{2·11}$$

where the arguments of d-sign indicate the integration variables, $[\ ]_{E0}$ means to put $t_1-t_2=t_1'-t_2'=0$ in $[\ ]_E$, and $(\Delta E)_m$ denotes the energy correction due to the vacuum polarization term in F_1 and F_2 of (1·1). In the present approximation we have only to consider the repetition of I_t for the second term in the bracket:

$$I'GI'=(4\pi ie^2)^2\gamma_t^{(1)}(\xi_1)\gamma_t^{(2)}(\xi_2)\,G_{12}^C\gamma_t^{(1)}(\xi_1')\gamma_t^{(2)}(\xi_2')\,D_t(\xi_1\xi_2)\,D_t(\xi_1'\xi_2'). \tag{2·12}$$

The interaction kernel given by (1·3) and (1·4) includes the effect of binding. This effect is appreciable only for processes containing virtual quanta with momenta smaller than binding energies ($\sim e^4m$) of electrons. In this case we can use the non-relativistic approximations for the Green functions. For processes involving virtual quanta with momenta larger than average momenta ($\sim e^2m$) of electrons, we can neglect the binding effect and decompose the Green functions into free propagators by means of (1·9). Thus we divide the region of momenta of virtual quanta into low-, intermediate- and high-frequency regions. For specifying these three regions, it is convenient to introduce two quantities, κ and K, such that $e^4m<\kappa<e^2m<K$. In the intermediate region, we may use either the non-relativistic approximations or the free propagators for the Green functions. It is to be noted that the specific values of κ and K are immaterial because the final result must be independent of them. In the respective frequency-regions the following method of approximations will be employed in order to calculate the required correction with an accuracy of $e^{10}m$.

In the high-frequency region most terms will be transformed into momentum integrals of the following type:

$$J=\int f(p_1, p_2, p_0)\,OF(p_1, p_2, p_0; p_1', p_2', p_0')\,\delta(p_1+p_2-p_1'-p_2')\,g(p_1', p_2', p_0')\,d(p_1, p_2, p_0, p_1', p_2', p_0'), \tag{2·13}$$

where O is an operator which is independent of integration variables, and F is a function which is nearly constant if the magnitudes of its independent variables are smaller than K. The functions, f and g, are the Fourier transforms of $\chi^*\Lambda_E^C$ and $\Lambda_E^C\chi$:

$$f(p_1, p_2, p_0)=\int[\chi^*\Lambda_E^C](x_1, x_2, t)\exp i(x_1p_1+x_2p_2-p_0t)\,d(x_1, x_2, t)/(2\pi)^{7/2}, \tag{2·14a}$$

$$g(\boldsymbol{p}_1, \boldsymbol{p}_2, p_0) = \int \exp -i(\boldsymbol{x}_1\boldsymbol{p}_1 + \boldsymbol{x}_2\boldsymbol{p}_2 - p_0 t)\,[\Lambda_E^C \chi](\boldsymbol{x}_1, \boldsymbol{x}_2, t)\, d(\boldsymbol{x}_1, \boldsymbol{x}_2, t)/(2\pi)^{7/2}, \tag{2·14b}$$

where $p_0 = (p_{01} - p_{02})/2$, χ is the solution of (2·5), Λ_E^C is defined by (2·1) and (2·2), and

$$[\chi^* \Lambda_E^C](\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \int \chi^*(\boldsymbol{x}_1', \boldsymbol{x}_2')\, \Lambda_E^C(\boldsymbol{x}_1', \boldsymbol{x}_2', 0;\ \boldsymbol{x}_1, \boldsymbol{x}_2, t)\, d(\boldsymbol{x}_1', \boldsymbol{x}_2'), \tag{2·15a}$$

$$[\Lambda_E^C \chi](\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \int \Lambda_E^C(\boldsymbol{x}_1, \boldsymbol{x}_2, t;\ \boldsymbol{x}_1', \boldsymbol{x}_2', 0)\, \chi(\boldsymbol{x}_1', \boldsymbol{x}_2')\, d(\boldsymbol{x}_1', \boldsymbol{x}_2'). \tag{2·15b}$$

The region in which the Fourier transforms, f and g, are of appreciable magnitudes is given by $|\boldsymbol{p}_1|$, $|\boldsymbol{p}_2|$, $|p_0| < K$ in view of the fact that the average momenta and binding energies of electrons are of orders $e^2 m$ and $e^4 m$ which are both smaller than K. Therefore main contribution to the value of J comes from the above-mentioned region in which F is nearly constant. Thus we can replace $F(\boldsymbol{p}_1, \boldsymbol{p}_2, p_0;\ \boldsymbol{p}_1', \boldsymbol{p}_2', p_0')$ with $F(0, 0, 0;\ 0, 0, 0)$. If this value of J is of order $e^{10} m$, the omitted part has no influence on our final result because our purpose is to calculate terms of orders up to $e^{10} m$. Then, by making use of (2·14) and the Fourier integral theorem, we have

$$\begin{aligned} J &= F(0, 0, 0;\ 0, 0, 0)\,(2\pi)^4 \int [\chi^* \Lambda_E^C](\boldsymbol{x}, \boldsymbol{x}, 0)\, \mathrm{O}\,[\Lambda_E^C \chi](\boldsymbol{x}, \boldsymbol{x}, 0)\, d(\boldsymbol{x}) \\ &= -(2\pi) F(0, 0, 0;\ 0, 0, 0)\, e^6 m^3 N \langle\!\langle \mathrm{O} \rangle\!\rangle, \end{aligned} \tag{2·16}$$

where

$$e^6 m^3 N = \int |\Lambda\chi(\boldsymbol{x}, \boldsymbol{x})|^2 d(\boldsymbol{x}), \tag{2·17a}$$

$$\langle\!\langle \mathrm{O} \rangle\!\rangle = \int \chi^*(\boldsymbol{x}, \boldsymbol{x})\, \Lambda \mathrm{O} \Lambda \chi(\boldsymbol{x}, \boldsymbol{x})\, d(\boldsymbol{x}) \Big/ \int |\Lambda\chi(\boldsymbol{x}, \boldsymbol{x})|^2 d(\boldsymbol{x}). \tag{2·17b}$$

The integrand means an inner product with respect to components of χ and it should be understood that $\chi^*(\boldsymbol{x}, \boldsymbol{x})\, \Lambda \mathrm{O} \Lambda \chi(\boldsymbol{x}, \boldsymbol{x}) = [\chi^*(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \Lambda \mathrm{O} \Lambda \chi(\boldsymbol{x}_1, \boldsymbol{x}_2)]_{\boldsymbol{x}_1 = \boldsymbol{x}_2}$. The value of $|\Lambda\chi|^2$ is distinct from zero in the domain whose volume is of order $e^{-1} m^{-3}$ (with respect to $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$). Therefore N is of order unity or less. In order to transform integrals into the form of (2·13) we often make use of the following relations which can be derived from the equation (2·5) :

$$\chi(\boldsymbol{x}_1, \boldsymbol{x}_2) = (W - \mathrm{H}_1 - \mathrm{H}_2)^{-1} (e^2/r_{12})\, \Lambda\chi(\boldsymbol{x}_1, \boldsymbol{x}_2), \tag{2·18a}$$

$$\begin{aligned} u(\boldsymbol{p}_1, \boldsymbol{p}_2) = [W - \mathrm{H}_1^0(\boldsymbol{p}_1) - \mathrm{H}_2^0(\boldsymbol{p}_2)]^{-1}\, 2\,(e/2\pi)^2 \int [&\Lambda u(\boldsymbol{p}_1 - \boldsymbol{q}, \boldsymbol{p}_2 + \boldsymbol{q}) \\ &- Zu(\boldsymbol{p}_1 - \boldsymbol{q}, \boldsymbol{p}_2) - Zu(\boldsymbol{p}_1, \boldsymbol{p}_2 - \boldsymbol{q})]\, d(\boldsymbol{q})/\boldsymbol{q}^2, \end{aligned} \tag{2·18b}$$

where $u(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is a momentum representation of $\chi(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

We next consider the following eigenvalue problem :

$$(\mathrm{H}_1 + \mathrm{H}_2 + \Lambda e^2/r_{12} - W_n)\, \varphi_n(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0, \quad (\varphi_n, \Lambda\varphi_n) = 1, \tag{2·19}$$

which is a little different from (2·5). If we multiply the first equation of (2·5) by Λ from the left and further replace $\Lambda\chi$ with φ we have an equation which is of the same form as (2·19). The normalization requirements for χ and φ_n are equivalent because Λ satisfies $\Lambda^3=\Lambda$. Therefore W_n's are real and one of them is equal to W. The corresponding φ is denoted by φ_W which is equal to $\Lambda\chi$. Thus N and $\langle\!\langle \mathrm{O} \rangle\!\rangle$ can be written in the form:

$$e^6 m^3 N = \int |\varphi_W(\boldsymbol{x}, \boldsymbol{x})|^2 d(\boldsymbol{x}), \tag{2·20a}$$

$$\langle\!\langle \mathrm{O} \rangle\!\rangle = \int \varphi_W^*(\boldsymbol{x}, \boldsymbol{x}) \mathrm{O} \varphi_W(\boldsymbol{x}, \boldsymbol{x}) d(\boldsymbol{x}) / \int |\varphi_W(\boldsymbol{x}, \boldsymbol{x})|^2 d(\boldsymbol{x}). \tag{2·20b}$$

In the case of low-frequency region we need a non-relativistic approximation of the Green function, G_{12}^C. We can easily show, from the definition (1·6c) and the relation (2·4), that the Green function satisfies

$$\begin{aligned}(i\partial/\partial T - \mathrm{H}_1 - \mathrm{H}_2 - \Lambda e^2/r_{12})\, \mathrm{G}_{12}^C(\boldsymbol{x}_1, \boldsymbol{x}_2\, 0, T;\ \boldsymbol{x}_1', \boldsymbol{x}_2', 0, T') \\ = -i\Lambda(\boldsymbol{x}_1, \boldsymbol{x}_2;\ \boldsymbol{x}_1', \boldsymbol{x}_2')\, \delta(T-T')\, \gamma_0^{(1)} \gamma_0^{(2)}.\end{aligned} \tag{2·21}$$

The contribution to G_{12}^C from negative-energy states is negligible in the low-frequency region. Therefore, we can write the solution of this equation in terms of φ_n as follows:

$$\begin{aligned}&\mathrm{G}_{12}^C(\boldsymbol{x}_1, \boldsymbol{x}_2, 0, T;\ \boldsymbol{x}_1', \boldsymbol{x}_2', 0, T')\gamma_0^{(1)}\gamma_0^{(2)} = 0 \qquad \text{for } T < T', \\ &= -\sum_{W_n>0} \varphi_n(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \varphi_n^*(\boldsymbol{x}_1', \boldsymbol{x}_2')\, e^{-iW_n(T-T')}\, \Lambda^2 \quad \text{for } T > T'.\end{aligned} \tag{2·22}$$

The eigenvalue problem of the relativistic Hamiltonian of the present system including the Coulomb interaction only is given by

$$(\mathrm{H}_1 + \mathrm{H}_2 + e^2/r_{12} - E_C)\, \phi = 0, \tag{2·23}$$

where E_C corresponds to E and W, and ϕ corresponds to φ_W and χ. The non-relativistic Hamiltonian of the system is given by

$$\mathrm{H}^{(nr)} = -(\Delta_1 + \Delta_2)/(2m) - Ze^2/r_1 - Ze^2/r_2 + e^2/r_{12}. \tag{2·24}$$

We denote the 8-component eigenfunctions of this Hamiltonian belonging to its eigenvalues E^0 and E_n^0 by $\psi_K(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $\psi_n(\boldsymbol{x}_1, \boldsymbol{x}_2)$:

$$|\mathrm{H}^{(nr)} - E^0|\psi_K = 0, \quad |\mathrm{H}^{(nr)} - E_n^0|\psi_n = 0, \tag{2·25}$$

where E^0 is a non-relativistic approximation of E_C and E, and ψ_K is that of ϕ, φ_W and χ.

The quantity, N, is given by (2·20a). If we replace φ_W in this equation with ψ_K, the error is of order e^2 at most as will be seen in the later section. N appears as a coefficient of quantities of order $e^{10}m$ at most. Therefore the error for the final result amounts to around $e^{12}m$ at most which is negligible in the present approximation. Thus we have

$$e^6 m^3 N = \int |\psi_E(\boldsymbol{x}, \boldsymbol{x})|^2 d(\boldsymbol{x}). \tag{2·26a}$$

Although N also appears in the $e^5 m$ correction it has a form of this integral from the outset. In the final stage of the calculation the same substitution can be done for the same reason. As will be seen in the later sections we have a term $\langle\!\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle\!\rangle$ in the final result. In the lower excited states of helium-like atoms, singlet-triplet mixing is very small. Therefore ψ_E is an eigenstate of $\sigma^{(1)} \cdot \sigma^{(2)}$ and we can replace $\langle\!\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle\!\rangle$ with the expectation value, $\langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle$, where the expectation value of any operator, O_p, is defined by

$$\langle O_p \rangle = \int \psi_E^*(\boldsymbol{x}_1, \boldsymbol{x}_2) O_p \psi_E(\boldsymbol{x}_1, \boldsymbol{x}_2) d(\boldsymbol{x}_1, \boldsymbol{x}_2), \tag{2·26b}$$

where the integrand means an inner product with respect to components of ψ_E.

Here we summarize the calculated result. The quantum-electrodynamical correction, $E - E_C$, can be written in terms of non-relativistic eigenfunctions in the following form:

$$E - E_C = \langle H^{(2)} + H^{(4)} \rangle. \tag{2·27}$$

$H^{(2)}$ and $H^{(4)}$ give the corrections of order $e^5 m$ and $e^{10} m$ respectively, and they correspond to the second and fourth order terms respectively. If we define the self spin-orbit coupling H_{so}^s, the mutual spin-orbit coupling H_{so}^m, and the spin-spin coupling H_{ss} by

$$H_{so}^s = 2\mu^2 Z (\mathbf{S}_1 \cdot \mathbf{L}_1 / r_1^3 + \mathbf{S}_2 \cdot \mathbf{L}_2 / r_2^3) - (2\mu^2 / r_{12}^3)(\mathbf{S}_1 \cdot \mathbf{L}_{12} + \mathbf{S}_2 \cdot \mathbf{L}_{21}), \tag{2·28a}$$

$$H_{so}^m = -(4\mu^2 / r_{12}^3)(\mathbf{S}_1 \cdot \mathbf{L}_{21} + \mathbf{S}_2 \cdot \mathbf{L}_{12}), \tag{2·28b}$$

$$H_{ss} = (4\mu^2 / r_{12}^3)[\mathbf{S}_1 \cdot \mathbf{S}_2 - (3/r_{12}^2)(\boldsymbol{x}_{12} \cdot \mathbf{S}_1)(\boldsymbol{x}_{12} \cdot \mathbf{S}_2)], \tag{2·28c}$$

we have

$$H^{(2)} = 2\mu^2 [r_{12}^{-1} \nabla_1 \cdot \nabla_2 + r_{12}^{-3} (\boldsymbol{x}_{12} (\boldsymbol{x}_{12} \cdot \nabla_1) \cdot \nabla_2)] + H_{so}^m + H_{ss} - (8\pi/3) e^8 m N \mathbf{S}_1 \cdot \mathbf{S}_2, \tag{2·29a}$$

$$H^{(4)} = (e^2/\pi)[H_{so}^s + (H_{so}^m/2) + H_{ss}] + e^{10} m N_0 Z [(76/45) - (8/3) \ln 2e^4] + e^{10} m N [(89/15) + (14/3) \ln e^2 - (20/3) \mathbf{S}_1 \cdot \mathbf{S}_2] - (14/3) e^{10} m Q - (2/3)(e^6/\pi) M, \tag{2·29b}$$

where μ is the Bohr magneton: $\mu = e/(2m)$. N is given by (2·26a). N_0, Q, and M are defined by

$$e^6 m^3 N_0 = \int |\psi_E(\boldsymbol{x}, 0)|^2 d(\boldsymbol{x}), \tag{2·30a}$$

$$Q = \lim_{a \to 0} [N \{\ln(e^2 ma) + C\} + \langle r_{12}^{-1}(a) \rangle / (4\pi e^6 m^3)], \tag{2·30b}$$

$$M = \sum_n (E_n^0 - E^0) |\boldsymbol{P}_{nE}|^2 \ln |(E_n^0 - E^0)/(e^4 m)|, \tag{2·30c}$$

where C is the Euler constant and

$$e^2 m \boldsymbol{P}_{nE} = (\psi_n, \boldsymbol{p}_1 \psi_E) + (\psi_n, \boldsymbol{p}_2 \psi_E) = -i \iint \psi_n^*(\boldsymbol{x}_1, \boldsymbol{x}_2)(\nabla_1 + \nabla_2) \psi_E(\boldsymbol{x}_1, \boldsymbol{x}_2) d(\boldsymbol{x}_1, \boldsymbol{x}_2), \tag{2·31a}$$

$$r_{12}^{-3}(a)=\begin{cases} r_{12}^{-3} & r_{12}>a \\ 0 & r_{12}<a. \end{cases} \tag{2·31b}$$

$\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are spin angular momenta. Various orbital angular momenta are defined by

$$\boldsymbol{L}_1=-i\boldsymbol{x}_1\times\boldsymbol{\Gamma}_1,\quad \boldsymbol{L}_2=-i\boldsymbol{x}_2\times\boldsymbol{\Gamma}_2,\quad \boldsymbol{x}_{12}=\boldsymbol{x}_1-\boldsymbol{x}_2,\quad \boldsymbol{L}_{12}=-i\boldsymbol{x}_{12}\times\boldsymbol{\Gamma}_1,\quad \boldsymbol{L}_{21}=i\boldsymbol{x}_{12}\times\boldsymbol{\Gamma}_2. \tag{2·32}$$

The terms containing N and N_0 mean the contact interactions between electrons and the nucleus respectively. These terms can be written in terms of $\delta(\boldsymbol{x}_{12})$, $\delta(\boldsymbol{x}_1)$ and $\delta(\boldsymbol{x}_2)$. We see that correction factors for spin-orbit and spin-spin couplings are in agreement with those derived by G. Araki[9] and that the second order correction, $\mathrm{H}^{(2)}$, involves a contact interaction term which gave rise to discussions by several authors.[14) 15) 16)] This is spin-dependent. We see that the fourth order correction involves a term of the same type which cannot be obtained by considering the anomalous magnetic moment of electrons[9] only as in the case of H_{so}^{m} and H_{ss}. Further $\mathrm{H}^{(2)}$ does not contain the Breit surplus term. This will be discussed, together with the contact interaction term in $\mathrm{H}^{(2)}$, in the fourth section.

§ 3. Corrections due to exchange of a photon

In the succeeding five sections we shall briefly explain the method of calculating the various energy corrections. The difference, $E-W$, is divided into $(\Delta E)_t$, $(\Delta E)_{tt}$, $(\Delta E)_{eL}$, $(\Delta E)_{nL}$, $(\Delta E)_{CC}$, $(\Delta E)_{ev}$, and $(\Delta E)_{nv}$ which come from I_t, $\mathrm{I}_{tt}+\mathrm{I}'\mathrm{GI}'$, I_{eL}, I_{nL}, I_{CC}, I_{ev} and $A_{0\,\mathrm{vac\,pol}}$ respectively. If we denote $W-E_C$ by $(\Delta E)_C$ we have

$$E-E_C=(\Delta E)_t+(\Delta E)_{tt}+(\Delta E)_{eL}+(\Delta E)_{nL}+(\Delta E)_{ev}+(\Delta E)_{nv}+(\Delta E)_C+(\Delta E)_{CC},$$

where E_t is the relativistic Coulomb energy defined by (2·23). In this section we shall calculate $(\Delta E)_t$ which means the level shift due to an exchange of a photon between two electrons. This correction consists of two parts which are proportional to e^8m and $e^{10}m$ respectively. Contributions form high-, intermediate-, and low-frequency regions will be calculated separately.

(a) *Contribution from high-frequency region,* $(\Delta E)_t^h$

We decompose the Green functions into free propagators:

$$\Lambda^C\gamma_0^{(1)}\gamma_0^{(2)}\mathrm{I}_t\Lambda^C=\mathrm{H}_t+\mathrm{H}_{Ct}, \tag{3·1}$$

where

$$\mathrm{H}_t=-4\pi ie^2(\mathrm{G}_1{}^0\gamma_0^{(1)}+\mathrm{G}_2{}^0\gamma_0^{(2)})\alpha_t^{(1)}\alpha_t^{(2)}(\mathrm{G}_1{}^0\gamma_0^{(1)}+\mathrm{G}_2{}^0\gamma_0^{(2)})D_t, \tag{3·2a}$$

$$\begin{aligned}\mathrm{H}_{Ct}=&-(4\pi ie^2)^2\Lambda^C[\alpha_t^{(1)}\mathrm{G}_1{}^0\gamma_0^{(1)}\mathrm{G}_2{}^0\gamma_0^{(2)}\alpha_t^{(2)}+(1\rightleftarrows 2)]\Lambda^C D_t D_C\\ &-4\pi iZe^4[\mathrm{G}_1{}^0\gamma_0^{(1)}r_1^{-1}\mathrm{G}_1{}^0\gamma_0^{(1)}+(1\rightleftarrows 2)]\alpha_t^{(1)}\alpha_t^{(2)}\Lambda^C D_t\\ &-4\pi iZe^4\Lambda^C\alpha_t^{(1)}\alpha_t^{(2)}[\mathrm{G}_1{}^0\gamma_0^{(1)}r_1^{-1}\mathrm{G}_1{}^0\gamma_0^{(1)}+(1\rightleftarrows 2)]D_t,\end{aligned} \tag{3·2b}$$

and α_t and D_t have the same meaning as γ_t and D_t in (1·8b). The last two lines of

H_{cr} give no contribution to $(\Delta E)_t$ in this region because r_1^{-1} or r_2^{-1} changes only a momentum of an electron. The correction due to the first line of H_{cr} can be transformed into the following form by means of (2·16) :

$$(\Delta E)_{cr}^{h} = -e^{10}m^3 N \int_{>K} d(\boldsymbol{k}) \int i dk_0 \sum \langle\!\langle \quad \rangle\!\rangle / [(\pi \boldsymbol{k})^2 (\boldsymbol{k}^2 - k_0^2)], \tag{3·3a}$$

where $>K$ means that the integration domain is confined to $|\boldsymbol{k}| > K$, and

$$\sum \langle\!\langle \quad \rangle\!\rangle = \sum_{j,l=1}^{3} \langle\!\langle \alpha_j^{(1)} [H_1^0(\boldsymbol{k}) - m + k_0]^{-1} [H_2^0(\boldsymbol{k}) - m + k_0]^{-1} \alpha_l^{(2)} + (1 \rightleftarrows 2) \rangle\!\rangle (\delta_{jl} - k_j k_l / k^2). \tag{3·3b}$$

The integration can readily be carried out. The result is given by

$$(\Delta E)_{cr}^{h} = (4/3) e^{10} m N \langle \sigma^{(1)} \cdot \sigma^{(2)} \rangle [1 + \ln(m/2K)]. \tag{3·4}$$

In order to evaluate the correction due to H_t we introduce the following integral :

$$w(a, b) = -[e^2/(2\pi^2)] \int_{(a,b)} B(\boldsymbol{k}) / k^2 d(\boldsymbol{k}), \tag{3·5a}$$

where (a, b) denotes the integration domain $a < |\boldsymbol{k}| < b$, and $B(\boldsymbol{k})$ is given by

$$B(\boldsymbol{k}) = \int \chi^*(\boldsymbol{x}_1, \boldsymbol{x}_2) [\alpha^{(1)} \cdot \alpha^{(2)} - (\boldsymbol{k} \cdot \boldsymbol{\alpha}^{(1)})(\boldsymbol{k} \cdot \alpha^{(2)}) / k^2] \chi(\boldsymbol{x}_1, \boldsymbol{x}_2) \exp i\boldsymbol{k} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2) d(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{3·5b}$$

The calculation of this integral will be explained in the next section. If we carry out all energy-integrations the correction due to H_t turns out

$$\begin{aligned}(\Delta E)_t^{h} - (\Delta E)_{cr}^{h} &= -e^2/(2\pi)^2 \int_{>K} d(\boldsymbol{k}) / |\boldsymbol{k}| \int u^*(\boldsymbol{p}_1, \boldsymbol{p}_2) \sum [\quad] u(\boldsymbol{p}_1', \boldsymbol{p}_2') \delta(\boldsymbol{p}_1 - \boldsymbol{k} - \boldsymbol{p}_1') \\ &\quad \times \delta(\boldsymbol{p}_2 + \boldsymbol{k} - \boldsymbol{p}_2') d(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2'), \end{aligned} \tag{3·6a}$$

$$\sum [\quad] = \sum_{i,j=1}^{3} \left[\frac{\Lambda_1^-(\boldsymbol{p}_1) \alpha_i^{(1)} \alpha_j^{(2)} \Lambda_2^+(\boldsymbol{p}_2')}{\varepsilon(\boldsymbol{p}_1) + \varepsilon(\boldsymbol{p}_2') + |\boldsymbol{k}| - 2m} - \frac{\Lambda_1^+(\boldsymbol{p}_1) \alpha_i^{(1)} \alpha_j^{(2)} \Lambda_1^-(\boldsymbol{p}_1')}{\varepsilon(\boldsymbol{p}_1) + \varepsilon(\boldsymbol{p}_1') + |\boldsymbol{k}|} + (+ \rightleftarrows -) + (1 \rightleftarrows 2) \right] \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2} \right), \tag{3·6b}$$

where $u(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is the momentum representation of χ,

$$\Lambda_1^{\pm}(\boldsymbol{p}) = [|H_1^0(\boldsymbol{p})| \pm H_1^0(\boldsymbol{p})] / |2H_1^0(\boldsymbol{p})|, \quad \varepsilon(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}, \tag{3·7}$$

and $(+ \rightleftarrows -)$ means the terms which are obtained by exchanging Λ^+ and Λ^- and further changing the sign of m of the denominator in the preceding terms.

We first separate $w(K, \infty)$ from the correction. As for the calculation of the remaining part we make use of the relation (2·18b), which is satisfied by $u(\boldsymbol{p}_1, \boldsymbol{p}_2)$,

and then (2·16). Further by means of the relations such as

$$\sum_{j,l=1}^{3} 4\lfloor \mathcal{E}(\boldsymbol{k}) \rceil^2 \langle\!\langle \alpha_s^{(1)} \alpha_t^{(2)} \Lambda_1^+(\boldsymbol{k}) \Lambda_2^+(-\boldsymbol{k}) \rangle\!\rangle (\delta_{jl} - k_j k_l/\boldsymbol{k}^2) = \langle\!\langle \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \boldsymbol{k}^2 - (\boldsymbol{k} \cdot \boldsymbol{\sigma}^{(1)})(\boldsymbol{k} \cdot \boldsymbol{\sigma}^{(2)}) \rangle\!\rangle, \tag{3·8}$$

we have

$$(JE)_t^h - (JE)_{Ct}^h = w(K, \infty) - (4/3) e^{10} mN \langle \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \rangle [2 + \ln(m/2K)], \tag{3·9}$$

where we omit terms proportional to Z in (2·18b) because they change only a momentum of an electron. The logarithmic term completely cancels that of (3·4), and we have

$$(JE)_t^h = w(K, \infty) - (4/3) e^{10} mN \langle \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \rangle. \tag{3·10}$$

(b) *Contribution from low-frequency region,* $(JE)_t'$

In this region the energy correction due to I, comes from the following terms only:

$$4\pi e^2 \int \chi^*(\boldsymbol{x}_1, \boldsymbol{x}_2) \lfloor \alpha_t^{(1)}(\xi_1) G_{12}^C \gamma_\nu^{(1)} \gamma_0^{(2)} \alpha_\nu^{(2)}(\xi_2') D_c(\xi_1 \xi_2') + (1 \rightleftarrows 2) \rceil_{x_0} \chi(\boldsymbol{x}_1', \boldsymbol{x}_2') \, d(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_1', \boldsymbol{x}_2'). \tag{3·11}$$

The terms, which give rise to pair processes or involve projections to negative energy states, give no significant correction. If we make use of (2·22) we have

$$\begin{aligned}(JE)_t' &= \frac{e^2}{2\pi^2} \lim_{\varepsilon \to 0} \int_{<\kappa} \frac{d(\boldsymbol{k})}{|\boldsymbol{k}|} \sum_{i,j=1}^{3} \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right) \sum_n \frac{\mathrm{Re}\lfloor \alpha_i^{(1)} \rceil_n \lfloor \alpha_j^{(2)} \rceil_n^*}{E - W_n - |\boldsymbol{k}| + i\varepsilon} \\ &= w(0, \kappa) + \frac{e^2}{2\pi^2} \lim_{\varepsilon \to 0} \int \frac{d(\boldsymbol{k})}{\boldsymbol{k}^2} \sum_{i,j=1}^{3} \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right) \sum_n \frac{(E - W_n) \mathrm{Re}\lfloor \alpha_i^{(1)} \rceil_n \lfloor \alpha_j^{(2)} \rceil_n^*}{E - W_n - |\boldsymbol{k}| + i\varepsilon},\end{aligned} \tag{3·12}$$

where $<\kappa$ means that the integration domain is confind to $|\boldsymbol{k}| < \kappa$, Re means 'the real part of', and

$$\begin{aligned}\lfloor \alpha_j^{(1)} \rceil_n &= \int \chi^*(\boldsymbol{x}_1, \boldsymbol{x}_2) \alpha_j^{(1)} \varphi_n(\boldsymbol{x}_1, \boldsymbol{x}_2) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}_1) \, d(\boldsymbol{x}_1, \boldsymbol{x}_2), \\ \lfloor \alpha_j^{(2)} \rceil_n &= \int \chi^*(\boldsymbol{x}_1, \boldsymbol{x}_2) \alpha_j^{(2)} \varphi_n(\boldsymbol{x}_1, \boldsymbol{x}_2) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}_2) \, d(\boldsymbol{x}_1, \boldsymbol{x}_2).\end{aligned} \tag{3·13}$$

In the low-frequency region the exponential factor in the integrand of (3·13) can approximately be assumed as unity because main contribution to the integral comes from the domain in which $|\boldsymbol{k} \cdot \boldsymbol{x}|$ is small compared with unity. For low energy, χ and φ_n can be expressed in terms of non-relativistic eigenfunctions as follows:

$$\begin{aligned}\varphi_n &= [1 + \boldsymbol{\alpha}^{(1)} \cdot \boldsymbol{p}_1/(2m)][1 + \boldsymbol{\alpha}^{(2)} \cdot \boldsymbol{p}_2/(2m)] \psi_n, \\ \chi &= [1 + \boldsymbol{\alpha}^{(1)} \cdot \boldsymbol{p}_1/(2m)][1 + \boldsymbol{\alpha}^{(2)} \cdot \boldsymbol{p}_2/(2m)] \psi_k.\end{aligned} \tag{3·14}$$

If we insert this in (3·12) we have

$$(\Delta E)_t^{I} = \{4e^2/(3\pi m^2)\} \sum_n (E - W_n) \operatorname{Re}(\boldsymbol{p}_{En}^{(1)} \boldsymbol{p}_{nE}^{(2)}) \ln|(E - W_n)/\kappa| + w(0, \kappa) + \text{imaginary terms}, \tag{3·15}$$

where

$$\boldsymbol{p}_{nE}^{(1)} = (\psi_E, \boldsymbol{p}_1 \psi_n), \quad \boldsymbol{p}_{nE}^{(2)} = (\psi_n, \boldsymbol{p}_2 \psi_E). \tag{3·16}$$

The meaning of the imaginary part will be discussed in the last section. The difference between E and E^0 or W_n and E_n^0 is at most e^4 times small as compared with themselves. If we neglect such a difference the error amounts to a quantity of order $e^{14}m$ because $(\Delta E)_t^{I}$ itself is of order $e^{10}m$. After replacing E and W_n with E^0 and E_n^0 respectively we have by means of the Parseval theorem,

$$2\sum_n (W_n - E) \operatorname{Re}(\boldsymbol{p}_{En}^{(1)} \boldsymbol{p}_{nE}^{(2)}) = 2\operatorname{Re}([\mathrm{H}^{(nr)} \boldsymbol{p}_1] \psi_E, \boldsymbol{p}_2 \psi_E) = 4\pi e^8 m^3 N, \tag{3·17}$$

where the bracket denotes a commutator. In this way (3·15) turns out

$$(\Delta E)_t^{I} = w(0, \kappa) - (8/3) e^{10} m N \ln(e^4 m/\kappa) - (2/3\pi) e^6 M' + \text{imaginary terms}, \tag{3·18a}$$

where

$$e^4 m^2 M' = 2\sum_n (E_n^0 - E^0) \operatorname{Re}(\boldsymbol{p}_{En}^{(1)} \boldsymbol{p}_{nE}^{(2)}) \ln|(E_n^0 - E^0)/(e^4 m)|. \tag{3·18b}$$

(c) *Contribution from intermediate-frequency region,* $(\Delta E)_t^{m}$

For the purpose of calculating the contribution from the intermediate-frequency region we can still employ the integrand of (3·12). In this case we should retain the exponential factor in the integrand of (3·13), while we can neglect $E - W_n$ in the denominator as compared with $|\boldsymbol{k}|$:

$$(\Delta E)_t^{m} = w(\kappa, K) + \frac{e^2}{2\pi^2} \int_{(\kappa, K)} \frac{d(\boldsymbol{k})}{|\boldsymbol{k}|^3} \sum_{j,l=1}^{3} \left(\delta_{jl} - \frac{k_j k_l}{\boldsymbol{k}^2}\right) \sum_n (W_n - E) \operatorname{Re}[\alpha_j^{(1)}]_n [\alpha_l^{(2)}]_n{}^*. \tag{3·19}$$

The series in the integrand can be summed as in the previous case:

$$m^2 \sum_n (W_n - E) \operatorname{Re}[\alpha_j^{(1)}]_n [\alpha_l^{(2)}]_n{}^* + (\boldsymbol{k} \to -\boldsymbol{k}) = e^2 \langle \cos(\boldsymbol{k} \cdot \boldsymbol{x}_{12}) (\boldsymbol{\nabla}_{1j} \cdot \boldsymbol{\nabla}_{2l} r_{12}^{-1}) \rangle + (4\pi/3) \delta_{jl} e^8 m^3 N, \tag{3·20}$$

where $(\boldsymbol{k} \to -\boldsymbol{k})$ means the term which is obtained from the preceding term by replacing $\boldsymbol{k}$ with $-\boldsymbol{k}$. The integration with respect to $\boldsymbol{k}$ gives

$$(\Delta E)_t^{m} = w(\kappa, K) + (8/3) e^{10} m N \ln(K/\kappa) + (e^4/m^2) \langle f(r_{12}) \rangle / \pi, \tag{3·21}$$

where

$$f(r) = \frac{2}{r^3} \left[\frac{\sin Kr}{(Kr)^3} - \frac{\cos Kr}{(Kr)^2} - \frac{\sin \kappa r}{(\kappa r)^3} + \frac{\cos \kappa r}{(\kappa r)^2} \right]. \tag{3·22}$$

For the calculation of the expectation of $f(r_{12})$, κr and Kr can be considered as very

small and very large respectively, and $f(r)$ is nearly equal to $-(2/3)r^{-3}$. Further we can make a following device. First we calculate the following difference:

$$\langle f\rangle - 4\pi e^6 m^3 N\int_0^\infty f(r)\,e^{-r}r^2\,dr$$

$$= -(2/3)\lim_{a\to 0}\left[\langle r_{12}^{-3}(a)\rangle + 4\pi e^6 m^3 N(\ln a + C)\right], \tag{3·23a}$$

where $r_{12}^{-3}(a)$ is given by (2·31b) and C is the Euler constant. The integral in the first line can be calculated in the following way:

$$4\pi\int_0^\infty f(r)\,e^{-r}r^2\,dr = 8\pi\int_0^{1/\varkappa} g(r)\,dr + (8\pi/3)\ln(\kappa/K) = (32\pi/9) - (8\pi/3)\ln K, \tag{3·23b}$$

where

$$g(r) = \int_0^\infty (\sin t - t\cos t)\,t^{-3}e^{-rt}\,dt = -\left[\{1 - r\,\mathrm{Tan}^{-1}(1/r)\}r - \mathrm{Tan}^{-1}(1/r)\right]/2. \tag{3·23c}$$

From (3·23) we have

$$\langle f(r_{12})\rangle/\pi = (8/3)e^6 m^3\left[(4/3)N + N\ln(e^2 m/K) - Q\right], \tag{3·24}$$

whence

$$(\Delta E)_t^{(m)} = w(\kappa, K) + (8/3)e^{10}m\left[(4/3)N + N\ln(e^2 m/\kappa) - Q\right], \tag{3·25}$$

where Q is defined by (2·30b).

The energy shift due to an exchange of a photon is obtained by adding (3·10), (3·18) and (3·25):

$$(\Delta E)_t = w(0,\infty) + (4/3)e^{10}mN\{(8/3) - \ln e^4 - \langle\sigma^{(1)}\cdot\sigma^{(2)}\rangle\} - (8/3)e^{10}mQ$$
$$- (2/3\pi)e^6 M' + \text{imaginary terms}. \tag{3·26}$$

§ 4. Breit correction

In this section we shall examine $w(0,\infty)$ where $w(a,b)$ is given by (3·5). If we carry out the k-integration we have

$$w(0,\infty) = \int \chi^*(x_1, x_2)\,H_B\,\chi(x_1, x_2)\,d(x_1, x_2), \tag{4·1a}$$

where

$$H_B = -\left[\alpha^{(1)}\cdot\alpha^{(2)} + (\alpha^{(1)}\cdot x_{12})(\alpha^{(2)}\cdot x_{12})/r_{12}^2\right]e^2/(2r_{12}). \tag{4·1b}$$

We shall refer to this integral as the Breit correction although this is not exactly identical with an expectation value of the Breit interaction. As we shall see, this is the $e^8 m$ part

of the energy correction in the present calculation. In order to write $w(0, \infty)$ in terms of the non-relativistic eigenfunction with an accuracy $e^{10}m$, we first derive approximate relations for χ from the exact equation (2·18) as follows:

$$(1+\beta^{(1)})\,(1+\beta^{(2)})\,\chi/4=[1-(\boldsymbol{p}_1^2+\boldsymbol{p}_2^2)/(8m^2)]\,\psi', \tag{4·2a}$$

$$(1+\beta^{(1)})\,(1-\beta^{(2)})\,\chi/4=[\sigma^{(2)}\cdot\boldsymbol{p}_2/(2m)]\,\psi', \tag{4·2b}$$

$$(1-\beta^{(1)})\,(1+\beta^{(2)})\,\chi/4=[\sigma^{(1)}\cdot\boldsymbol{p}_1/(2m)]\,\psi', \tag{4·2c}$$

$$(1-\beta^{(1)})\,(1-\beta^{(2)})\,\chi/4=[(\sigma^{(1)}\cdot\boldsymbol{p}_1)\,(\sigma^{(2)}\cdot\boldsymbol{p}_2)/(2m)^2]\,\psi', \tag{4·2d}$$

$$\psi'=[E^0-(\boldsymbol{p}_1^2+\boldsymbol{p}_2^2)/(2m)]^{-1}[(1+\beta^{(1)})\,(1+\beta^{(2)})/4][(e^2/r_{12})\Lambda-(Ze^2/r_1)-(Ze^2/r_2)]\,\chi. \tag{4·2e}$$

Speaking in terms of a momentum representation, neglected terms in these equations are e^4 times smaller than retained ones for small and intermediate momenta of electrons and e^2 times for large momenta if $|\boldsymbol{p}_1|-|\boldsymbol{p}_2|<K$. If $|\boldsymbol{p}_1|-|\boldsymbol{p}_2|$ is greater than K, we should add $(\boldsymbol{p}_1^2-\boldsymbol{p}_2^2)^2/(16m^3)$ to the denominator of the last equation, but this case need not be considered in the present calculation. The second term in the bracket of (4·2a) can be neglected for small and intermediate momenta. This is retained, however, because it can not be neglected for large momenta.

For small and intermediate momenta the difference between ψ_E and $(1+\beta^{(1)})\,(1+\beta^{(2)})\,\chi/4$ or $(1+\beta^{(1)})\,(1+\beta^{(2)})\,\Lambda\chi/4$ is e^4 times smaller than ψ_E itself as is seen from the equations which they satisfy. In this case we can neglect the difference, and the equations (2·25) and (4·2e) show that $\psi'=\psi_E$. In the right-hand side of (4·2e), main contribution from χ to ψ' comes from the intermediate-momentum part of χ in any case. Therefore $\psi'=\psi_E$ is valid for any momentum. Making use of (4·2) we have

$$4m\chi^*\alpha_j^{(1)}\alpha_i^{(2)}\chi=\psi_E^*\sigma_j^{(1)}(\sigma^{(1)}\cdot\mathbf{p}_1)\,\sigma_i^{(2)}(\sigma^{(2)}\cdot\mathbf{p}_2)\,\psi_E+(\sigma^{(1)}\cdot\mathbf{p}_1\psi_E)^*\sigma_j^{(1)}\sigma_i^{(2)}(\sigma^{(2)}\cdot\mathbf{p}_2)\,\psi_E$$
$$+\text{complex conjugate}, \tag{4·3}$$

where $\mathbf{p}_1$ in the second term operates on the first ψ_E only. The second term of the right side of (4·2a) gives no significant contribution to (4·3) for the reason that, in this calculation, $(1+\beta^{(1)})\,(1+\beta^{(2)})\,\chi/4$ is accompanied with a factor $(1-\beta^{(1)})\,(1-\beta^{(2)})\,\chi/4$. If we introduce the relation (4·3) into (4·1) we have

$$w(0, \infty)=\langle \mathrm{H}^{(2)}\rangle, \tag{4·4}$$

where $\mathrm{H}^{(2)}$ is given by (2·29a).

For the derivation of the last result we might adopt another procedure. We might express small components of χ in terms of its large component neglecting the Coulomb terms, then replace the latter with ψ_E, and finally derive (4·4). Although such a procedure has conventionally been employed for calculating corrections of order e^8m, this is not free from doubt when an accuracy of $e^{10}m$ is required. This procedure is certainly valid for evaluating contributions from the regions of small and intermediate momenta, but may not be for the evaluation of that from the region of large momenta, because

$1/r$ can not be considered as small in the latter case. In fact (4·2a) shows that the behavior of $(1+i\hat{\beta}^{(1)})(1+i\hat{\beta}^{(2)})\chi/4$ is quite different from that of ψ_E and the former can not be replaced with the latter in the case of large momenta. The first mentioned method is free from such a deficiency, and we can obtain (4·4) on the firm basis.

As for the derivation of the contact interaction term there were some confusions[14),15),16)] and it may be necessary to add a note here. This term comes from the following integral in the present calculation:

$$-\int r_{12}^{-1}\left[\{(\sigma^{(1)}\times\nabla_1)\cdot(\sigma^{(2)}\times\nabla_2)\psi_E\}^*\psi_E+\psi_E^*(\sigma^{(1)}\times\nabla_1)(\sigma^{(2)}\times\nabla_2)\psi_E\right.$$
$$\left.+2(\sigma^{(1)}\times\nabla_1\psi_E)^*\cdot(\sigma^{(2)}\times\nabla_2\psi_E)\right]d(\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{4·5}$$

where ∇_1 and ∇_2 in the first term of the integrand operate on ψ_E in the brace only, and ∇_1 in the last term operates on the first ψ_E only. The integrand of (4·5) can be transformed into

$$\nabla_1\cdot\left[r_{12}^{-1}\{\sigma^{(1)}\times(\sigma^{(2)}\times\nabla_2\psi_E)\}^*\psi_E+r_{12}^{-1}\psi_E^*\sigma^{(1)}\times(\sigma^{(2)}\times\nabla_2\psi_E)\right]$$
$$+\nabla_2\cdot\left[r_{12}^{-3}\psi_E^*\{\sigma^{(1)}(\sigma^{(2)}\cdot\boldsymbol{x}_{12})-\boldsymbol{x}_{12}(\sigma^{(1)}\cdot\sigma^{(2)})\}\psi_E\right]$$
$$+\psi_E^*r_{12}^{-3}\left[\sigma^{(1)}\cdot\sigma^{(2)}-3(\sigma^{(1)}\cdot\boldsymbol{x}_{12})(\sigma^{(2)}\cdot\boldsymbol{x}_{12})/r_{12}^2\right]\psi_E, \tag{4·6}$$

for $r_{12}\neq 0$. The integrand of (4·5) has a singular point at $r_{12}=0$. Therefore the integral means a limit of an integral, over a domain excluding the vicinity of the singular point, as the vicinity uniformly vanishes. If we apply the Gauss theorem for such integrations the first and second lines of (4·6) turn out surface integrals over the surface of the restricted domain. The contribution from the first line vanishes and the second line gives rise to the contact interaction term in question. From the third line of (4·6) we get the spin-spin coupling, $\langle H_{ss}\rangle$. (As for point at infinity a similar procedure should also be applied. In this case both the surface integrals vanish). In this connection Glover and Chraplyvy[16)] made a mistake. They thought that $-(\sigma^{(1)}\times\nabla_1)\cdot(\sigma^{(2)}\times\nabla_2)r_{12}^{-1}$ contains the contact interaction, applying the theorem to an integral over a domain including a singular point of its integrand, but the theorem is not valid for such a domain, as is well known. $-(\sigma^{(1)}\times\nabla_1)\cdot(\sigma^{(2)}\times\nabla_2)r_{12}^{-1}$ contains no contact interaction, and it gives rise to the third line of (4·6) only.

It has been known for energy levels of a two-electron system that the Breit interaction gives rise to a second-order perturbation term of order e^2m which is of the same order as first-order perturbation terms of the interaction.[13),17),19)] It was shown by Breit[17)] that this second-order term can not be reconcilable with the observed intervals of He $2\,^3P$. Of course the reason why the incorrect term appears lies in an incorrect method of derivation.[13),16),18)] In the correct method of the present calculation such an incorrect term does not appear, although there is a kind of a second-order term of an interaction transmitted by a single photon, namely I_2GI_1 in (2·12). The contribution from this term to the energy correction is, however, of order $e^{10}m$ but not of order e^2m as will be shown in the next section. The reason for the result is that small factors arise not only

from the coupling constant but also from the electron and photon Green functions through pair processes. The effect of these Green functions is similar to that of energy denominators in the conventional perturbation theory which were discussed by Araki and Huzinaga.[13]

§ 5. Corrections due to exchange of two photons

In this section we shall calculate $(\Delta E)_{II}$ which comes from the transmission of two photons between two electrons, namely from I_{II} and I_tGI_t. The low-frequency region gives no contribution to this correction. According to the method of (2·16) we have the contribution from the high-frequency region as follows:

$$(\Delta E)_{II}^{h}=e^{10}m^3N(i/\pi^2)\int_{>K}\sum_{ijln}^{3}(\delta_{ij}-k_ik_j/k^2)\,(\delta_{ln}-k_lk_n/k^2)\,Y_{ijln}(\boldsymbol{k})\,d(k)\,, \tag{5·1a}$$

where

$$Y_{ijln}(\boldsymbol{k})=\int\langle\gamma_i^{(1)}\gamma_n^{(2)}[H_1^{0}(\boldsymbol{k})-(m+k_0)]^{-1}\gamma_0^{(1)}[H_2^{0}(\boldsymbol{k})-(m+k_0)]^{-1}\gamma_0^{(2)}\gamma_l^{(1)}\gamma_j^{(2)} + (j\rightleftarrows n)\rangle\,dk_0/(\boldsymbol{k}^2-k_0^2)^2. \tag{5·1b}$$

If we make use of the relations

$$\langle\gamma_i^{(1)}[H_1^{0}(\boldsymbol{k})+m+k_0]\gamma_0^{(1)}\gamma_l^{(1)}\rangle=-\langle k_0\gamma_i^{(1)}\gamma_l^{(1)}\rangle=\langle k_0\alpha_i^{(1)}\alpha_l^{(1)}\rangle, \tag{5·2a}$$

$$\sum_{ijln}(\delta_{ij}-k_ik_j/\boldsymbol{k}^2)\,(\delta_{ln}-k_lk_n/\boldsymbol{k}^2)\,\langle\alpha_i^{(1)}\alpha_l^{(1)}\alpha_n^{(2)}\alpha_j^{(2)}\rangle=2\langle 1+(\sigma^{(1)}\cdot\boldsymbol{k})\,(\sigma^{(2)}\cdot\boldsymbol{k})/\boldsymbol{k}^2\rangle, \tag{5·2b}$$

$$\sum_{ijln}(\delta_{ij}-k_ik_j/\boldsymbol{k}^2)\,(\delta_{ln}-k_lk_n/\boldsymbol{k}^2)\,\langle\alpha_i^{(1)}\alpha_l^{(1)}\alpha_j^{(2)}\alpha_n^{(2)}\rangle=2\langle 1-(\sigma^{(1)}\cdot\boldsymbol{k})\,(\sigma^{(2)}\cdot\boldsymbol{k})/\boldsymbol{k}^2\rangle, \tag{5·2c}$$

we obtain

$$(\Delta E)_{II}^{h}=e^{10}mN[1+2\ln(2K/m)+(1/3)\langle\sigma^{(1)}\cdot\sigma^{(2)}\rangle]. \tag{5·3}$$

The contribution from the intermediate-frequency region arises only from the two-pair process. Since the time dependence of Λ_E^C can be neglected in this region, we can replace Λ_E^C with $i\Lambda$. Thus, integrating over $\boldsymbol{k}$, we have

$$(\Delta E)_{II}^{m}=-\frac{e^4}{32\pi^4m^2}\int_{<K}d(k)\int\frac{d(q)}{\boldsymbol{k}^2|\boldsymbol{q}|+\boldsymbol{q}^2|\boldsymbol{k}|}\sum_{ijln}^{3}\left(\delta_{ij}-\frac{k_ik_j}{\boldsymbol{k}^2}\right)\left(\delta_{ln}-\frac{q_lq_n}{\boldsymbol{q}^2}\right)U_{ijln}, \tag{5·4a}$$

where U_{ijln} is given, in terms of the momentum representation of χ, by

$$U_{ijln}=\int u^*(\boldsymbol{p}_1,\boldsymbol{p}_2)\,\Lambda\alpha_i^{(1)}\alpha_l^{(1)}\,(\alpha_n^{(2)}\alpha_j^{(2)}+\alpha_j^{(2)}\alpha_n^{(2)})\,\Lambda u(\boldsymbol{p}_1',\boldsymbol{p}_2')\,\delta(\boldsymbol{p}_1'+\boldsymbol{p}_2'-\boldsymbol{p}_1-\boldsymbol{p}_2) \times\delta(\boldsymbol{p}_1'+\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{p}_1)\,d(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_1',\boldsymbol{p}_2'). \tag{5·4b}$$

The low-frequency region is included in the k-integration of (5·4a) because it gives no

significant contribution. If we come back to the configuration-space representation of χ the integral can be written in the form

$$(JE)_{tt}^{m} = -\{e^4/(2\pi m^2)\}\int \varphi_W^*(\boldsymbol{x}_1, \boldsymbol{x}_2) F(K, r_{12}) \varphi_W(\boldsymbol{x}_1, \boldsymbol{x}_2) d(\boldsymbol{x}_1, \boldsymbol{x}_2), \qquad (5\cdot 5a)$$

where

$$F(K, r_{12}) = \int_{<K} \frac{d(\boldsymbol{k})}{(2\pi)^3} \int d(\boldsymbol{q}) \quad \frac{1+(\boldsymbol{k}\cdot\boldsymbol{q})^2/(\boldsymbol{k}^2\boldsymbol{q}^2)}{\boldsymbol{k}^2|\boldsymbol{q}| + \boldsymbol{q}^2|\boldsymbol{k}|} \exp i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x}_{12}. \qquad (5\cdot 5b)$$

Since the integral is invariant for the space rotation, $F(K, r_{12})$ depends on r_{12} only besides K. The value of the integral, (5·5a), is for the most part given by the contribution from the region in which $Kr_{12} > 1$. In this region $F(K, r_{12})$ is nearly proportional to r_{12}^{-3}. Its coefficient can be obtained by integrating $F(k, r)$ over $\boldsymbol{x}$ after multiplying it by $\exp(-\boldsymbol{a}\cdot\boldsymbol{x}) - \exp(-\boldsymbol{b}\cdot\boldsymbol{x})$. In this way we have

$$F(\infty, r) = r^{-3}. \qquad (5\cdot 5c)$$

As is seen from (5·5), $(JE)_{tt}^{m}$ is proportional to $e^{10}m$. Therefore the integral of (5·5a) can be replaced with the expectation value defined by (2·26b) :

$$(JE)_{tt}^{m} = -(e^4/m^2)\langle F(K, r_{12})\rangle/(2\pi). \qquad (5\cdot 6)$$

This expectation value can be calculated in the same way as that of $f(r_{12})$ in § 3 (c). Corresponding to (3·23a) we have

$$\langle F(k, r_{12})\rangle - 4\pi e^6 m^3 N J(K) = 4\pi e^6 m^3 \{Q - N\ln(e^2 m)\}, \qquad (5\cdot 7a)$$

where Q is given by (2·30b), and $J(K)$ is given by

$$J(K) = \int_0^\infty F(K, r) e^{-r} r^2 dr. \qquad (5\cdot 7b)$$

If we substitute (5·5b) for $F(K, r)$ in this integrand, $J(K)$ becomes an integral with respect to $\boldsymbol{k}$, $\boldsymbol{q}$, and r. If we omit e^{-r} from the integrand of this integral and confine the domain of $\boldsymbol{k}$-integration to $\delta \cdot |\boldsymbol{k}| \cdot K$, the integration can immediately be carried out. We see that the value of the integral is unchanged if we confine the domains of $\boldsymbol{k}$- and $\boldsymbol{q}$-integrations to $\delta \cdot |\boldsymbol{k}|$ and $|\boldsymbol{q}| \cdot 2K - |\boldsymbol{k}|$ respectively. Further we can show that if we replace e^{-r} of (5·7b) with $e^{-r} - 1$, the value of the integral over the former domain is still equal to that over the latter domain for large K. Therefore, if we add these two integrals and let δ vanish, we get $J(K)$ for large K as an integration over the domains $\delta \cdot |\boldsymbol{k}|$ and $|\boldsymbol{q}| \cdot 2K - |\boldsymbol{k}|$. This can be evaluated in the following way.[7] First we integrate with respect to r and angular parts of $\boldsymbol{k}$ and $\boldsymbol{q}$. Then we transform the integration variables into x and u as $x = (|\boldsymbol{k}| + |\boldsymbol{q}|)/\sqrt{2}$ and $xu = (|\boldsymbol{k}| - |\boldsymbol{q}|)/\sqrt{2}$, and finally integrating with respect to x and u, we have

$$J(K) = (7/3)(\ln 2 - 1) + \ln K. \qquad (5\cdot 8)$$

From this result together with (5·7a) we get

$$\langle F(K, r_{12})\rangle/(2\pi) = 2e^6m^3[Q + (7/3)N(\ln 2 - 1) - N\ln(e^2m/K)], \quad (5\cdot 9)$$

whence

$$(\Delta E)^m_{tt} = 2e^{10}m[(7/3)N(1-\ln 2) + N\ln(e^2m/K) - Q]. \quad (5\cdot 10)$$

Adding (5·3) and (5·10) we have

$$(\Delta E)_{tt} = e^{10}m[N\{(17/3) - (8/3)\ln 2 + \ln e^4 + (1/3)\langle\sigma^{(1)}\cdot\sigma^{(2)}\rangle\} - 2Q]. \quad (5\cdot 11)$$

§ 6. Vertex and vacuum polarization corrections

(a) *Vertex correction to electron-electron interaction,* $(\Delta E)_{eL}$

We first calculate $(\Delta E)_{eL}$. The contribution from high- and intermediate-frequency regions can be obtained, by evaluating in the same way as in Fulton and Martin's calculation, as follows:

$$(\Delta E)^{hm}_{eL} = (2/3)e^{10}mN[4\ln(2\kappa/m) - (10/3) - \langle\sigma^{(1)}\cdot\sigma^{(2)}\rangle] + (e^2/\pi)\langle H^{se}_{so} + (H^m_{so}/2) + H_{ss}\rangle, \quad (6\cdot 1)$$

where H^m_{so} and H_{ss} are mutual spin-orbit and spin-spin couplings defined by (2·28), and H^{se}_{so} is the electronic-interaction part of the self spin-orbit coupling [see (2·28a)]:

$$H^{se}_{so} = -(2\mu^2/r_{12}^3)(\mathbf{S}_1\cdot\mathbf{L}_{12} + \mathbf{S}_2\cdot\mathbf{L}_{21}). \quad (6\cdot 2)$$

The corrections proportional to H^m_{so}, H_{ss} and $\sigma^{(1)}\cdot\sigma^{(2)}$ in (6·1) can be obtained from the similar terms in $H^{(2)}$ by multiplying $\mathbf{S}_1$ and $\mathbf{S}_2$ by $e^2/(2\pi)$, in other words, by taking into account the anomalous magnetic moment of electrons.

The contribution from a low-frequency region to the vertex correction to the electron-electron interaction arises from the Coulomb interaction between electrons and a transverse interaction in the vertex part only because the Coulomb interaction in the vertex part works in association with pair processes and the transverse interaction between electrons is accompanied with odd operators. Λ^C_L can be replaced with $G_1{}^C\gamma^{(1)}_0$ for the part including F_1 and with $G_2{}^C\gamma^{(2)}_0$ for that including F_2. Thus the correction comes from

$$4\pi e^2\int\chi^*(\boldsymbol{x}_1, \boldsymbol{x}_2)[G_2{}^C\gamma^{(1)}_0\gamma^{(1)}_t(\tilde{\xi}_1)F_2(G_{12}{}^C - G_1{}^CG_2{}^C)F_2\gamma^{(1)}_t(\tilde{\xi}_1')D_t(\tilde{\xi}_1\tilde{\xi}_1')G_2{}^C\gamma^{(2)}_0 + (1\rightleftarrows 2)]_{E0}\chi(\boldsymbol{x}_1', \boldsymbol{x}_2')\,d(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_1', \boldsymbol{x}_2'). \quad (6\cdot 3)$$

The same method of calculation as in the case of $(\Delta E)^l_t$ in § 3 (b) yields

$$(\Delta E)^l_{eL} = (2/3\pi)(e/m)^2(\boldsymbol{p}_1\psi_E, (E^0 - H^{(nr)})\ln|(H^{(nr)} - E^0)/\kappa|\,\boldsymbol{p}_1\psi_E) + (\boldsymbol{p}_1\rightarrow\boldsymbol{p}_2) - [H^{(nr)}\rightarrow(H^{(nr)} - e^2/r_{12})] + \text{imaginary terms}. \quad (6\cdot 4)$$

where $(\boldsymbol{p}_1\rightarrow\boldsymbol{p}_2)$ means a term obtained by replacing $\boldsymbol{p}_1$ with $\boldsymbol{p}_2$ from the preceding one and an arrow in the bracket has a similar meaning.

(b) *Vertex correction to the external Coulomb field,* $(\Delta E)_{nL}$

We next calculate $(\Delta E)_{nL}$. We see from (1·4f), (1·9b) and (2·4) that the

contribution from high- and intermediate-frequency regions to the correction comes from the following term:

$$(4\pi e^2)^2\int\chi^*(\boldsymbol{x}_1,\boldsymbol{x}_2)\,(H_1+H_2-E)\,[G_1''\gamma^{(1)}(\hat{\xi}_1)\,G_1^0\gamma^{(1)}(Z/r'')\,G_1''\gamma^{(1)}(\hat{\xi}')\,D(\hat{\xi}_1\hat{\xi}_1')\,I''$$
$$+(1\rightleftarrows 2)\,]_{EO}\chi(\boldsymbol{x}_1',\boldsymbol{x}_2')\,d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_1',\boldsymbol{x}_2')+\text{renormalization terms.}\qquad(6\cdot5)$$

If we carry out energy integrations we see that contributions of some terms are negligible due to the factor H_1+H_2-E. Calculating as in the case of $(\Delta E)_{eL}$, we have

$$(\Delta E)_{nL}^{hm}=(8/3)\,e^{10}mZN_0[\,(5/6)-\ln(2\kappa/m)\,]+(e^2/\pi)\langle H_{so}^{sn}\rangle,\qquad(6\cdot6)$$

where N_0 is defined by

$$2e^6m^3N_0=\int\{|\chi(\boldsymbol{x},0)|^2+|\chi(0,\boldsymbol{x})|^2\}\,d(\boldsymbol{x}),\qquad(6\cdot7)$$

and H_{so}^{sn} is the nuclear-Coulomb-field part of the self spin-orbit coupling [see (2·28a)]:

$$H_{so}^{sn}=2\mu^2Z(\mathbf{S}_1\cdot\mathbf{L}_1/r_1^3+\mathbf{S}_2\cdot\mathbf{L}_2/r_2^3).\qquad(6\cdot8)$$

As was explained the equation (2·30a) is valid in the present approximation.

The contribution from the low-frequency region is calculated in the same way as in the case of $(\Delta E)_{eL}^{l}$. The result is

$$(\Delta E)_{nL}^{l}=(2/3\pi)\,(e/m)^2(\boldsymbol{p}_1\psi_E,\ (E^0-H^{(nr)}+e^2/r_{12})\ln|(H^{(nr)}-E^0-e^2/r_{12})/\kappa|\boldsymbol{p}_1\psi_E)$$
$$+(\boldsymbol{p}_1\rightarrow\boldsymbol{p}_2)+\text{imaginary terms.}\qquad(6\cdot9)$$

Adding (6·4) and (6·9) we have

$$(\Delta E)_L^l=(\Delta E)_{eL}^l+(\Delta E)_{nL}^l=(8/3)\,e^{10}m(N-ZN_0)\ln(e^4m/\kappa)-(2/3\pi)\,e^6M''$$
$$+\text{imaginary terms,}\qquad(6\cdot10)$$

where M'' is defined by

$$e^4m^2M''=\sum_n{}'(E_n^0-E^0)\,(|\boldsymbol{p}_{nE}^{(1)}|^2+|\boldsymbol{p}_{nE}^{(2)}|^2)\ln|(E_n^0-E^0)/e^4m|.\qquad(6\cdot11)$$

Further, adding (6·1), (6·6), and (6·10), we have the vertex corrections as follows:

$$(\Delta E)_L=(\Delta E)_{eL}+(\Delta E)_{nL}=(8/3)\,e^{10}m(ZN_0-N)[\,(5/6)-\ln(2e^4)\,]-(2/3\pi)\,e^6M''$$
$$+(e^2/\pi)\langle H_{so}^{s}+(H_{so}^{m}/2)+H_{ss}\rangle-(8/3)\,e^{10}mN\langle\mathbf{S}_1\cdot\mathbf{S}_2\rangle+\text{imaginary terms.}\qquad(6\cdot12)$$

The term containing M'' in this correction together with that containing M' in the correction due to an exchange of a photon gives the term containing M in $H^{(1)}$ because $M=M'+M''$. [Compare (2·29b), (2·30c), (3·18b), (6·11) and (6·12)].

(c) *Vacuum-polarization correction*

As is well known, I_r is given by[20]

$$I_r=4ie^2\gamma^{(1)}\gamma^{(2)}(2\pi)^{-4}\iint c(\boldsymbol{k}^2-k_0^2)\exp ik(\hat{\xi}_1-\hat{\xi}_2)\,d(\boldsymbol{k})\,dk_0/(\boldsymbol{k}^2-k_0^2),\qquad(6\cdot13)$$

where $c(\boldsymbol{k}^2-k_0^2)/(\boldsymbol{k}^2-k_0^2)=e^2/(15m^2)$ for $\boldsymbol{k}^2-k_0^2\ll m^2$. By means of (2·16), we get

the vacuum-polarization correction to the one-quantum exchange as follows:

$$(\Delta E)_{e}=(4/15)\,e^{10}mN. \tag{6·14}$$

The vacuum-polarization correction to the nuclear Coulomb field is given by

$$(\Delta E)_{nr}=-(8/15)\,e^{10}mZN_0. \tag{6·15}$$

§ 7. Coulomb corrections

The eigenvalue problem for the relativistic Hamiltonian of the two-electron system including only the Coulomb interaction is given by (2·23). We first calculate the difference between the eigenvalue of this equation and that of (2·5): $(\Delta E)_{C}=W-E_{C}$. The equation (2·5) can be written as follows:

$$(\mathrm{H}_1+\mathrm{H}_2+e^2/r_{12}-E_C)\,\chi=\{W-E_C+(e^2/r_{12})\,(1-\Lambda)\}\,\chi. \tag{7·1}$$

Since the left side of this equation is the same as that of (2·23), the right side is orthogonal to the solution of (2·23):

$$(W-E_C)\,(\phi,\,\chi)=\int\phi^*\,(e^2/r_{12})\,(\Lambda-1)\,\chi\,d\,(\boldsymbol{x}_1,\,\boldsymbol{x}_2). \tag{7·2}$$

If we replace ϕ with χ in the right side the error is at most e^6 times the integral itself. This has no influence on our final result because the integral itself is as small as $e^{10}m$. We can also replace χ with ϕ in the left side. Thus we have

$$(\Delta E)_{C}=W-E_C=-\int\chi^*\,(e^2/r_{12})\,(\Lambda_{C1}^{+}\Lambda_{C2}^{-}+\Lambda_{C1}^{-}\Lambda_{C2}^{+}+2\Lambda_{C1}^{-}\Lambda_{C2}^{-})\,\chi\,d\,(\boldsymbol{x}_1,\,\boldsymbol{x}_2). \tag{7·3}$$

If we replace χ in front of the d-sign with the right side of (2·18a) we see that the main contribution comes from a large momentum part. Therefore we can omit the nuclear Coulomb potential from Λ_C's. Making use of (2·16), we have

$$\begin{aligned}(\Delta E)_{C}&=(2/\pi)\,e^{10}m^3N\int\left\langle\!\!\left\langle\frac{\Lambda_1^{+}(\boldsymbol{k})\,\Lambda_2^{-}(\boldsymbol{k})+\Lambda_1^{-}(\boldsymbol{k})\,\Lambda_2^{+}(\boldsymbol{k})+2\Lambda_1^{-}(\boldsymbol{k})\,\Lambda_2^{-}(\boldsymbol{k})}{\mathrm{H}_1^{0}(\boldsymbol{k})+\mathrm{H}_2^{0}(\boldsymbol{k})-2m}\right\rangle\!\!\right\rangle\frac{d(\boldsymbol{k})}{k^4}\\&=-(10/3)\,e^{10}mN,\end{aligned} \tag{7·4}$$

where $\Lambda_1^{\pm}$ and $\Lambda_2^{\pm}$ are defined by (3·7).

Next we calculate $(\Delta E)_{CC}$. The main contribution to this correction comes from the high-frequency region for the same reason as in the previous case. Therefore we replace G_1^{C} and G_2^{C} with G_1^{0} and G_2^{0} respectively. By means of (2·16) we have

$$\begin{aligned}(\Delta E)_{CC}&=(i/\pi^2)\,e^{10}m^3N\iint\langle[\mathrm{H}_1^{0}(\boldsymbol{k})-p_0]^{-1}[\mathrm{H}_2^{0}(\boldsymbol{k})-p_0]^{-1}\rangle\,dp_0\,d\,(\boldsymbol{k})/k^4\\&=2e^{10}mN.\end{aligned} \tag{7·5}$$

§ 8. Imaginary part

We shall finally consider the meaning of the imaginary part of the energy correction

which appeared in (3·26) and (6·12). This seems to be deeply connected with the essential nature of the Bethe-Salpeter equation as we shall see. The fact that the energy eigenvalue includes an imaginary part invalidates the very intention of the present paper because we can not apply the Gell-Mann and Low procedure to the derivation of the Bethe-Salpeter equation nor consider the Fourier transform, (2·9), in this case. It is thus necessary to examine whether the energy correction obtained in the present calculation really gives the correct energy shift for *excited states*. Of course in the case of the ground state the imaginary part vanishes. Therefore the present result has no contradiction in this case. In what follows we shall give some considerations on the justification of the present result.

It seems to have been believed that the Bethe-Salpeter equation of a two-body system gives real values to energy levels of excited bound states as well as of a ground state of the system. As for the method of deriving the equation for excited states, however, it is not free from doubt because the excited bound states are by no means stationary states of a system consisting of two bodies and quanta but they are unstable. Of course standing-wave solutions without an incident wave exist in a simple case. They correspond to poles of the reaction matrix[21] and consequently to resonance energies in the case of scattering, for example, of a photon by the two-electron system in the ground state. The existence of such solutions seems to indicate a possibility that the Bethe-Salpeter equation generally gives the resonance energy in scattering.

It has recently be shown by Kita and Wakano[22] in an exact way that the Bethe-Salpeter equation gives no excited level corresponding to a stationary state of a hydrogen atom. At first sight this seems to contradict with the conclusion of Wick[23] and Cutkosky.[24] They considered the Bethe-Salpeter equation of a two-boson system in which an interaction between two bosons is transmitted by scalar photons. Assuming the ladder approximation they obtained higher excited levels, some of which approach to well-known levels of a hydrogen atom in a non-relativistic limit. We can easily disclose the reason for the contradiction. At the same time we can see how the situation is connected with the occurrence of the imaginary energy-correction in the present calculation.

In terms of the Feynman diagram, the ladder approximation means to take into account only graphs of type (a) in Fig. 1. Such graphs include no real process in their intermediate states if the total energy is less than $2m$. Therefore they give rise to no imaginary part in the calculation of energy levels of the system. Even any graph of type (b) includes no real process in its intermediate states. If we sum up graphs of such a type we can obtain a graph of type (c) which includes a real process in its intermediate states. In this case an imaginary part must appear in general when we calculate energy of an excited bound state so far as we adopt an outgoing-wave boundary condition for the Green function G^C. This is the reason for the discrepancy between Kita-Wakano and Wick-Cutkosky and also the reason why the present result includes an imaginary part.

In the previous calculation of the Lamb shift for the hydrogen atom[25] or positronium[7] the energy correction included no imaginary part. The reason was that a principal value

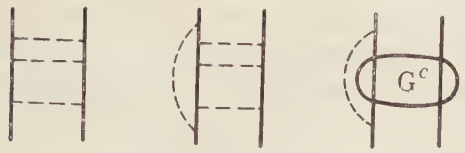

Fig. 1

for K-integration was adopted for the correction arising from the exchange of a photon or for the vertex correction. The basis for this assumption was Heitler's damping theory.[26] The result of the present calculation has the form which is expected either from the conventional perturbation theory for bound electrons or from the covariant perturbation theory for free electrons, as was noted by Fulton and Martin[7] in the case of positronium. If we would consider this fact as an evidence for the justification of the present result we might still be in self-contradiction to discard the Bethe-Salpeter equation as the basis.

We shall next examine the imaginary part in question. If we define

$$\Gamma = (4/3)\, e^6 \sum_{>0} (E^0 - E_n^{\,0})\, |\boldsymbol{P}_{nE}|^2, \tag{8·1}$$

the imaginary part of the energy correction is given by

$$\mathrm{Im}\,(\Delta E) = -\Gamma/2, \tag{8·2}$$

where >0 means that the summation should be extended over positive summands: $E^0 - E_n^{\,0} > 0$; Im means 'an imaginary part of', and $\boldsymbol{P}_{nE}$ is defined by (2·31a). The definition (8·1) shows that Γ is just equal to the first order perturbation term of the total width of the level in question. This fact suggests that the energy eigenvalue in the present paper is a complex pole of the scattering matrix. In order to examine this expectation we shall recapitulate the present evaluation. The Bethe-Salpeter equation of the present system in the momentum representation is given by

$$[1 - S'_{F1} S'_{F1} \mathrm{I}]\, \Psi = 0, \tag{8·3}$$

where S'_{F1} is the modified propagator of the first electron and S'_{F2} is that of the second electron which can be replaced with $G_1^{(c)}$ and $G_2^{(c)}$ respectively in the present approximation. We first eliminate Ψ from this equation by means of a momentum representation of χ, and then solve the equation for the total energy E by the iteration method. This procedure is formally equivalent to the evaluation of the pole of the complete Green function for the two-electron system. This Green function is given by

$$G = [1 - S'_{F1} S'_{F2} \mathrm{I}]^{-1} S'_{F1} S'_{F2}. \tag{8·4}$$

When E is real, as it should be for the ground state, this is a real pole of G and corresponds to a stationary state. When E is complex it may be a complex pole of the extended Green function which is obtained by an analytical continuation of G in the complex energy-momentum space.[28] In the case of scattering of photons by the two-electron system in its ground state, G is contained in the scattering matrix. Therefore

it may be inferred that the real and imaginary parts of the complex E give an energy level and half a reciprocal life (except for its sign) of a metastable state of the two-electron system respectively. In order to show this situation correctly a time-dependent consideration on the phenomenon is necessary. A full discussion for it will be given elsewhere.

The relations between the energy level in Heitler's damping theory, the real part of the pole of the scattering matrix, and the pole of the reaction matrix were discussed by Araki and coworkers[28] in the case of unstable particles. They showed that these three are the same within an accuracy of the width. The conclusion seems to hold rather in general and this suggests that the result of the present calculation is correct for the excited state as well as the ground state. If we could obtain a relativistically covariant wave equation whose eigenvalue gives a pole of the reaction matrix, this equation might be equivalent to the Bethe-Salpeter equation within the accuracy considered here except that the electron and photon Green functions satisfy the standing-wave boundary condition. If the formulation of such a covariant equation will succeed the situation will become clearer, although it is now an open question.

In conclusion the author wishes to express his sincere thanks to Professor Hideki Yukawa for his interest taken in this work and also to Professor Gentaro Araki for his kindness of suggesting the present calculation and of giving many valuable suggestions.

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The Penetration of the Diffuse Nuclear Potential Barrier

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The effect of the diffuse nuclear potential on the penetration of the potential barrier is examined using a simplified model. The transmission coefficient for slow neutrons is treated in an exact way employing an exponential diffuse potential and is found to be larger by a factor 2 than in the usual calculation (§ 2). The Coulomb penetrability is investigated on the basis of the W-K-B method, and the diffuseness makes the penetration factor much larger than that obtained in the conventional calculation (§ 3). An application to the evaporation process is developed (§ 4) and the energy spectrum of α-particles from the $^{27}Al(n;\alpha)$ ^{24}Na reaction is found to be explained by the corrected penetrability in a satisfactory way.

§ 1. Introduction

In the last decade the study on nuclear reactions has been performed on the basis of the Bohr hypothesis and the evaporation theory developed by Weisskopf has been a useful weapon in analysing an enormous amount of experimental data. Recent experiments, however, mainly on the inelastic scattering of charged particles have suggested that the compound nucleus formalism might be insufficient to account for them and all sorts of non-compound processes should be taken into account. Among various processes to be considered, the direct processes occurring in the nucleus and at the nuclear surface have been investigated by a number of authors,[1),2)] almost all of these concerned with the angular distribution of scattered particles. Then it has been found that these non-compound processes play an important role in describing the nuclear reactions when residual nuclei are in discrete states. Nevertheless, several points seem to be left unaccountable even by introducing other models of direct interactions than the collision complex which forgets how it was formed. As one of these, we can present the energy spectrum of protons scattered inelastically into the backward direction. These spectra have resulted in decreasing of nuclear temperature with increasing excitation energy, if interpreted in terms of the evaporation theory.[3)] This difficulty on the evaporation spectrum hardly seems to be due to a contribution from the nuclear direct interactions, because protons directly scattered are confined in the forward direction in usual cases and they may fail to yield a larger contribution at such backward angles. The small difference in proton yields between the forward and the backward directions in the 18 Mev proton inelastic scattering[4)] indicates that the compound nucleus process predominates over other processes in this energy region, especially when residual nuclei are left highly excited. Recently the experiment[5)] on the bombardment of medium weight nuclei by 20 Mev protons has

shown a larger yield of low energy protons than that expected by the conventional calculation based upon the evaporation model. Though the evaporation of charged particles is favoured by depression of the Coulomb barrier due to the expansion of highly excited nuclei, yet this effect is negligible at temperatures concerned here.[6] Now the larger yield of low energy protons is closely related with the strange tendency of the nuclear temperature mentioned above, since the latter is due to the concave upwards trend of the observed energy spectrum divided by the product of the particle energy and the transmission coefficient.

In view of above considerations, the purpose of this work is to investigate the emission of charged particles from the compound nucleus on a little different basis, namely on the analysis of the penetration of the nuclear potential barriers. In the conventional calculations the nuclear potential has been generally assumed to be a square well. Detailed investigations on the elastic scattering angular distribution of protons[7] and α-particles[8] have predicted that the actual form of the nuclear potential is not a square well with sharp boundary but rather has a diffuse nuclear surface. The diffuseness of nuclear surface makes the penetration of the potential barrier easier, as it should be. Further, if we interpret the larger yield of charged particles simply by introducing a larger choice of nuclear radii, the magnitude necessary to fit the experiments must be much larger than that usually employed, as can be seen from later discussions on the effective barrier height. In § 2 we shall obtain a rough estimate of the transmission coefficient for slow neutrons, assuming an exponential diffuse potential outside the square well. In § 3 are derived the transmission coefficient for charged particles and the lowering down of the effective barrier height. When the Coulomb barrier exists, since a great complexity is introduced in the exact calculation by such a diffuse boundary, we adopt the W-K-B method appropriate for a crude discussions. The transmission coefficient corrected by the diffuse potential will be applied to the evaporation model in § 4; the increase of the charged particle emission width and the analysis of the $^{27}\mathrm{Al}(n;\alpha)\,^{24}\mathrm{Na}$ reaction[9] will be undertaken.

§ 2. Qualitative discussions on slow neutrons

In this section we investigate the effect of the diffuse potential on slow neutrons. For neutrons with high energies the nuclear surface effect plays a smaller role than for slow neutrons. Then we concentrate our efforts to examine the effect of the nuclear diffuse boundary on S-wave neutrons.

Consider the nuclear potential of a diffuse surface* given by

$$V(r) = \begin{cases} -V_0 & \text{for } r < R \quad (\mathrm{I}) \qquad (1\cdot 1) \\ -V_0 \exp -\varepsilon(r-R) & \text{for } r > R \quad (\mathrm{II}) \qquad (1\cdot 2) \end{cases}$$

* A similar potential was treated by H. A. Bethe in introducing the absorption potential (Phys. Rev. 57 (1940), 1125).

where $\mathcal{E}$ is the diffuseness parameter. This potential gives the radial Schrödinger equations

$$d^2u/dr^2+K^2u=0 \qquad \text{(I)} \qquad (2\cdot1)$$

$$d^2u/dr^2+[k^2+K_0^2e^{-\mathcal{E}(r-R)}]u=0 \qquad \text{(II)} \qquad (2\cdot2)$$

with

$$k^2=2ME/\hbar^2, \quad K_0^2=2MV_0/\hbar^2, \quad K^2=k^2+K_0^2,$$

where u is r times the radial factor of the wave function Ψ. Obviously the solutions of $(2\cdot1)$ are $\exp(\pm iKr)$. The equation $(2\cdot2)$ is solved by introducing the independent variable

$$x=\frac{2K_0}{\mathcal{E}}e^{-\mathcal{E}(r-R)/2} \qquad (3)$$

then we have the Bessel equation of λ-th order

$$\frac{d^2u}{dx^2}+\frac{1}{x}\cdot\frac{du}{dx}+\left(1-\frac{\lambda^2}{x^2}\right)u=0 \qquad (4)$$

with $$\lambda=(2k/\mathcal{E})\,i$$

whose solutions regular at $x=2K_0/\mathcal{E}$, i.e. $r=R$, are $J_\lambda(x)$ and $J_{-\lambda}(x)$.
The solution u_1 in the region (I) can be written as

$$u_1=e^{iKr}+Ae^{-iKr} \quad r<R \qquad \text{(I)} \qquad (5\cdot1)$$

consisting of an incident wave $\exp(iKr)$ of unit amplitude and a reflected wave of unknown amplitude A. For $r>R$ we have a solution u_2

$$u_2=B\,J_\lambda(x) \qquad r>R \qquad \text{(II)} \qquad (5\cdot2)$$

representing a wave transmitted through the potential jump. The boundary conditions at $r=R$ give

$$A=\frac{K-k\{1+2f(\mathcal{E})\}}{K+k\{1+2f(\mathcal{E})\}}e^{2iKR} \qquad (6)$$

where $f(\mathcal{E})$ is

$$f(\mathcal{E})=\frac{M_{-\lambda-1}(2K_0/\mathcal{E})-M_{-\lambda}(2K_0/\mathcal{E})}{1+M_{-\lambda}(2K_0/\mathcal{E})} \qquad (7)$$

with

$$M_\nu(z)=\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(\nu+1)}{n!\,\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2n}. \qquad (8)$$

It can be seen from (6) that the reflection amplitude A may be obtained replacing k by $k(1+2f)$ in the amplitude formula for the square well potential. $f(\mathcal{E})$ is a function of the diffuseness parameter $\mathcal{E}$ and vanishes for the square well potential, i.e. $\mathcal{E}$ becomes infinite. For $\mathcal{E}\gg K_0$, neglecting higher orders of $(K_0/\mathcal{E})$, (7) goes over into

$$f(\varepsilon) \sim (K_0/\varepsilon)^2 - (K_0^2/2k\varepsilon)\,i, \tag{9}$$

when the transmission coefficient T is written as

$$T = 1 - |A|^2 \sim \frac{4kK\{1+(4K_0^2/\varepsilon^2)\}}{[K+k\{1+(4K_0^2/\varepsilon^2)\}]^2+(K_0^4/\varepsilon^2)} \tag{10}$$

$$\sim 4k_e K/(K+k_e)^2 \tag{11}$$

where

$$k_e = k\{1+(4K_0^2/\varepsilon^2)\}. \tag{12}$$

In actual nuclei, the magnitude of ε^{-1} is very close to 0.5×10^{-13} cm and K_0^{-1} is the order of 1×10^{-13}cm ; these choice of ε and K_0 increases the value of T by about a factor 2.

§ 3. Transmission of the Coulomb barrier

In the case of charged particles, let us consider the following potential

$$V(r) = V_n(r) + V_c(r) \quad \text{for} \quad r > R \tag{13}$$

where V_n is the nuclear potential and V_c the Coulomb potential. They are, respectively, given by

$$V_n(r) = -V_0\left[1+\exp\frac{r-R}{d}\right]^{-1} \tag{14}$$

$$V_c(r) = ZZ'e^2/r. \tag{15}$$

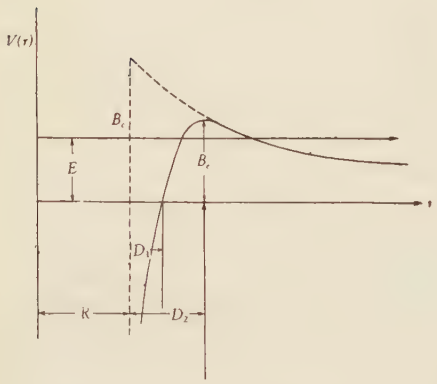

Fig. 1. The schematic illustration for the diffuse nuclear potential.
The solid curve indicates the diffuse potential given by (13) and the dashed curve indicates the potential barrier conventionally employed.

This potential is illustrated in Fig. 1. The dashed curve is the usual potential, the square well plus the Coulomb potential. Theoretical calculations on the elastic scattering of both protons and α-particles showed that the magnitude of V_0 is nearly 40 Mev. and d is about 0.5×10^{-13}cm. In most cases, since V_0 is much larger than the Coulomb barrier height $B_c = ZZ'e^2/R$, $V(r)$ is negative at $r=R$ and increases rapidly with increasing r, then becomes zero at $r=R+D_1$ as shown in Fig. 1. The effective barrier height B_e and the corresponding radial distance $r=R+D_2$ can be determined by solving the equation $dV(r)/dr=0$. Since D_2 lies between 1.5 and 2×10^{-13}cm, neglecting the nuclear potential near at $r=R+D_2$ yields a rough estimate for the effective barrier height as

$$B_e = V(R+D_2) \sim B_c(1+D_2/R)^{-1} \tag{16}$$

$$\sim B_c(1-1.7/R) \qquad (R \text{ in } 10^{-13}\text{cm}). \tag{17}$$

As is easily seen from the above formula, the lowering down of the barrier height is of less important heavier nuclei. The exact value of B_e is shown in Fig. 2.

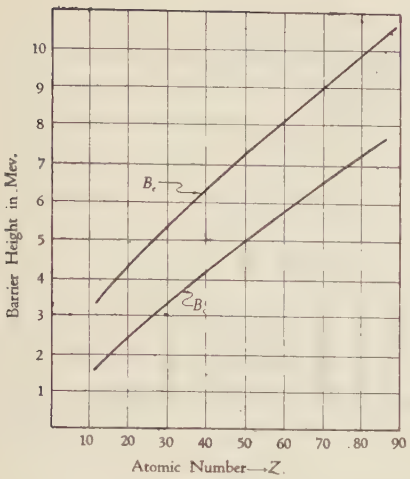

Fig. 2. Barrier height, *v.s.* atomic number Z. B_c is the Coulomb barrier height given by $ZZ'e^2/S$. B_e is the maximum of the diffuse potential (13), as illustrated in Fig. 1.

An appropriate calculation of the transmission coefficients may be performed on the basis of the W-K-B approximation. By the W-K-B method the transmission coefficient $T(E)$ for the diffuse Coulomb potential and $T_c(E)$ for the usual case are given by phase integrals respectively as

$$T(E) = \exp\left\{-2(2M/\hbar^2)^{1/2}\int_{r_1}^{r_2}\sqrt{V(r)-E}\,dr\right\} \tag{18}$$

and

$$T_c(E) = \exp\left\{-2(2M/\hbar^2)^{1/2}\int_{R}^{r_2'}\sqrt{V_c(r)-E}\,dr\right\} \tag{19}$$

where r_1 and r_2 are determined by $V(r)=E$ and r_2' by $V_c(r_2')=E$. The nuclear potential $V_n(r)$ is only subsidiary for small values of r, then we can take $r_2' \approx r_2$. In crude estimates, the transmission coefficient $T(E)$ may be approximated by

$$T(E) \sim \exp\left\{-2(2M/\hbar^2)^{1/2}\int_{R+D}^{r_2}\sqrt{V_c(r)-E}\,dr.\right\} \tag{20}$$

Here D is some distance between D_1 and D_2. If the particle energy, E, is sufficiently small in comparison with B_e and B_c; the ratio of $T(E)$ and $T_c(E)$ is roughly

$$T(E)/T_c(E) \sim \exp 2D(2MB_c/\hbar^2)^{1/2} \tag{21}$$

$$= \exp 0.44D\sqrt{B_c} \quad (B_c \text{ in Mev and } D \text{ in } 10^{-13}\text{cm}). \tag{22}$$

The transmission coefficient T differs from T_c only by a factor dependent on the Coulomb barrier height, B_c, and is nearly independent of the particle energy E. This trend can be found from Fig. 3. The values of integrals (18) and (19) for protons of different energies are given in Fig. 3. The calculation of the integral (19) can be obtained in an exact way through an elementary calculations and (18) is performed by graphical integrations. In our work nuclear radius parameter r_0 is taken as 1.35×10^{-13} cm. The ratio $T(E)/T_c(E)$ becomes maximum near $E=B_e$, since in the W-K-B method the transmission coefficient equals unity for particle energies larger than the barrier height.

In the above treatment, the angular momentum barrier has not been taken into account, which is of importance only for particles below the barrier height. As the centrifugal potential with its r^{-2} dependence presents a thinner barrier to the protons than the Coulomb repulsion proportional to r^{-1}, the diffuse nuclear boundary seems to make the ratio $T(E)/T_c(E)$ much larger for particles with high angular momentum than for

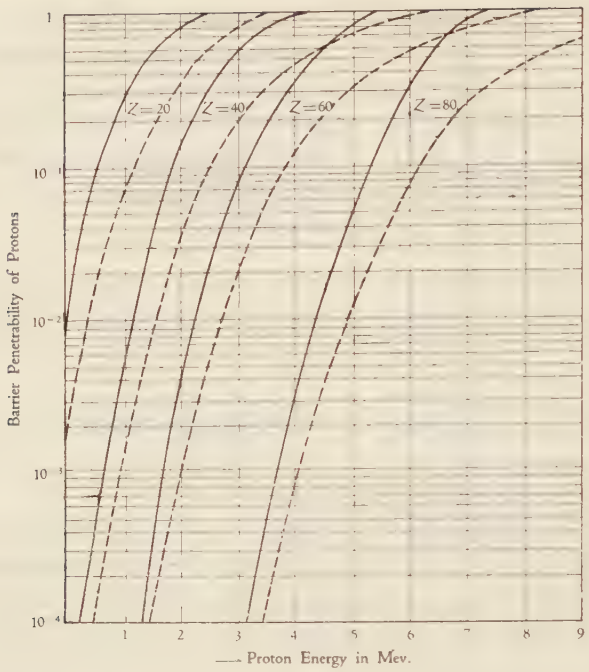

Fig. 3. Transmission coefficient of protons for different nuclei $Z=20$, 40, 60 and 80. The solid and the dashed curves indicate the transmission coefficicnts for the diffuse and the usual barriers respectively.

those with zero angular momentum.

§ 4. Application to the evaporation process

In this section we shall apply the transmission coefficient obtained in the last section to several problems of the evaporation process. From the formula for the partial width for emitting a particle i with energy E, one obtains

$$dI'_i/dE = C_i E \sigma_c(E) \exp(-E/\theta) \tag{23}$$

where $\sigma_c(E)$ is the cross section for the formation of the compound nucleus and θ is the temperature of the compound nucleus and C_i a constant determined by the mass and the spin of emitted particle. It should be noted that the formula (23) is valid only for low energy-emitted particles, i.e. when the temperature of the residual nucleus left after the emission of i differs by a small magnitude of negligible order from that of the compound nucleus. For neutrons $\sigma_c(E)$ is reasonably assumed to be constant σ_0, and consequently, we obtain a Maxwellian distribution for the neutron spectrum. For charged particles we may take a classical approximation that $\sigma_c(E)$ is equal to be σ_0 for E larger than the barrier height B, otherwise zero. With the above assumptions, we have the width I'_i in a convenient form as

$$I'_i = C_i \sigma_0 \cdot \theta (B+\theta) \exp(-(B+S_i)/\theta). \tag{24}$$

S_i is the separation energy of i from the compound nucleus. In the formula (25) the neutron width Γ_n can be obtained by putting $B=0$. The formula (24) facilitates us to discuss the effect of the lowering down of the potential barrier due to the diffuse boundary. The ratio of the emission probabilities for charged particles in the diffuse potential and the usual case is

$$\Gamma(B_e)/\Gamma(B_c) \approx (B_e+\Theta)/(B_c+\Theta)\exp\{(B_c-B_e)/\Theta\}. \quad (25)$$

For medium weight nuclei at temperature of 2～3 Mev, this ratio yields about 2～5 and the cross section for proton emission processes, such as $(n;p)$ or $(p;p')$ reactions, may be multiplied by the same factor.

Next we shall study the energy spectrum of α-particles emitted from Alminium nuclei bombarded by D-T neutrons[9]. The transmission coefficient for α-particles concerned here is calculated by the W-K-B method and shown in Fig. 4.

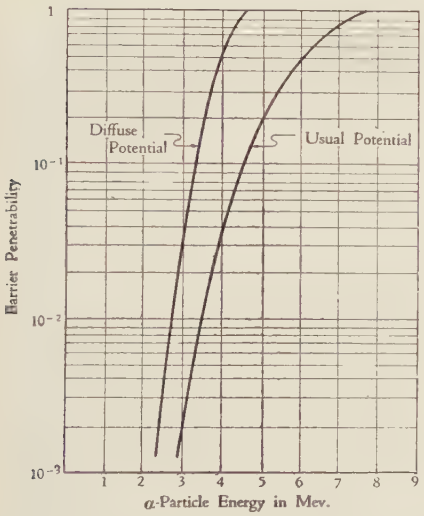

Fig. 4. Transmission coefficient of α-particles for ^{24}Na nucleus *vs.* α-particle energy. The two curves correspond to the two different potentials. The nuclear radius is taken as 4.2×10^{-13} cm.

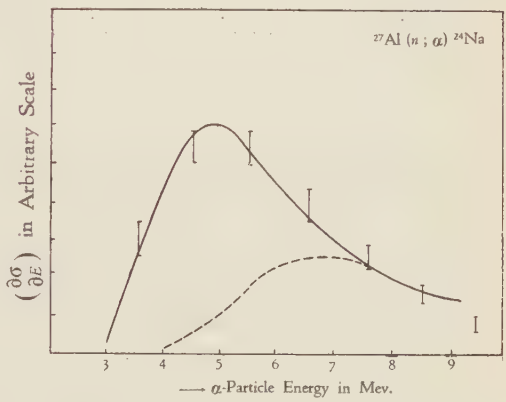

Fig. 5. The energy spectrum of α-particles from ^{27}Al$(n;\alpha)^{24}$Na reaction. I's indicate the experimental results. The solid and the dashed curves are the calculated ones referred to the diffuse and the usual potential respectively.

Taking the nuclear radius $R=4.2\times10^{-13}$cm, the barrier height B_c is nearly 7.6 Mev and B_e, 4.5 Mev. Assuming the nuclear temperature Θ of 2 Mev, the calculated energy spectra are shown in Fig. 5 for both cases, i.e. the diffuse potential and the usual case. As can easily be seen, if we employ the usual potential, i.e. the square well plus the Coulomb potential, the observed spectrum may hardly be explained.

Now let us discuss the behaviour of the nuclear temperature. The argument with α-particles mentioned above can be applied to the proton energy spectra observed in nuclear reactions initiated by protons with moderate energies. In the evaporation model, the energy spectrum divided by the product of the particle energy and the penetrability is

itself the relative level density of the residual nucleus ; we expect it to show a parabolic behaviour in view of the Fermi gas model. In order to explain the strange behaviour of the apparent nuclear temperature several attempts[10~13] have been made, partly improving the level density formula and partly considering the effect of two steps evaporations or of other non-compound mechanisms. None of them has succeeded in describing the unexpected behaviour of nuclear level density in a satisfactory manner. The relative level density of residual nucleus can be obtained, in more reasonable manner, by multiplying the ratio T_c/T on the $(d^2\sigma/dEd\Omega)/E\sigma_c$ as conventionally presented in a number of papers. The factor T_c/T has its minimum for particles with energies near B_c and approaches unity for high energies. Therefore the concave upwards curves of the relative level density can be improved to the expected parabolic curves by multiplying the ratio T_c/T. As the nuclear temperature depends upon the details of the behaviour of the level density, it is desirable to make a more exact calculation of the transmission coefficient.

The author is deeply grateful to Professors Y. Fujimoto and S. Yoshida for their valuable comments on this problem. His thanks are also due to Drs. A. Arima and Dr. M. Sano for their stimulating discussions. Thanks are due to Dr. I. Kumabe for sending us his experimental data before publication.

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The Photon-Nucleon Scattering

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The general expressions, the differential cross section, and the polarization of the recoil nucleon of the photon-nucleon scattering are given. Then experimental data of the photon-nucleon scattering at high energies are analyzed in terms of the shadow scattering of the photo-pion production.

§ 1. Introduction

For the study of the pion-nucleon interaction and the electromagnetic properties of the nucleon, the photon-nucleon scattering is an important process as well as the pion-nucleon scattering and the photo-pion production. Nevertheless there had been few experimental data of the photon-nucleon scattering mainly because of technical difficulties in the observation. Recently, Illinois and other groups[1] have managed to perform experiments of this process. On the other hand there have been many theoretical works[2)-12)] on this process. In some of them the anomalous magnetic moment of the nucleon has been taken into account, and in others the nucleon isobar with the spin 3/2 and the isotopic spin 3/2 has been assumed or the Chew-Low formalism is used.[7]

In this paper, however, we would like to examine this process from a different standpoint. One of the authors (S.M.) previously investigated the pion-nucleon scattering and the photo-pion production kinematically,[13) 14)] and pointed out remarkable symmetry properties in the differential cross sections of those reactions. We can expect such a symmetry property also in the photon-nucleon scattering. Therefore, in § 2, we first develop kinematical treatment in a manner analogous to ref. 13 and give the general expression for the differential cross section. Then we can easily show the symmetry property. Next, the explicit forms of the differential cross sections in the states with the total angular momenta up to $J=3/2$ are shown in Table 1 in order to facilitate analysis of experimental results.

It was shown in the previous paper[14] that in the pion-nucleon scattering and the photo-pion production processes, the symmetry properties are lost when the polarization of the recoil nucleon is observed. To investigate the situation in the case of the photon-nucleon scattering, the expression for the polarization of the recoil nucleon is obtained in § 3. According to this result, we can see, contrary to the former two cases, that the symmetry property still remains in the expression for the polarization and the degeneracy cannot be removed by the kinematical argument only.

In § 4, we first investigate the relationship between the phases of the amplitudes

of the pion-nucleon scattering, the photo-pion production and the photon-nucleon scattering using the unitarity of the S-matrix. Next, applying the results thus obtained, we try to analyze the experimental data of the photon-proton scattering. In this analysis, we are particularly interested in the energy region corresponding to the resonance of the $(3/2, 3/2)$ state well known in the case of the pion-nucleon scattering and the photo-pion production. Therefore we can assume that the $(3, 2, 3, 2)$ state of the pion-nucleon system plays a predominant role also in this process.

In § 5, we examine the origin of the fore-aft asymmetry of the angular distribution obtained by the experiments. Since the experiments were performed only for the energies lower than the resonance energy, we cannot depend on our experience. Therefore in § 6, we introduce another assumption that the angular distribution of the photon-proton scattering changes at or about the resonance energy from the one peaked backward in lower energies to the one peaked forward in higher energies. This is one of the results obtained by one of the authors (Y.Y.) in his investigation[7] of the photon-nucleon scattering based on the Chew-Low formalism. However, whether this assumption is acceptable or not is one of the problems which must be checked by experiments at higher energies, and we are anxious for such experiments.

If we accept the above assumption we can obtain $\delta^{+}_{M1}=0$ for the matrix element of R-matrix $\gamma^{+}_{M1}e^{2i\delta^{+}_{M1}}-1$ corresponding to the magnetic dipole state with $J=3/2$, which is considered to give the main contribution to the process (here γ_{M1} represents the absorption due to the photo-pion production). Therefore, we can describe the process as a "shadow" scattering process due to the absorption. As γ_{M1} can be calculated from the experimental values of the photo-pion production cross section $\sigma_{\gamma-\pi}$, we can calculate the cross section of the photon-nucleon scattering by making use of the experimental values for $\sigma_{\gamma-\pi}$. The result thus obtained shows a fairly good agreement with the experimental value of the Illinois groups.

§ 2. The differential cross section of the photon-nucleon scattering

The differential cross section of the photon-nucleon scattering can easily be calculated using the formula given by Morita et al.[15] For the photon-nucleon system with a specified resultant angular momentum l of the spin and the orbital angular momentum of the photon, there are two states* with the total angular momenta $J=l+1/2$ and $J=l-1/2$. Moreover we distinguish between the two states which belong to the same l but have different parities, i.e., magnetic l-pole and electric l-pole, by suffices Ml and El. Then we obtain the differential cross section for the unpolarized photon beam in the centre of mass system as follows

$$d\sigma/d\omega=(1/8k^2)\sum_{l,l'}\sum_{\substack{N,N'\\ =l \text{ or } M}}\sum_{\substack{\epsilon,\epsilon'\\ =+\text{ or }-}}(R^{\epsilon}_{Nl})^{*}(R^{\epsilon'}_{N'l'})F(Nl\epsilon,N'l'\epsilon';\theta). \tag{1}$$

Where

$$R^{\epsilon}_{Nl}=\eta^{\epsilon}_{Nl}\exp(2i\delta^{\epsilon}_{Nl})-1 \tag{2}$$

*) The superscripts $\epsilon=\pm$ are used to specify the states of $J=l\pm1/2$ respectively.

is a matrix element of the so-called R-matrix, and γ^{ϵ}_{Nl} represents the absorption due to the photo-pion production. θ is the scattering angle of the photon and explicit form of the factor $F(Nl\epsilon,\ N'l'\epsilon';\ \theta)$ can easily be obtained[15]. As $F(Nl\epsilon,\ N'l'\epsilon';\ \theta)=F(N'l\epsilon,\ Nl'\epsilon',\ \theta)$, $d\sigma/d\omega$ does not change when R^{ϵ}_{El} are interchanged with R^{ϵ}_{Ml} concerning all l's. This, similar to the case of the photo-pion production, comes from the fact that the photon can take only two values of its magnetic quantum number in spite of its unit spin. We show in Table 1 the expression F for the states with $J\leqslant 3/2$ to facilitate analysis of the experiments. Here the result that the dipole and quadrupole scattering with $J=3/2$ give the same angular distribution $(7+3\cos^2\theta)/4$ is an accidental coincidence due to the special value of l.

Table 1. Differential cross sections for photon-proton scattering (up to $J=3/2$).

$(8/\lambda^2)\,d\sigma/d\omega$	$M1(J=1/2)$ R_{M1}^{-}	$E1(J=1/2)$ R_{E1}^{-}	$M1(J=3/2)$ R_{M1}^{+}	$E1(J=3/2)$ R_{E1}^{+}	$M_2(J=3/2)$ R_{M2}^{-}	$E2(J=3/2)$ R_{E2}^{-}
$M1(J=1/2)$ R_{M1}^{-*}	1	$\cos\theta$	$(3\cos^2\theta-1)/4$	$\frac{1}{2}\cos\theta$	$\frac{3}{2}\cos\theta$	$\frac{3}{4}(3\cos^2\theta-1)$
$E1(J=1/2)$ R_{E1}^{-*}	$\cos\theta$	1	$\frac{1}{2}\cos\theta$	$(3\cos^2\theta-1)/4$	$\frac{3}{4}(3\cos^2\theta-1)$	$\frac{3}{2}\cos\theta$
$M1(J=3/2)$ R_{M1}^{+*}	$(3\cos^2\theta-1)/4$	$\frac{1}{2}\cos\theta$	$(7+3\cos^2\theta)/4$	$\frac{5}{2}\cos\theta$	$\frac{5}{2}\cos\theta\,(2\cos^2\theta-1)$	$\frac{3}{4}(3\cos^2\theta-1)$
$E1(J=3/2)$ R_{E1}^{+*}	$\frac{1}{2}\cos\theta$	$(3\cos^2\theta-1)/4$	$\frac{5}{2}\cos\theta$	$(7+3\cos^2\theta)/4$	$\frac{3}{4}(3\cos^2\theta-1)$	$\frac{3}{2}\cos\theta\,(2\cos^2\theta-1)$
$M2(J=3/2)$ R_{M2}^{-*}	$\frac{3}{2}\cos\theta$	$\frac{3}{4}(3\cos^2\theta-1)$	$\frac{5}{2}\cos\theta\,(2\cos^2\theta-1)$	$\frac{5}{4}(3\cos^2\theta-1)$	$(7+3\cos^2\theta)/4$	$\cos\theta(8\cos^2\theta-3)/2$
$E2(J=3/2)$ R_{E2}^{-*}	$\frac{3}{4}(3\cos^2\theta-1)$	$\frac{5}{2}\cos\theta$	$\frac{3}{4}(3\cos^2\theta-1)$	$\frac{3}{2}\cos\theta\,(2\cos^2\theta-1)$	$\cos\theta(8\cos^2\theta-3)/2$	$(7+3\cos^2\theta)/4$

§ 3. Polarization of the recoil nucleon

Let us calculate the polarization of the recoil nucleon in a way similar to the pion-nucleon scattering and the photo-pion production. For the unpolarized initial photon, the polarization $\boldsymbol{P}$ of the recoil nucleon is given as follows:

$$\boldsymbol{P}=X\boldsymbol{n}/Y. \tag{3}$$

Where $\boldsymbol{n}$ is the unit vector perpendicular to the scattering plane and $Y=d\sigma/d\omega$, eq. (1). X is given as

$$X=(1/8k^2)\sum_{N,N'}\sum_{l,l'}\sum_{\epsilon,\epsilon'}(R^{\epsilon}_{Nl})^{*}(R^{\epsilon'}_{N'l'})G(Nl\epsilon,\ N'l'\epsilon';\ \theta). \tag{4}$$

The explicit forms for the factors G can easily be calculated (see Table 2).

We can again find that G is invariant under the simultaneous interchanges $R_{Ml}^{-}\rightleftarrows R_{El}^{+}$ and $R_{Ml}^{-}\rightleftarrows R_{El}^{-}$ concerning all l's. This is quite different from the cases of the pion-nucleon scattering and the photo-pion production. Thus, in the case of the photon-nucleon scattering, even though the polarization of the recoil nucleons is observed, we cannot avoid the ambiguity

in choosing R'_{EI} and R'_{MI} solely by the kinematics. To distinguish between them we should use a specified dynamical theory.

Table 2. Polarization of recoil nucleons for photon-proton scattering (up to $J=3/2$).

$(8/i\lambda^2 \sin\theta)X$	$M1(J=1/2)$ R_{M1}^{-*}	$E1(J=1/2)$ R_{E1}^{-*}	$M1(J=3/2)$ R_{M1}^{+*}	$E1(J=3/2)$ R_{E1}^{+*}	$M2(J=3/2)$ R_{M2}^{-*}	$E2(J=3/2)$ R_{E2}^{-*}
$M1(J=1/2)$ R_{M1}^{-}	0	0	$\frac{3}{4}\cos\theta$	3/4	$-3/4$	$-\frac{3}{4}\cos\theta$
$E1(J=1/2)$ R_{E1}^{-}	0	0	3/4	$\frac{3}{4}\cos\theta$	$-\frac{3}{4}\cos\theta$	$-3/4$
$M1(J=3/2)$ R_{M1}^{+}	$-\frac{3}{4}\cos\theta$	$-3/4$	0	0	$-3(\cos^2\theta-1)/2$	0
$E1(J=3/2)$ R_{E1}^{+}	$-3/4$	$-\frac{3}{4}\cos\theta$	0	0	0	$-3(\cos^2\theta-1)/2$
$M2(J=3/2)$ R_{M2}^{-}	3/4	$\frac{3}{4}\cos\theta$	$3(\cos^2\theta-1)/2$	0	0	0
$E2(J=3/2)$ R_{E2}^{-}	$\frac{3}{4}\cos\theta$	3/4	0	$3(\cos^2\theta-1)/2$	0	0

§ 4. The relationship between the amplitudes of the photon-nucleon scattering, the pion-nucleon scattering, and the photo-pion production

Next we examine the relationship between the phases of the amplitudes of the pion-nucleon scattering, photo-pion production and the photon-nucleon scattering. As is well known,

$$SR^{+}=-R \tag{5}$$

by virtue of the unitarity of the S-matrix, $S=1+R$.

Therefore, in energy regions where the production of two pions or more can be neglected, one can obtain the relations;

$$(\pi|S|\pi)(\pi|R^{+}|\pi)+(\pi|S|\gamma)(\gamma|R^{+}|\pi)=-(\pi|R|\pi), \tag{6}$$

$$(\pi|S|\pi)(\pi|R^{+}|\gamma)+(\pi|S|\gamma)(\gamma|R^{+}|\gamma)=-(\pi|R|\gamma). \tag{7}$$

Where $|\pi)$ or $|\gamma)$ represents a state which consists of a pion and a nucleon or a photon and a nucleon, respectively. We employ the representation in which the total angular momentum, its z-component, and the parity are diagonal.

For simplicity, we assume that the (3/2, 3/2) state of the pion-nucleon system plays such an important role that we have only to take this state into account.* In the following we indicate with a suffix 3 the p-state with $J=3/2$ in the pion-nucleon

*) The cases in which other states are not negligible can also be treated in a similar way, but we leave it out now.

system or the magnetic dipole state with $J=3/2$ in the case of the photon-nucleon system. Then, if we denote

$$\left.\begin{aligned}(\pi|S|\pi)&=\xi_3\exp(2i\beta_3),\\(\pi|S|\gamma)&=(\pi|R|\gamma)=a_3\exp(2i\alpha_3),\\(\gamma|S|\gamma)&=\eta^+_{M1}\exp(2i\delta^+_{M1}),\end{aligned}\right\}\tag{8}$$

we obtain approximately

$$(\xi_3)^2=1-(a_3)^2,\tag{9}$$

$$\eta^+_{M1}=\xi_3=\sqrt{1-(a_3)^2},\tag{10}$$

$$\delta^+_{M1}=2\alpha_3-\beta_3\ \text{(more generally, } \delta^+_{M1}=2\alpha_3-\beta_3+n\pi,\ n=0,\ \pm1,\ \pm2,\ldots\ldots),\tag{11}$$

from the relations (6), (7) and (8). Here, in eqs. (9), (10) and (11), $(a_3)^2$ represents the resultant effect of the photo-$\pi^\pm$ and the photo-π^0 production. Here $(a_3)^2$ and δ^-_{M1} are of the order of e^2. Neglecting these small quantities, we have the well-known relation between the phases of the pion-nucleon scattering and the photo-pion production,

$$2\alpha_3=\beta_3.$$

It must be noted here that δ^+_{M1} is small compared with α_3 and β_3 so that, even in the resonance energy region where $2\alpha_3$ and β_3 are near $90°$, δ^-_{M1} will still be small. This circumstance will be utilized in later discussions.

§ 5. On the forward-backward asymmetry of the angular distribution of the photon-proton scattering

The asymmetry of the angular distribution of the photon-proton scattering is due to cross terms. Since we have assumed that the large contribution to the process comes from $M1(J=3/2)$, we examine interferences between this state and others. If we restrict our consideration to the states with $l\leqslant 1$, we only have $E1(J=1/2)$ and $E1(J=3/2)$ to take into account (see Table 1)*. The contribution from interferences of $M1(J=3/2)$ and $E1(J=1\pm1/2)$ are expressed in the forms of**

$$(\text{products of } R\text{-matrix elements})\times(\text{a positive constant})\times\cos\theta.$$

Therefore, to find out asymmetry of the angular distribution, we have only to examine the sign of the products of the R-matrix elements.

In general, for a process with absorption,

$$R=\eta e^{2i\delta}-1=(\eta-1)+2i\eta e^{i\delta}\sin\delta\tag{12}$$

and we have

---

*) The interference term between $M1(J=1/2)$ and $M1(J=3/2)$ does not contain the odd powers of $\cos\theta$ and is neglected here.

**) $M1$ and $E1$ denote the magnetic dipole and the electric dipole states respectively.

$$(R^{-}_{M1})^{*}R^{\pm}_{E1}+(R^{\pm}_{E1})^{*}R^{-}_{M1}=2(\eta^{-}_{M1}-1)(\eta^{\pm}_{E1}-1)+\{-4\eta^{\pm}_{E1}(\eta^{-}_{M1}-1)\sin^{2}\delta^{\pm}_{E1}\}$$
$$+\{-4\eta^{-}_{M1}(\eta^{\pm}_{E1}-1)\sin^{2}\delta^{-}_{M1}\}+8\eta^{-}_{M1}\eta^{\pm}_{E1}\sin\delta^{-}_{M1}\sin\delta^{\pm}_{E1}\cos(\delta^{-}_{M1}-\delta^{\pm}_{E1}). \quad (13)$$

Since $0\leqq\eta_{M1}\leqq 1$, $0\leqq\eta^{\pm}_{E1}\leqq 1$, the first, second, and the third terms of eq. (13) are positive definite. Accordingly, contributions from them give an angular distribution peaked forward. At energies lower than the resonance (100 Mev$\sim$240 Mev), however, the experiments give the angular distribution peaked backward. Hence the contribution from the fourth term must surpass the contributions from the others, and its sign must be negative. Here δ^{-}_{M1} and $\delta^{\pm}_{E1}$ are of the order of e^2 as mentioned in § 4, so that we can consider $\cos(\delta^{-}_{M1}-\delta^{\pm}_{E1})>0$. Accordingly, the product $\sin\delta_{M1}\sin\delta^{\pm}_{E1}$ must be negative. In the following we discuss about this fourth term.

§ 6. The "shadow" scattering due to the absorption process, the photo-pion production

As mentioned in § 1, we here adopt an important assumption that the angular distribution of the photon-proton scattering changes at or about the resonance energy from the one peaked backward in lower energies to the one peaked forward in higher energies. We attribute this change of the distribution to a change of the sign of $\sin\delta_{M1}\sin\delta^{\pm}_{E1}$ mentioned above. It is more natural to consider that δ_{M1} changes its sign at the resonance energy than to consider that both of δ^{-}_{E1} and δ^{-}_{E1} do. A result calculated with the Chew-Low formalism[7] is also consistent with this assumption. Thus $\delta_{M1}=0$ at the resonance energy, so that

$$R^{-}_{M1}=\eta^{-}_{M1}-1 \qquad \text{(at resonance)}. \quad (14)$$

Accordingly we obtain the cross section of the photon-proton scattering in $M1(J=3/2)$ as follows*

$$d\sigma'_{\gamma-p}/d\omega=(1/32k^2)(\eta^{+}_{M1}-1)^2(7+3\cos^2\theta)=(1/32k^2)(a_3)^4(7+3\cos^2\theta)/4. \quad (15)$$

Also the cross sections of the photo-pion production in $M1(J=3/2)$ are[13]

$$d\sigma'_{\gamma-\pi^0}/d\omega=(1/24k^2)(a_3)^2(5-3\cos^2\theta), \quad (16)$$

$$d\sigma'_{\gamma-\pi^+}/d\omega=(1/2)(1/24k^2)(a_3)^2(5-3\cos^2\theta), \quad (17)$$

where the factor $\frac{1}{2}$ in the right-hand side of eq. (17) arises because we take only the state with the total isotopic spin $I=3/2$ into account.**

It is well known that in the process of photo-π^0 production the $M1(J=3/2)$ state plays such a predominant role that one can neglect contributions from other states. Therefore, we obtain $(d\sigma'_{\gamma-p}/d\omega)_{90^\circ}$ by equating $(d\sigma'_{\gamma-\pi^0}/d\omega)_{90^\circ}$ (eq. (16)) to the experimental value of $(d\sigma_{\gamma-\pi^0}/d\omega)_{90^\circ}$. Although experimental data of the photo-π^0 production

*) $d\sigma'/d\omega$ denotes the partial cross section in the state $M1(J=3/2)$ (with the total isospin $I=3/2$).

**) It must be noted that the effect of the shadow scattering due to the photo-production of both π^+ and π^0 are taken into account.

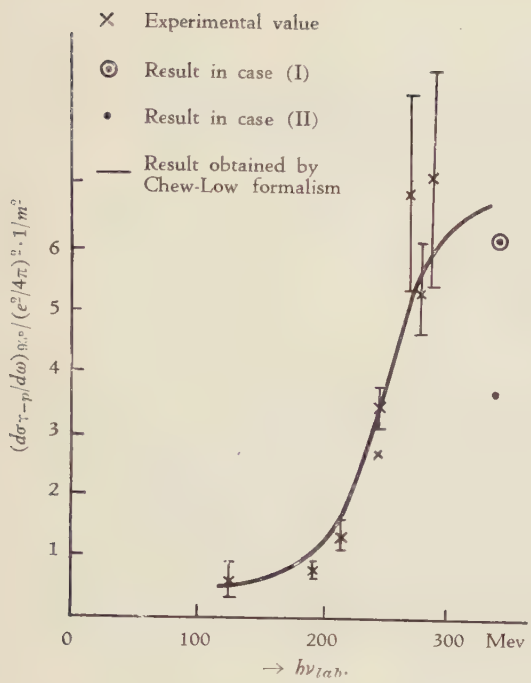

Fig. 1.

near the resonance energy region have been well accumulated, there are still some uncertainties.* Hence we calculate $(d\sigma'_{\gamma-p}/d\omega)_{90°}$ adopting two values for $(d\sigma_{\gamma-\pi^0}/d\omega)_{90°}$ (c.f., Fig. 1)

(I)

$$(d\sigma_{\gamma-\pi^0}/d\omega)_{90°}=2.6\times 10^{-29}\ \mathrm{cm}^2/\mathrm{sterad}$$
$$(\text{or } \sigma_{\gamma-\pi^0}=2.6\times 10^{-28}\ \mathrm{cm}^2),$$

then

$$(d\sigma'_{\gamma-p}/d\omega)_{90°}=14.9\times 10^{-32}\ \mathrm{cm}^2/\mathrm{sterad}$$
$$=(e^2/4\pi)^2\times(1/m^2)\times(6.34), \quad (18)$$

(II)

$$(d\sigma_{\gamma-\pi^0}/d\omega)_{90°}=2\times 10^{-29}\ \mathrm{cm}^2/\mathrm{sterad}$$
$$(\text{or } \sigma_{\gamma-\pi^0}=2\times 10^{-28}\ \mathrm{cm}^2),$$

then

$$(d\sigma'_{\gamma-p}/d\omega)_{90°}=8.82\times 10^{-32}\ \mathrm{cm}^2/\mathrm{sterad}$$
$$=(e^2/4\pi)^2\times(1/m^2)\times(3.75), \quad (19)$$

where $e^2/4\pi \doteqdot 1/137$, and m = nucleon mass.

§ 7. Summary

As is seen in § 6, our calculated value of $\sigma'_{\gamma-p}$ strongly depends on the value of $\sigma_{\gamma-\pi^0}$, and we cannot draw a definite conclusion until we have the accurate value of the latter. However, we can qualitatively conclude that the steep rise of the cross section of the photon-proton scattering about 300 Mev can be attributed mostly to the "shadow" scattering due to the photo-pion production.

*) Although, in the photo-pion production, the photon energy E_R corresponding to the resonance energy (where the kinetic energy of the pion = 190 Mev) of the pion-nucleon scattering is 340 Mev, experiments show the maximum at somewhat lower energies. Here, however, we calculate the cross sections for E_R = 340 Mev (k = 260 Mev/c).

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Note on the Ferromagnetic Resonance Absorption

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The theory of ferromagnetic resonance absorption is discussed in Hamiltonian form. The general formula of magnetic resonance absorption in a ferromagnetic single crystal is obtained when the applied magnetic field has arbitrary intensity and direction. This formula reduces to Kittel's formula in the case of infinitely strong applied field. When a magnetic field is applied in a direction near the hard axis, an additional resonance is possible in the non-aligned state of magnetization vector to the direction of the applied field. It is concluded, however, that the mechanism described above is not responsible for double peaks in iron single crystal which are reported by Kip-Arnold and in cobalt ferrite by Tannenwald. The resonance frequencies for antiferromagnetic crystals of ordinary two sublattices type and that of triangular lattice are obtained by a similar method.

§ 1. Introduction

Kittel[1] pointed out that in the theory of ferromagnetic resonance absorption, demagnetizing field played an important role. He has further discussed that for single crystals the magnetic field for resonance should be dependent on the angle which the external field makes with the crystal axis, owing to the effect of anisotropy. In the experiment of an iron single crystal by Kip and Arnold,[2] Kittel's theory agrees well with the result observed at 23675 Mc/sec, but it deviates systematically from the expected angular variation at 9260 Mc/sec. A similar deviation is observed in cobalt ferrite single crystal by Tannenwald.[3] In Kittel's theory, it is assumed that the external applied field is always great enough so that the magnetization is saturated and oriented parallel to the external field. In the case of lower frequencies, however, the torque of crystalline anisotropy energy prevents alignment of the direction of magnetization to the applied field for a certain range of angle of the applied field, so that Kittel's formula must be corrected. In this paper, the general formula of ferromagnetic resonance absorption will be given by the method of spin deviation.

In § 2, the theory of ferromagnetic resonance absorption is discussed in the Hamiltonian form which is expressed in the operators of 2nd order. The frequency of resonance absorption can be determined from the difference of eigenvalues of the Hamiltonian, we can give the real and imaginary parts of the susceptibility by treating the oscillating magnetic field as the time dependent perturbation. In § 3, the anisotropy energy is taken into account, and the general formula of magnetic resonance absorption in a ferromagnetic single crystal can be obtained when the applied magnetic field has

arbitrary intensity and direction. In § 4, the results obtained in § 3 are compared with the experimental results of the ferromagnetic resonance absorption in single iron crystal by Kip-Arnold, and with that in cobalt ferrite by Tannenwald. In § 5, the antiferromagnetic resonance absorption in an ordinary two-sublattices model is discussed. In § 6, the theory is applied to the antiferromagnetic crystal in triangular lattice, and three resonance frequencies can be obtained. One of them may belong to the ordinary microwave range. The others are still shorter waves, and are degenerate approximately. In § 7, some discussions are given.

§ 2. Ferromagnetic resonance absorption I

In ferromagnetic crystal, all spins are coupled by exchange interactions, so that we may regard them as a giant molecule which has spin S. S is the sum of effective spin of atoms in unit volume and it is defined, connecting with spontaneous magnetization M_s, as

$$M_s = g\mu S, \tag{2·1}$$

where g is spectroscopic splitting factor and μ is Bohr magneton.

We consider the resonance condition for a general ellipsoid with principal axes parallel to the x, y, z axes of the coordinate system. The demagnetizing factors are N_x, N_y, N_z. The static magnetic field is H_z; the oscillating field is $H_x = H_x^0 e^{i\omega t}$. First of all, we neglect the anisotropy energy, whose effect will be considered in the next section.

The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= -(H_x - \tfrac{1}{2}N_x M_x)M_x - (-\tfrac{1}{2}N_y M_y)M_y - (H_z - \tfrac{1}{2}N_z M_z)M_z \\ &= -(H_x - \tfrac{1}{2}N_x g\mu S_x)g\mu S_x - (-\tfrac{1}{2}N_y g\mu S_y)g\mu S_y - (H_z - \tfrac{1}{2}N_z g\mu S_z)g\mu S_z . \end{aligned} \tag{2·2}$$ *

The method of spin deviation can well be applied to obtain the resonance condition, because when the system absorbs an oscillating field quantum, z component of spin decreases by a unity. We write spin angular momentum operators in terms of the spin deviation operators introduced by Holstein and Primakoff ;[4]

$$\begin{aligned} S_x &= (S/2)^{1/2}\{(1 - a^* a/2S)^{1/2} a + a^*(1 - a^* a/2S)^{1/2}\}, \\ S_y &= -i(S/2)^{1/2}\{(1 - a^* a/2S)^{1/2} a - a^*(1 - a^* a/2S)^{1/2}\}, \\ S_z &= S - a^* a, \end{aligned} \tag{2·3}$$

and

$$[a,\ a^*] = 1, \tag{2·4}$$

where a and a^* are respectively destruction and creation operators of spin deviation. In

* The minus sign should be added to the Zeeman terms in (2·2), because the ratio of magnetic moment to angular momentum is negative for electron spin. For convenience' sake, however, we will omit it.

the present problem, S is so large that we can safely neglect the term $a^* a/2S$ in the square root. Then we can write them as follows :

$$\begin{aligned} S_x &= (S/2)^{1/2}(a+a^*), \\ S_y &= -i(S/2)^{1/2}(a-a^*), \\ S_z &= S-a^* a. \end{aligned} \tag{2·5}$$

We substitute (2·5) in (2·2) and omit the terms of lower order of S. We have

$$\begin{aligned} \mathcal{H} = &-H_x g\mu (S/2)^{1/2}(a+a^*) + \tfrac{1}{4} SN_x (g\mu)^2 (a+a^*)^2 - \tfrac{1}{4} SN_y (g\mu)^2 (a-a^*)^2 \\ &+ (H_z - N_z g\mu S) g\mu a^* a. \end{aligned} \tag{2·6}$$

If we introduce the real operators q and p defined as

$$a = 2^{-1/2}(q+ip), \quad a^* = 2^{-1/2}(q-ip), \tag{2·7}$$

we obtain the Hamiltonian form,

$$\begin{aligned} \mathcal{H} = &\tfrac{1}{2}(g\mu)\{H_z + (N_x - N_z) M_s\} q^2 + \tfrac{1}{2}(g\mu)\{H_z + (N_y - N_z) M_s\} p^2 \\ &- H_x g\mu S^{1/2} q. \end{aligned} \tag{2·8}$$

From this we can get the canonical equations of motion, which are well-known Bloch-Kittel's equation of motion.

In order to obtain the eigenvalues of (2·6), we use the following transformation,

$$\begin{aligned} a &= \cosh\theta\, b + \sinh\theta\, b^* \\ a^* &= \sinh\theta\, b + \cosh\theta\, b^* \end{aligned} \tag{2·9}$$

$$\tanh 2\theta = M_s (N_y - N_x) / \{2H_z + M_s (N_x + N_y - 2N_z)\}. \tag{2·10}$$

Then the Hamiltonian is written

$$\begin{aligned} \mathcal{H} = &-H_x g\mu (S/2)^{1/2} [\{H_z + (N_y - N_z) M_s\} / \{H_z + (N_x - N_z) M_s\}]^{1/4} (b+b^*) \\ &+ g\mu [\{H_z + (N_x - N_z) M_s\} \{H_z + (N_y - N_z) M_s\}]^{1/2} b^* b. \end{aligned} \tag{2·11}$$

We regard the linear term of the operator as the perturbation to the second order term of the operator. The selection rule for transition can determine the resonance frequency such as

$$\omega_0 = (g\mu/\hbar) [\{H_z + (N_x - N_z) M_s\} \{H_z + (N_y - N_z) M_s\}]^{1/2}. \tag{2·12}$$

This is the famous Kittel's formula. We calculate the transition probability w by the time dependent perturbation,

$$w = (\pi/\hbar^2) H_x^{0^2} g\mu\chi_0 [\{H_z + (N_x - N_z) M_s\} \{H_z + (N_y - N_z) M_s\}]^{1/2} \delta(\omega - \omega_0), \tag{2·13}$$

where

$$\chi_0 = M_s / \{H_z + (N_x - N_z) M_s\}. \tag{2·14}$$

The power absorbed A is defined as

$$A=2\omega\chi'' H_x^{0^2}=\hbar\omega_0 w. \tag{2·15}$$

From (2·13) and (2·15), we can obtain the imaginary part of the susceptibility χ'',

$$\chi''=\pi/2\cdot\omega_0^2/\omega\cdot\chi_0\delta(\omega-\omega_0). \tag{2·16}$$

Now, we use the Kramers-Kronig relation;

$$\chi'(\omega)=\chi_0-\omega\int_0^\infty\phi(\tau)\sin\omega\tau\, d\tau, \tag{2·17}$$

$$\chi''(\omega)=\omega\int_0^\infty\phi(\tau)\cos\omega\tau\, d\tau, \tag{2·18}$$

where $\phi(\tau)$ is the relaxation function.[5] First, we calculate $\phi(\tau)$ from (2·16) and (2·18);

$$\phi(\tau)=\chi_0\cos\omega_0\tau. \tag{2·19}$$

Then we can determine from (2·17), the real part of the susceptibility:

$$\chi'(\omega)=\frac{\chi_0}{1-(\omega/\omega_0)^2}. \tag{2·20}$$

This result is also in agreement with that by Kittel.

We have now demonstrated that the results obtained from the Hamiltonian form as proposed in the present paper are equivalent to the solutions of Bloch-Kittel's equation of motion. In the next section, we will treat the general case in a ferromagnetic crystal in the Hamiltonian form.

3. Ferromagnetic resonance absorption II

In this section we will consider the effect of the anisotropy energy in a cubic crystal. The anisotropy energy is written by

$$\mathcal{H}_{an}=K_1(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)+K_2\alpha_1^2\alpha_2^2\alpha_3^2\cdots\cdots \tag{3·1}$$

where K_1, K_2 are anisotropy constants and α_i are the direction cosines of the magnetization vector to the crystalline axes. We will write α_i as follows:

$$\alpha_1=S_x/S,\ \alpha_2=S_y/S,\ \alpha_3=S_z/S, \tag{3·2}*$$

where S_x, S_y and S_z are Cartesian components of operator $\mathbf{S}$. The dipolar interaction energy is divided into self energies of Lorentz field and demagnetizing field. Since the former is isotropic, we will neglect it, because it gives a homogeneous shift and does not change the level spacing in the same way as exchange energy does. The latter is

$$\mathcal{H}_{dem}=\tfrac{1}{2}(g\mu)^2(N_xS_x^2+N_yS_y^2+N_zS_z^2). \tag{3·3}$$

The Zeeman energy of the external field is

$$\mathcal{H}_f=-g\mu H\cdot S, \tag{3·4}$$

[1] We can neglect the non-commutability of operators, because it gives the terms of lower order of S.

where $\boldsymbol{H}$ is the external field and its direction is denoted by (Θ, Φ) in polar coordinates taking a crystalline axis as the polar axis.

Thus the total Hamiltonian is the sum of three parts;

$$\mathcal{H}=\mathcal{H}_f+\mathcal{H}_{an}+\mathcal{H}_{dem}. \tag{3·5}$$

The equilibrium direction of magnetization can be determined by minimizing the total energy. Let this direction be denoted by (θ, φ) in polar angles. We will consider this direction as the axis of quantization of the spins, and denote it as ζ axis. ξ and η axes are taken perpendicularly to it. Then we have the following relations,

$$\begin{aligned}
S_x&=\cos\theta\cos\varphi\, S_\xi-\sin\varphi\, S_\eta+\sin\theta\cos\varphi\, S_\zeta,\\
S_y&=\cos\theta\sin\varphi\, S_\xi+\cos\varphi\, S_\eta+\sin\theta\sin\varphi\, S_\zeta,\\
S_z&=-\sin\theta\, S_\xi+\cos\theta\, S_\zeta.
\end{aligned} \tag{3·6}$$

As explained in the previous section, we introduce the spin deviation operators,

$$\begin{aligned}
S_\xi&=(S/2)^{1/2}(a+a^*),\\
S_\eta&=-i(S/2)^{1/2}(a-a^*),\\
S_\zeta&=S-a^*a.
\end{aligned} \tag{3·7}$$

We substitute (3·6) and (3·7) in (3·5) and omit the terms of small quantity. We have

$$\mathcal{H}=D_0+D_1a+D_1^*a^*+D_2a^2+D_2^*a^{*2}+D_{\cdot\cdot}a^*a, \tag{3·8}$$

where

$$\begin{aligned}
D_0=&\tfrac{1}{4}K_1(\sin^4\theta\sin^2 2\varphi+\sin^2 2\theta)+(1/16)K_2\sin^2\theta\sin^2 2\theta\sin^2 2\varphi\\
&-g\mu SH\{\sin\theta\sin\Theta\cos(\varphi-\Phi)+\cos\theta\cos\Theta\}\\
&+\tfrac{1}{2}(g\mu S)^2(N_x\sin^2\theta\cos^2\varphi+N_y\sin^2\theta\sin^2\varphi+N_z\cos^2\theta),
\end{aligned} \tag{3·9a}$$

$$\begin{aligned}
D_1=&\tfrac{1}{2}(K_1/S)(S/2)^{1/2}(\sin 4\theta+\sin^2\theta\sin 2\theta\sin^2 2\varphi-i\sin^3\theta\sin 4\varphi)\\
&+\tfrac{1}{4}(K_2/S)(S/2)^{1/2}\{\sin^2\theta\sin 2\theta\sin^2 2\varphi(2-3\sin^2\theta)\\
&\qquad-\tfrac{1}{2}i\sin\theta\sin^2 2\theta\sin 4\varphi\}\\
&-g\mu(S/2)^{1/2}H\{\cos\theta\sin\Theta\cos(\varphi-\Phi)-\sin\theta\cos\Theta+i\sin\Theta\sin(\varphi-\Phi)\\
&+\tfrac{1}{2}(g\mu)^2S(S/2)^{1/2}\{\sin 2\theta(N_x\cos^2\varphi+N_y\sin^2\varphi-N_z)\\
&\qquad+i\sin\theta\sin 2\varphi(N_x-N_y)\},
\end{aligned} \tag{3·9b}$$

$$\begin{aligned}
D_2=&\tfrac{3}{4}(K_1/S)(-\sin^2 2\theta+2\sin^2\theta\sin^2 2\varphi-\sin^4\theta\sin^2 2\varphi\\
&\qquad-i\sin^2\theta\cos\theta\sin 4\varphi)\\
&+(1/8)(K_2/S)\{-\sin^2 2\theta+\sin^2\theta(12-26\sin^2\theta+15\sin^4\theta)\sin^2 2\varphi\\
&\qquad+2i\sin^2\theta\cos\theta(-3+5\sin^2\theta)\sin 4\varphi\}
\end{aligned}$$

$$+\tfrac{1}{4}(g\mu)^2 S\{N_x(\cos^2\theta\cos^2\varphi-\sin^2\varphi+i\cos\theta\sin 2\varphi)$$
$$+N_y(\cos^2\theta\sin^2\varphi-\cos^2\varphi-i\cos\theta\sin 2\varphi)$$
$$+N_z\sin^2\theta\}, \tag{3·9c}$$

$$D_0=\tfrac{1}{2}(K_1/S)(4-5\sin^2 2\theta-5\sin^4\theta\sin^2 2\varphi)$$
$$+\tfrac{1}{4}(K_2/S)(\sin^2 2\theta-20\sin^4\theta\sin^2 2\varphi+21\sin^6\theta\sin^2 2\varphi)$$
$$+g\mu H\{\cos\theta\cos\Theta+\sin\theta\sin\Theta\cos(\varphi-\Phi)\}$$
$$+\tfrac{1}{2}(g\mu)^2 S\{N_x(1-3\sin^2\theta\cos^2\varphi)+N_y(1-3\sin^2\theta\sin^2\varphi)$$
$$+N_z(1-3\cos^2\theta)\}. \tag{3·9d}$$

We will determine the values θ and φ. This can be done by minimizing D_0 with regard to θ and φ. But it is proved[6] that they are equivalent to the condition of eliminating the linear terms of the operators. We obtain

Real Part $D_1=0$:

$$\tfrac{1}{2}K_1(\sin 4\theta+\sin^2\theta\sin 2\theta\sin^2 2\varphi)$$
$$+\tfrac{1}{4}K_2\sin^2\theta\sin 2\theta\sin^2 2\varphi(2-3\sin^2\theta)$$
$$-M_s H\{\cos\theta\sin\Theta\cos(\varphi-\Phi)-\sin\theta\cos\Theta\}$$
$$+\tfrac{1}{2}M_s^2\sin 2\theta(N_x\cos^2\varphi+N_y\sin^2\varphi-N_z)=0. \tag{3·10a}$$

Imaginary Part $D_1=0$:

$$K_1\sin^3\theta\sin 4\varphi+\tfrac{1}{4}K_2\sin\theta\sin^2 2\theta\sin 4\varphi$$
$$+2M_s H\sin\Theta\sin(\varphi-\Phi)-M_s^2\sin\theta\sin 2\varphi(N_x-N_y)=0. \tag{3·10b}$$

If the magnitude of the external field increases infinitely, θ and φ will be equal to Θ and Φ respectively, and the magnetization will be parallel to the external field.

In order to diagonalize (3·8), we introduce real operators q and p by the relations,

$$a=\xi 2^{-1/2}(q+ip),\quad a^*=2^{-1/2}\xi^{-1}(q-ip),\quad \xi^4=D_2^*/D_2, \tag{3·11}$$
$$[q,\ p]=i. \tag{3·12}$$

The Hamiltonian becomes

$$\mathcal{H}=D_0-\tfrac{1}{2}D_3+\tfrac{1}{2}(D_3+2D_2\xi^2)q^2+\tfrac{1}{2}(D_3-2D_2\xi^2)p^2, \tag{3·13}$$

and eigenvalues are given by

$$E_n=D_0-\tfrac{1}{2}D_3+(D_3^2-4|D_2|^2)^{1/2}(n+\tfrac{1}{2}),\quad n=0,\ 1,\ 2,\cdots \tag{3·14}$$

Hamiltonian of an oscillating field applied to x direction is

$$\mathcal{H}_{osc}=-g\mu H_x S_x=-g\mu H_x\sin\theta\cos\varphi(S+\tfrac{1}{2})$$
$$-g\mu H_x(S/2)^{1/2}\{(\cos\theta\cos\varphi+i\sin\varphi)\xi 2^{-1/2}(q+ip)$$
$$+(\cos\theta\cos\varphi-i\sin\varphi)2^{-1/2}\xi^{-1}(q-ip)\}$$
$$+\tfrac{1}{2}g\mu H_x\sin\theta\cos\varphi(q^2+p^2). \tag{3·15}$$

(3·15) is the perturbation to (3·13), and the selection rule for transition is

$$\Delta n = \pm 1, \tag{3·16}$$

$$\Delta n = \pm 2. \tag{3·17}$$

But the intensity of absorption line by (3·17) is negligibly small compared with that by (3·16), so that we can neglect (3·17).

According to (3·14) and (3·16), the resonance frequency is

$$\omega = \hbar^{-1}(D_3^2 - 4|D_2|^2)^{1/2}. \tag{3·18}$$

This is the general formula of ferromagnetic resonance absorption. In the special case, (3·18) reduces to the formula given by Kittel[1] and Bickford[7] and Artman[8]*.

4. Comparison with experiments in single crystal

Kip and Arnold[2] observed the ferromagnetic resonance absorption in (001) plane disk of iron single crystal, at various angles Φ between the external field and the [100] direction in the plane of the crystal. In this case, $N_x = N_y = 0$, $N_z = 4\pi$, $\theta = \pi/2$. If the external field is strong enough, we may put $\varphi = \Phi$ and $\theta = \pi/2$, and obtain from (3·18)

$$\omega = (g\,e/2\,mc)\left[\left(\frac{2K_1}{M_s}\cos 4\Phi + H\right)\left\{\frac{1}{4M_s}(6K_1 + K_2) + \frac{1}{4M_s}(2K_1 - K_2)\cos 4\Phi + H + 4\pi M_s\right\}\right]^{1/2}. \tag{4·1}$$

This is in good agreement with the experimental results at 23675 Mc/sec, when the values of constants are taken such as

$$K_1 = 3.0 \times 10^5 \text{ erg/c.c.}, \quad K_2 = 1.5 \times 10^5 \text{ erg/c.c.}$$
$$M_s = 1575 \text{ Gauss}, \quad g = 2.14. \tag{4·2}$$

In the case of 9260 Mc/sec, the assumption $\varphi = \Phi$ is no more right and the deviations from (4·1) can be seen in Fig. 1. (broken line). Instead of $\varphi = \Phi$, φ must be determined from the following equation,

$$(K_1/M_s)\sin 4\varphi = 2H\sin(\Phi - \varphi), \tag{4·3}$$

and the resonance frequency is

$$\omega = (g\,e/2\,mc)\left[\left\{\frac{2K_1}{M_s}\cos 4\varphi + H\cos(\varphi - \Phi)\right\}\left\{\frac{1}{4M_s}(6K_1 + K_2) + \frac{1}{4M_s}(2K_1 - K_2)\cos 4\varphi + H\cos(\varphi - \Phi) + 4\pi M_s\right\}\right]^{1/2}. \tag{4·4}$$

* The author thanks Professor P. E. Tannenwald for calling his attention to the work of Artman.

We can determine ω as the function of Φ from (4·3) and (4·4), and the results agree satisfactorily with the experiments as shown in Fig. 1. (Full line). In experimental results, however, a second resonance occurs at angles near [110] direction, and we will consider it later.

Tannenwald[3] measured the ferromagnetic resonance absorption at 23800 Mc/sec on cobalt ferrite single crystal as a function of magnetic field applied in various direction θ in the (110) plane. The results are shown in Fig. 2. In order to evaluate the values of g-factor, K_1/M_s and K_2/M_s, we utilize the data at [001], [111] and [110] directions of applied field, because in these cases, we easily see that $\varphi=\Phi=\pi/4$ and $\theta=\Theta$ are the solutions and the resonance frequency is

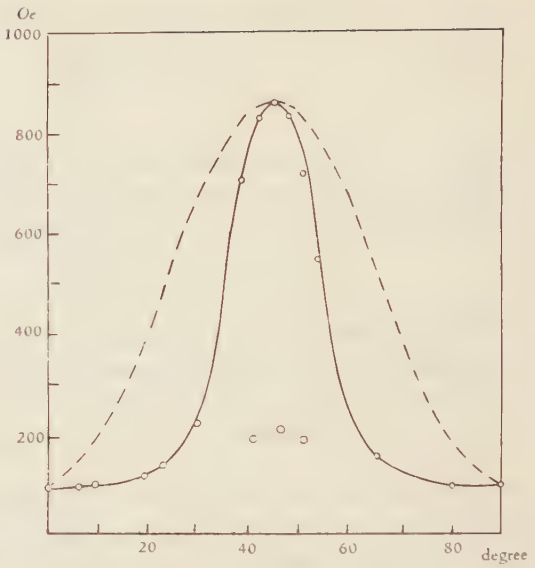

Fig. 1. Resonance magnetic field *vs.* angle between applied field and [100] direction in (001) plane disk of iron single crystal. The broken line represents Kittel's formula (4·1). The full line represents the result of the present theory, (4·3) and (4·4). Small circles represent the experimental result by Kip and Arnold at 9260 Mc/sec.

$$\omega=(g e/2mc)\left[\left\{\frac{2K_1}{M_s}(1-2\sin^2\theta-\frac{3}{8}\sin^2 2\theta)-\frac{1}{8}\frac{K_2}{M_s}\sin^2 2\theta(2+3\sin^2\theta)+H\right\}\right.$$
$$\times\left\{\frac{K_1}{M_s}(2-\sin^2\theta-3\sin^2 2\theta)+\frac{1}{2}\frac{K_2}{M_s}\sin^2\theta\left(6-5\sin^2\theta\right.\right.$$
$$\left.\left.-\frac{9}{2}\sin^2 2\theta\right)+H\right\}\Big]^{1/2}. \qquad (4\cdot5)$$

The experimental data[3] are

magnetic field direction	resonance field intensity
[001]	2900 Oe
[111]	8000 Oe
[110]	10000 Oe.

(4·6)

From (4·5) and (4·6), we can obtain the following values,

$$g=2.54,\ K_1/M_s=1900,\ K_2/M_s=1750. \qquad (4\cdot7)$$

(4·5) is the Kittel's formula. If we use (4·5) for other angles except [001], [111] and [110], the deviations from experimental values can be seen in Fig. 2. (broken line).

In this case we must use (3·10) and (3·18) instead of (4·5), and the result obtained is in satisfactory agreement with the experimental data as shown in Fig. 2. (full line).

As mentioned above, a second resonance occurred at angles near the [110] direction in the experiment by Kip and Arnold,[1] and they thought that it was the result of incomplete alignment to the applied field. We can calculate resonance frequencies from (4·3) and (4·4). The result obtained is the full line in Fig. 3. Since in the experiment the operating frequency was 9260 Mc/sec, the mechanism described above would not be responsible for the double peaks. In cobalt ferrite too, the second peak observed by Tannenwald is not the resonance in non-aligned state as easily seen in Fig. 4.

One of the origins of the double peaks is an effect of the dynamical surface charges induced on the domain walls.[9] If we apply weak magnetic field in [110] direction in iron sheet of (001) plane, several domain structures may be possible to exist. We will

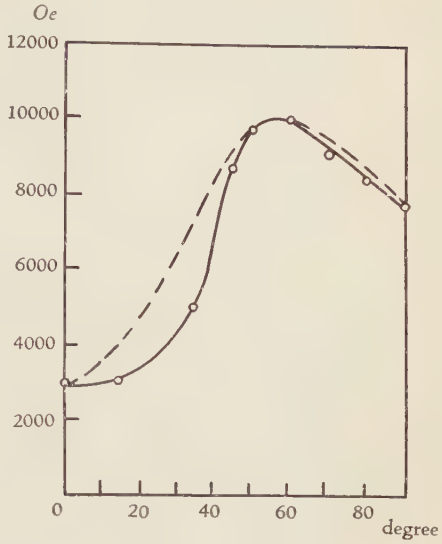

Fig. 2. The resonance magnetic field *vs.* the angle between the applied field and [001] direction in (110) plane in cobalt ferrite single crystal. The broken line represents Kittel's formula, (4·5). The full line represents the result of the present theory, (3·10) and (3·18). Small circles represent the experimental result by Tannenwald at 23800 Mc/sec. at 90°C.

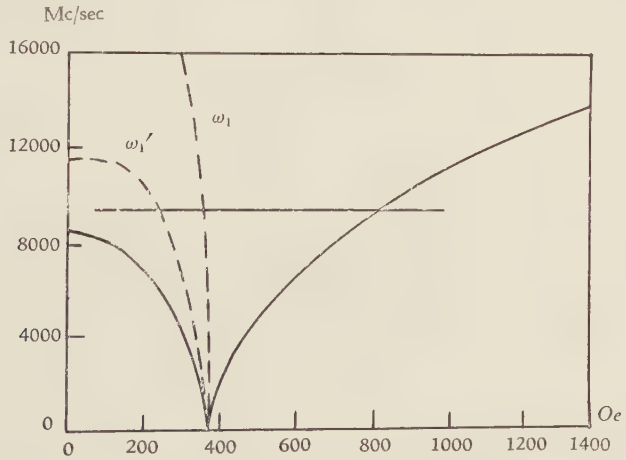

Fig. 3. Resonance condition in (001) plane in iron plate crystal. The external field is applied in [110] direction. The full line is the result calculated using (4·3) and (4·4) in which $\phi=\pi/4$. Broken lines are the results of (4·17) and (4·20). The horizontal line represents the operating frequency 9260 Mc/sec.

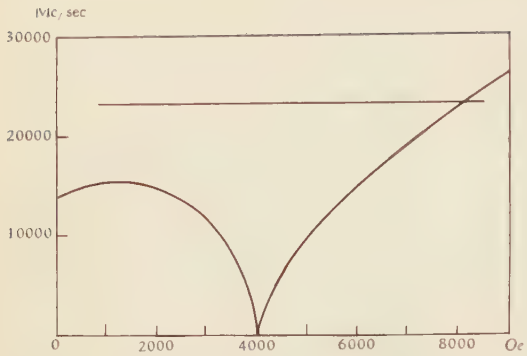

Fig. 4. Resonance condition in cobalt ferrite single crystal. The external field is applied in [110] direction. The full line is the result calculated using (3·10) and (3·18) in which $\phi=\pi/4$ and $\theta=\Theta=\pi/2$. The horizontal line represents the operating frequency 23800 Mc/sec.

consider only two types among them. In (A) type, as shown in Fig. 5, the specimen is divided into parallel striped domain with their walls perpendicular to the external field. This domain structure may be one of the lowest energy states. In (B) type, the specimen is divided into flat domains parallel to the disk plane. In reality, more complex domain structure between (A) type and (B) type may exist, and at the edge of any specimen, there must in general be a system of closure domains and these may have serious effects on the following treatment. So the calculations hereafter have only qualitative meaning.

At first, we will consider the case of (A) type. Let two kinds of domains be denoted by j and k respectively. The demagnetizing field from wall charge points to x' direction and its magnitude is $2\pi(g\mu)(S_{jx'}-S_{kx'})$, if the interval between walls is small enough compared to the thickness of the disk. On the other hand, macroscopic demagnetizing field is approximately the mean value for two kinds of domains. Thus the Hamiltonian of demagnetizing field is

$$\mathcal{H}_{dem}=\pi(g\mu)^2\left[\tfrac{1}{2}(uS_{j\eta}+vS_{j\zeta}+uS_{k\eta}-vS_{k\zeta})^2+(S_{j\zeta}+S_{k\zeta})^2\right], \qquad (4\cdot8)$$

where

$$u=\cos\varphi-\sin\varphi, \quad v=\cos\varphi+\sin\varphi. \qquad (4\cdot9)$$

Now, we put

$$\begin{aligned} &S_{j\xi}=(S/2)^{1/2}(a+a^*), \quad S_{k\xi}=(S/2)^{1/2}(b+b^*),\\ &S_{j\eta}=-i(S/2)^{1/2}(a-a^*), \quad S_{k\eta}=-i(S/2)^{1/2}(b-b^*),\\ &S_{j\zeta}=S-a^*a, \quad S_{k\zeta}=S-b^*b. \end{aligned} \qquad (4\cdot10)$$

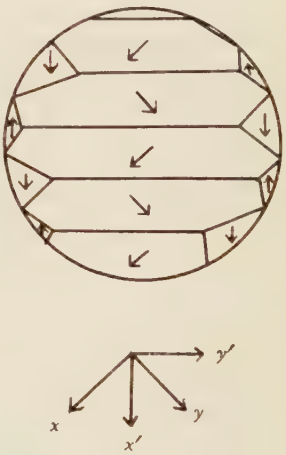

Fig. 5. (A) type domain structure in (001) plane disk of iron single crystal when the magnetic field is applied in [110] direction.

Total Hamiltonian is

$$\mathcal{H}=D_2'(a^2+a^{*2})+D_3'a^*a+D_2'(b^2+b^{*2})+D_3'b^*b+\mathcal{H}_{dem}, \qquad (4\cdot11)$$

where D_2' and D_3' are the residual part excluding the term of demagnetizing factor from D_2 and D_3 in (3·9).

The condition for determining the direction of magnetization is

$$(K_1/M_s)\sin 4\varphi = 2H\sin(\pi/4-\varphi). \tag{4·12}$$

We transform the operators by

$$\begin{aligned} a &= 2^{-1/2}(q+ip), \quad b = 2^{-1/2}(r+is), \\ a^* &= 2^{-1/2}(q-ip), \quad b^* = 2^{-1/2}(r-is), \end{aligned} \tag{4·13}$$

and further,

$$\begin{aligned} q &= 2^{-1/2}(Q_1+Q_2), \quad r = 2^{-1/2}(Q_1-Q_2), \\ p &= 2^{-1/2}(P_1+P_2), \quad s = 2^{-1/2}(P_1-P_2), \end{aligned} \tag{4·14}$$

$$[Q_1,\ P_1]=i, \quad [Q_2,\ P_2]=i. \tag{4·15}$$

The Hamiltonian becomes

$$\begin{aligned} \mathcal{H} = {} & (\tfrac{1}{2}D_3'+D_2'+2\pi M_s)Q_1^2+(\tfrac{1}{2}D_3'-D_2'+u^2\pi M_s)P_1^2 \\ & +(\tfrac{1}{2}D_3'+D_2')Q_2^2+(\tfrac{1}{2}D_3'-D_2')P_2^2. \end{aligned} \tag{4·16}$$

We have two resonance frequencies,

$$\begin{aligned} \omega_1 = (g\,e/2\,mc)\Bigg[& \left\{\frac{K_1}{M_s}(2-\sin^2 2\varphi)+\frac{K_2}{2M_s}\sin^2 2\varphi + H\cos\left(\frac{\pi}{4}-\varphi\right)+4\pi M_s\right\} \\ & \times\left\{\frac{2K_1}{M_s}\cos 4\varphi + H\cos\left(\frac{\pi}{4}-\varphi\right)+2\pi M_s(\cos\varphi-\sin\varphi)^2\right\}\Bigg]^{1/2}, \end{aligned} \tag{4·17}$$

$$\begin{aligned} \omega_2 = (g\,e/2\,mc)\Bigg[& \left\{\frac{K_1}{M_s}(2-\sin^2 2\varphi)+\frac{K_2}{2M_s}\sin^2 2\varphi + H\cos\left(\frac{\pi}{4}-\varphi\right)\right\} \\ & \times\left\{\frac{2K_1}{M_s}\cos 4\varphi + H\cos\left(\frac{\pi}{4}-\varphi\right)\right\}\Bigg]^{1/2}. \end{aligned} \tag{4·18}$$

ω_2 is so small that it is out of question. ω_1 is shown in Fig. 3.

Next, we will consider the case of type (B). The Hamiltonian of the demagnetizing field is

$$\mathcal{H}_{dem} = 2\pi(g\mu)^2(S_{jz}-S_{kz})^2+\pi(g\mu)^2(S_{jz}+S_{kz})^2. \tag{4·19}$$

In the same way as before, the resonance frequencies are

$$\begin{aligned} \omega_1' = (g\,e/2\,mc)\Bigg[& \left\{\frac{K_1}{M_s}(2-\sin^2\varphi)+\frac{1}{2}\frac{K_2}{M_s}\sin^2 2\varphi + H\cos(\varphi-\Phi)+4\pi M_s\right\} \\ & \times\left\{\frac{2K_1}{M_s}\cos 4\varphi + H\cos(\varphi-\Phi)\right\}\Bigg]^{1/2}, \end{aligned} \tag{4·20}$$

$$\begin{aligned} \omega_2' = (g\,e/2\,mc)\Bigg[& \left\{\frac{K_1}{M_s}(2-\sin^2\varphi)+\frac{1}{2}\frac{K_2}{M_s}\sin^2 2\varphi + H\cos(\varphi-\Phi)+8\pi M_s\right\} \\ & \times\left\{\frac{2K_1}{M_s}\cos 4\varphi + H\cos(\varphi-\Phi)\right\}\Bigg]^{1/2}. \end{aligned} \tag{4·21}$$

The equilibrium condition is

$$(K_1/M_s)\sin 4\varphi = 2H\sin(\Phi-\varphi). \tag{4·22}$$

ω_1' is also shown in Fig. 3.

Since the experimental value of low reasonance field is about 200 Oe, the calculated values ω_1 and ω_1' are not in good agreement with it. But considering that the low resonance field depends delicately on the domain structure, we cannot hope the quantitative agreement between the experimental value and the calculated values for ideal domain structure.*

5. Antiferromagnetic resonance absorption I

In antiferromagnetic substance, we regard each sublattice as a giant molecule with effective spin S respectively. We need not consider the demagnetizing effect but the exchange interaction. Anisotropy energy is written by second order terms as often done. z axis is the preferred direction. If we apply external field along x axis, the total Hamiltonian is

$$\mathcal{H} = 2Jz(\mathbf{S}_j\cdot\mathbf{S}_k) - g\mu H_x(S_{jx}+S_{kx}) - (K/S^2)(S_{jz}^2+S_{kz}^2), \tag{5·1}$$

where $J=|J'|/n$ and J' is the exchange integral, n is the number of spins, and z is the number of the nearest neighbors. The equilibrium directions of spins are inclined at $\pm\theta$ from z axis in xz plane. For j-sublattice, we use the transformation (3·6), in which we put $\varphi=0$. And for k-sublattice, we use (3·6), in which $\varphi=0$ and $\theta=-\theta$. Moreover we use the spin deviation operators,

$$\begin{aligned} &S_{j\xi} = (S/2)^{1/2}(a+a^*),\ S_{j\eta} = -i(S/2)^{1/2}(a-a^*),\ S_{jz} = S-a^*a,\\ &S_{k\xi} = (S/2)^{1/2}(b+b^*),\ S_{k\eta} = -i(S/2)^{1/2}(b^*-b),\ S_{kz} = -S+b^*b. \end{aligned} \tag{5·2}$$

We substitute (5·2) in (5·1), and neglect the terms of order 1 S. Elimination of linear term of operators gives

$$\sin\theta = M_sH/2(2JzS^2+K),\quad M_s=g\mu S. \tag{5·3}$$

By similar transformation as (4·13) and (4·14), (5·1) reduces to

$$\begin{aligned} \mathcal{H} = &(2JzS\cos 2\theta+\tfrac{1}{2}g\mu H\sin\theta+K/S\cdot\cos 2\theta)Q_1^2\\ &+(-2JzS\sin^2\theta+\tfrac{1}{2}g\mu H\sin\theta+K/S\cdot\cos^2\theta)P_1^2\\ &+(\tfrac{1}{2}g\mu H\sin\theta+K/S\cdot\cos 2\theta)Q_2^2\\ &+(2JzS\cos^2\theta+\tfrac{1}{2}g\mu H\sin\theta+K/S\cdot\cos^2\theta)P_2^2. \end{aligned} \tag{5·4}$$

From (5·4), the resonance frequencies can be determined as follows;

* Artman[8] independently calculated the double peaks in cobalt ferrite by the same mechanism as that described here. But his calculation is semi-quantitative, and the numerical calculation by the author shows that the theoretical value for the simple domain structure is not in good agreement with the experimental value.

$$\omega_1=(g\,e/2\,mc)\Big[\Big\{-\frac{4JzS^2\cos 2\theta}{M_s^2}+H\sin\theta+\frac{2K}{M_s}\cos 2\theta\Big\}$$
$$\times\Big\{-\frac{4JzS^2}{M_s}\sin^2\theta+H\sin\theta+\frac{2K}{M_s}\cos^2\theta\Big\}\Big]^{1/2}$$
$$=\hbar^{-1}(8JzK)^{1/2}. \tag{5·5}$$

$$\omega_2=(g\,e/2\,mc)\Big[\Big\{-\frac{4JzS^2\cos^2\theta}{M_s^2}+H\sin\theta+\frac{2K}{M_s}\cos^2\theta\Big\}\Big\{H\sin\theta+\frac{2K}{M_s}\cos 2\theta\Big\}\Big]^{1/2}$$
$$=\hbar^{-1}\{8JzK+(g\mu H)^2\}^{1/2}. \tag{5·6}$$

(5·5) and (5·6) agree exactly with the classical ones derived by Nagamiya[10] and Keffer-Kittel[11] and the method described above is essentially the same as that of antiferromagnetic spin wave theory by Kanamori-Yosida.[6]

When the applied field is in z direction, the Hamiltonian is

$$\mathcal{H}=2Jz(\boldsymbol{S}_j\cdot\boldsymbol{S}_k)-g\mu H_z(S_{jz}+S_{kz})-(K/S^2)(S_{jz}^2+S_{kz}^2). \tag{5·7}$$

In this case, the magnetizations do not rotate but are induced by the applied field. So we put

$$S_{jx}=(S_1/2)^{1/2}(a+a^*),\ S_{jy}=-i(S_1/2)^{1/2}(a-a^*),\ S_{jz}=S_1-a^*a,$$
$$S_{kx}=(S_2/2)^{1/2}(b+b^*),\ S_{ky}=-i(S_2/2)^{1/2}(b^*-b),\ S_{kz}=-S_2+b^*b,$$
$$S_1=S+\Delta S,\ S_2=-\Delta S. \tag{5·8}$$

By (4·13), we obtain

$$\mathcal{H}=(zJS_2+(K/S^2)S_1+\tfrac{1}{2}g\mu H_z)(q^2+p^2)$$
$$+(zJS_1+(K/S^2)S_2-\tfrac{1}{2}g\mu H_z)(r^2+s^2)$$
$$+2zJ(S_1S_2)^{1/2}(qr-ps). \tag{5·9}$$

Besides, we use the following transformation,

$$p=P_1\,\mathrm{ch}\,\theta+P_2\,\mathrm{sh}\,\theta,\quad r=-Q_1\,\mathrm{sh}\,\theta+Q_2\,\mathrm{ch}\,\theta,$$
$$q=Q_1\,\mathrm{ch}\,\theta-Q_2\,\mathrm{sh}\,\theta,\quad s=P_1\,\mathrm{sh}\,\theta+P_2\,\mathrm{ch}\,\theta,$$
$$\mathrm{th}\,2\theta=\frac{2zJ(S_1S_2)^{1/2}}{zJ(S_1+S_2)+(K/S^2)(S_1+S_2)}, \tag{5·10}$$

the Hamiltonian becomes

$$\mathcal{H}=\{-zJ(S_1S_2)^{1/2}\,\mathrm{sh}\,2\theta+(zJS_2+(K/S^2)S_1)\,\mathrm{ch}^2\theta$$
$$+(zJS_1+(K/S^2)S_2)\,\mathrm{sh}^2\theta+\tfrac{1}{2}g\mu H_z\}(Q_1^2+P_1^2)$$
$$+\{-zJ(S_1S_2)\,\mathrm{sh}\,2\theta+(zJS_2+(K/S^2)S_1)\,\mathrm{sh}^2\theta$$
$$+(zJS_1+(K/S^2)S_2)\,\mathrm{ch}^2\theta-\tfrac{1}{2}g\mu H_z\}(Q_2^2+P_2^2). \tag{5·11}$$

The resonance frequencies are

$$\omega_{1,2}=\hbar^{-1}[\{(zJ+K/S^2)^2(S_1+S_2)^2-S_1S_2(2zJ)^2\}^{1/2} \pm\{(zJ-K/S^2)(S_1-S_2)-g\mu H_z\}]$$
$$=\hbar^{-1}[\{8zJK+(zJ\chi_{//}H_z/g\mu)^2\}^{1/2}\pm(zJ\chi_{//}H_z/g\mu-g\mu H_z)], \quad (5\cdot12)$$

where

$$\chi_{//}=2g\mu JS/H_z. \quad (5\cdot13)$$

(5·12) is also in agreement with the classical one.

Similar calculations have been carried out for ferrimagnetism and we have obtained the same result as that by Wangsness.[12]

6. Antiferromagnetic resonance absorption II

If the exchange integral is antiferromagnetic sign in a crystal of triangular lattice, spin super-structure is not ordinary two sublattices but three sublattices, on which the spins incline at an angle of 120° one another, as easily understood from the consideration of the total energies. In fact, some antiferromagnetic crystals of NiAs type, in which the exchange interactions between triangular lattice planes are negligible, seem to be three sublattices antiferromagnetic mentioned above. We will consider the antiferromagnetic resonance in these crystals.

We regard each sublattice as a giant molecule with effective spin S respectively. We denote each sublattice by A, B and C. A Hamiltonian is the sum of exchange part $\mathcal{H}_{ex}$, anisotropy part $\mathcal{H}_{an}$ and Zeeman part $\mathcal{H}_f$. $\mathcal{H}_{ex}$ is

$$\mathcal{H}_{ex}=J\{(\boldsymbol{S}^A\cdot\boldsymbol{S}^B)+(\boldsymbol{S}^B\cdot\boldsymbol{S}^C)+(\boldsymbol{S}^C\cdot\boldsymbol{S}^A)\}, \quad (6\cdot1)$$

where $J=9|J'|/n$ and J' is the exchange integral and n is the number of total sites. We take the direction of A spin as z axis and take y axis perpendicular to the plane as shown in Fig. 6. In each sublattice, we use individual coordinate systems defined as follows,

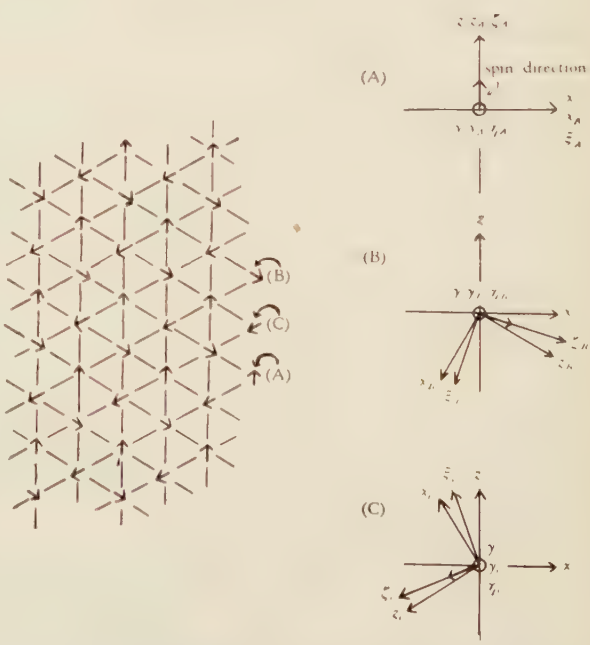

Fig. 6. Three sublattices and spin superstructure in triangular lattice and individual coordinates for A, B and C sublattices.

for A sublattice
$$x=x_A,\quad y=y_A,\quad z=z_A. \tag{6·2a}$$
for B sublattice
$$x=-\tfrac{1}{2}x_B+\sqrt{\tfrac{3}{4}}\,z_B,\quad y=y_B,\quad z=-\sqrt{\tfrac{3}{4}}\,x_B-\tfrac{1}{2}z_B. \tag{6·2b}$$
for C sublattice
$$x=-\tfrac{1}{2}x_C-\sqrt{\tfrac{3}{4}}\,z_C,\quad y=y_C,\quad z=\sqrt{\tfrac{3}{4}}\,x_C-\tfrac{1}{2}z_C. \tag{6·2c}$$

If we apply the external field along z axis, the spins of A sublattice are induced small quantity ΔS and that of B and C sublattices turn small angle θ toward external field. We turn the coordinate axes $+\theta$ and $-\theta$ about y axis respectively and define the following coordinate systems :

for A sublattice
$$x_A=\xi_A,\quad y_A=\eta_A,\quad z_A=\zeta_A. \tag{6·3a}$$
for B sublattice
$$x_B=\cos\theta\,\xi_B-\sin\theta\,\zeta_B,\quad y_B=\eta_B,\quad z_B=\sin\theta\,\xi_B+\cos\theta\,\zeta_B, \tag{6·3b}$$
for C sublattice
$$x_C=\cos\theta\,\xi_C+\sin\theta\,\zeta_C,\quad y_C=\eta_C,\quad z_C=-\sin\theta\,\xi_C+\cos\theta\,\zeta_C. \tag{6·3c}$$

The exchange Hamiltonian are represented by these coordinate systems as follows,

$$\begin{aligned}\mathcal{H}_{ex}=J[&S_\xi^A\{(\sqrt{\tfrac{3}{4}}\cos\theta+\tfrac{1}{2}\sin\theta)S_\zeta^B+(\sqrt{\tfrac{3}{4}}\sin\theta-\tfrac{1}{2}\cos\theta)S_\xi^B\}+S_\eta^AS_\eta^B\\
&+S_\zeta^A\{(-\tfrac{1}{2}\cos\theta+\sqrt{\tfrac{3}{4}}\sin\theta)S_\zeta^B+(-\tfrac{1}{2}\sin\theta-\sqrt{\tfrac{3}{4}}\cos\theta)S_\xi^B\}\\
&+S_\xi^A\{-(\sqrt{\tfrac{3}{4}}\cos\theta+\tfrac{1}{2}\sin\theta)S_\zeta^C+(\sqrt{\tfrac{3}{4}}\sin\theta-\tfrac{1}{2}\cos\theta)S_\xi^C\}+S_\eta^AS_\eta^C\\
&+S_\zeta^A\{(-\tfrac{1}{2}\cos\theta+\sqrt{\tfrac{3}{4}}\sin\theta)S_\zeta^C+(\tfrac{1}{2}\sin\theta+\sqrt{\tfrac{3}{4}}\cos\theta)S_\xi^C\}\\
&+\{(\sqrt{\tfrac{3}{4}}\cos\theta+\tfrac{1}{2}\sin\theta)S_\zeta^B+(\sqrt{\tfrac{3}{4}}\sin\theta-\tfrac{1}{2}\cos\theta)S_\xi^B\}\\
&\times\{-(\sqrt{\tfrac{3}{4}}\cos\theta+\tfrac{1}{2}\sin\theta)S_\zeta^C+(\sqrt{\tfrac{3}{4}}\sin\theta-\tfrac{1}{2}\cos\theta)S_\xi^C\}+S_\eta^BS_\eta^C\\
&+\{(-\tfrac{1}{2}\cos\theta+\sqrt{\tfrac{3}{4}}\sin\theta)S_\zeta^B+(-\tfrac{1}{2}\sin\theta-\sqrt{\tfrac{3}{4}}\cos\theta)S_\xi^B\}\\
&\times\{(-\tfrac{1}{2}\cos\theta+\sqrt{\tfrac{3}{4}}\sin\theta)S_\zeta^C+(\tfrac{1}{2}\sin\theta+\sqrt{\tfrac{3}{4}}\cos\theta)S_\xi^C\}].\end{aligned} \tag{6·4}$$

Anisotropy energy is composed of two parts; one is a uniaxial part $\mathcal{H}_{an}^{(1)}$ which keeps the spin direction in the plane, and the other is $\mathcal{H}_{an}^{(2)}$ which has six-fold symmetry in the plane.

$$\begin{aligned}\mathcal{H}_{an}&=\mathcal{H}_{an}^{(1)}+\mathcal{H}_{an}^{(2)}\\&=K\{(S_y^A/S')^2+(S_y^B/S)^2+(S_y^C/S)^2\}\\&\quad-K'[\{32(S_z^A/S')^6-48(S_z^A/S')^4+18(S_z^A/S')^2-1\}\\&\quad+\{(32/S^6)(-\sqrt{\tfrac{3}{4}}S_x^B-\tfrac{1}{2}S_z^B)^6-(48/S^4)(-\sqrt{\tfrac{3}{4}}S_x^B-\tfrac{1}{2}S_z^B)^4\\&\quad+(18/S^2)(-\sqrt{\tfrac{3}{4}}S_x^B-\tfrac{1}{2}S_z^B)^2-1\}\\&\quad+\{(32/S^6)(\sqrt{\tfrac{3}{4}}S_x^C-\tfrac{1}{2}S_z^C)^6-(48/S^4)(\sqrt{\tfrac{3}{4}}S_x^C-\tfrac{1}{2}S_z^C)^4\\&\quad+(18/S^2)(\sqrt{\tfrac{3}{4}}S_x^C-\tfrac{1}{2}S_z^C)^2-1\}].\end{aligned}\tag{6·5}$$

Zeeman part $\mathcal{H}_f$ is written as

$$\begin{aligned}\mathcal{H}_f&=-Hg\mu(S_z^A+S_z^B+S_z^C)\\&=-Hg\mu[S_\zeta^A+(-\tfrac{1}{2}\cos\theta+\sqrt{\tfrac{3}{4}}\sin\theta)S_\zeta^B+(-\tfrac{1}{2}\sin\theta-\sqrt{\tfrac{3}{4}}\cos\theta)S_\xi^B\\&\quad+(-\tfrac{1}{2}\cos\theta+\sqrt{\tfrac{3}{4}}\sin\theta)S_\zeta^C+(\tfrac{1}{2}\sin\theta+\sqrt{\tfrac{3}{4}}\cos\theta)S_\xi^C].\end{aligned}\tag{6·6}$$

We define spin deviation operators as follows

$$\begin{aligned}&S_\xi^A=(S'/2)^{1/2}(a+a^*),\ S_\xi^B=(S/2)^{1/2}(b+b^*),\ S_\xi^C=(S/2)^{1/2}(c+c^*)\\&S_\eta^A=-i(S'/2)^{1/2}(a-a^*),\ S_\eta^B=-i(S/2)^{1/2}(b-b^*),\ S_\eta^C=-i(S/2)^{1/2}(c-c^*),\\&S_\zeta^A=S'-a^*a,\ S_\zeta^B=S-b^*b,\ S_\zeta^C=S-c^*c,\end{aligned}\tag{6·7}$$

where

$$S'=S+\Delta S=S+\chi' H.\tag{6·8}$$

We substitute (6·7) in (6·4), (6·5) and (6·6), and neglect the terms more than third order of spin deviation operators. We can determine θ by means of eliminating the coefficient of linear term of operators;

$$\begin{aligned}&(\tfrac{1}{2}\sin\theta+\sqrt{\tfrac{3}{4}}\cos\theta)\{-JS'+JS(\cos\theta-\sqrt{3}\sin\theta)+Hg\mu\}\\&\qquad-(36K'/S)\sin\theta=0.\end{aligned}\tag{6·9}$$

Considering that K' is small quantity, the last term in (6·9) can be neglected and (6·9) is reduced to

$$\theta=(g\mu-J\chi')H/(\sqrt{3}JS).\tag{6·10}$$

The second order terms of operators in the Hamiltonian are

$$\begin{aligned}\mathcal{H}=&-\tfrac{1}{4}J(S'S)^{1/2}(1-\sqrt{3}\theta)(a+a^*)\{(b+b^*)+(c+c^*)\}\\&-\tfrac{1}{4}JS(1+2\sqrt{3}\theta)(b+b^*)(c+c^*)\\&-\tfrac{1}{2}J(SS')^{1/2}(a-a^*)\{(b-b^*)+(c-c^*)\}-\tfrac{1}{2}JS(b-b^*)(c-c^*)\end{aligned}$$

$$+\left(JS'+36\frac{K'}{S}\right)a^*a+JS(b^*b+c^*c)+(9K'/S)\{(b+b^*)^2+(c+c^*)^2\}$$

$$-\frac{1}{2}K\left\{\frac{1}{S'}(a-a^*)^2+\frac{1}{S}(b-b^*)^2+\frac{1}{S}(c-c^*)^2\right\}. \qquad (6\cdot11)$$

In (6·11), we used the relation (6·10). By the following transformation,

$$\begin{aligned} &2^{-1/2}(a+a^*)=q_1, \quad 2^{-1/2}(a-a^*)=ip_1, \\ &b+b^*=q_2+q_3, \quad b-b^*=i(p_2+p_3), \\ &c+c^*=-q_2+q_3, \quad c-c^*=i(-p_2+p_3). \end{aligned} \qquad (6\cdot12)$$

The Hamiltonian (6·11) becomes

$$\begin{aligned} \mathcal{H}=&\left(\frac{5}{4}JS-\frac{1}{2}JS'+18\frac{K'}{S}+\frac{1}{2}Hg\mu\right)q_2^2+\frac{K}{S}p_2^2 \\ &+\left(\frac{1}{2}JS'+18\frac{K'}{S}\right)q_1^2+\left(\frac{1}{2}JS'+\frac{K}{S'}+18\frac{K'}{S}\right)p_1^2 \\ &+\left(-\frac{1}{4}JS+\frac{1}{2}JS'+18\frac{K'}{S}-\frac{1}{2}Hg\mu\right)q_3^2+\left(JS+\frac{K}{S}\right)p_3^2 \\ &-(\tfrac{1}{2}S'/S)^{1/2}(JS'-Hg\mu)q_1q_3+J(2SS')^{1/2}p_1p_3. \end{aligned} \qquad (6\cdot13)$$

In (6·13), the mode 2 is separated, and the resonance frequency ω_2 by the mode 2 is

$$\omega_2=\hbar^{-1}\left[3JK\left\{1+\frac{2}{3}\left(\frac{g\mu}{J}-\chi'\right)\frac{H}{S}+24\frac{K'}{JS^2}\right\}\right]^{1/2}. \qquad (6\cdot14)$$

We exclude the terms of the mode 2 from the Hamiltonian, and transform the rest terms by following transformation,

$$\begin{aligned} &q_1=\cos\varphi Q_1+\sin\varphi Q_3, \quad q_3=-\sin\varphi Q_1+\cos\varphi Q_3, \\ &p_1=\cos\varphi P_1+\sin\varphi P_3, \quad p_3=-\sin\varphi P_1+\cos\varphi P_3, \end{aligned} \qquad (6\cdot15)$$

$$\tan 2\varphi=\frac{J(2SS')^{1/2}}{\left(JS+\frac{K}{S}\right)-\left(\frac{1}{2}JS'+\frac{K}{S'}+18\frac{K}{S}\right)}. \qquad (6\cdot16)$$

Then,

$$\mathcal{H}=AQ_1^2+BQ_3^2+CQ_1Q_3+DP_1^2+EP_3^2, \qquad (6\cdot17)$$

where

$$A=\frac{3}{4}JS+18\frac{K'}{S}+\frac{1}{2}(2J\chi'-g\mu)H,$$

$$B=18\frac{K'}{S},$$

$$C=\frac{\sqrt{2}}{6}\left(3Hg\mu-\frac{2K\chi' H}{S^2}+\frac{6K'}{S}\right), \tag{6·18}$$

$$D=\frac{K}{S}-\frac{2K\chi' H}{3S^2}+\frac{12K'}{S},$$

$$E=\frac{3}{2}JS+\frac{K}{S}+\frac{1}{2}J\chi' H-\frac{K\chi' H}{3S^2}+\frac{6K'}{S}.$$

Moreover, we use the following transformation

$$Q_1=(D/E)^{1/2}(\cos\phi Q_1'+\sin\phi Q_3'),\quad Q_3=-\sin\phi Q_1'+\cos\phi Q_3',$$

$$P_1=(E/D)^{1/2}(\cos\phi P_1'+\sin\phi P_3'),\quad P_3=-\sin\phi P_1'+\cos\phi P_3', \tag{6·19}$$

$$\tan 2\phi=C(DE)^{1/2}/\{BE-AD\}. \tag{6·20}$$

Thus the Hamiltonian can be written as

$$\mathcal{H}=\frac{1}{2}\left[\left(\frac{AD}{E}+B\right)-\left\{\left(\frac{AD}{E}-B\right)^2+\frac{C^2D}{E}\right\}^{1/2}\right]Q_1'^2+EP_1'^2$$

$$+\frac{1}{2}\left[\left(\frac{AD}{E}+B\right)+\left\{\left(\frac{AD}{E}-B\right)^2+\frac{C^2D}{E}\right\}^{1/2}\right]Q_3'^2+EP_3'^2. \tag{6·21}$$

Using (6·18) and (6·21), the resonance frequencies are

$$\omega_1=\hbar^{-1}(27JK')^{1/2}, \tag{6·22}$$

$$\omega_3=\hbar^{-1}\left[3JK\left\{1-\frac{2}{3}\left(\frac{g\mu}{J}-\chi'\right)\frac{H}{S}+\frac{24K'}{JS^2}+\frac{12K'}{K}\right\}\right]^{1/2}. \tag{6·23}$$

We have obtained three resonance frequencies (6·14), (6·22) and (6·23). Considering that $K'\ll K$, ω_1 is far smaller than ω_2 and ω_3. ω_2 and ω_3 are degenerate approximately, but this degeneracy is removed by the external field and anisotropy K'. If we tentatively assume that $K=10^5$ erg/c.c., $K'=10^2$ erg/c.c., $J=9|J'|/n=10^{-36}$ erg c.c., the wave lengths $\lambda_2=\lambda_3=0.4$ cm and $\lambda_1=2$ cm.

Though the calculations described above are carried out in the case that the external field is applied along one of the easiest directions, we can conclude that even in the case of applying the field along arbitrary direction, one resonance frequency is smaller than the other two resonance frequencies which are degenerate approximately.

7. Discussion

The method of determining the ferromagnetic resonance frequency by means of the Hamiltonian form is equivalent to the method of solving the equation of Bloch-Kittel as often done. But the former is sometimes easier than the latter when the external field has arbitrary intensity and direction. In addition to that, the Hamiltonian form is superior in respect that it can be directly connected with the perturbation theory.

When we want to determine the ferromagnetic resonance frequency in the spin wave

theory, we must obtain creation energy of the spin wave with zero wave number. It is the sum of Zeeman energy, anisotropy energy and dipolar interaction energy. The spin wave with zero wave number is the mode in which every spin moves in phase in classical picture. So the resonance frequencies by our theory and by the spin wave theory completely agreed at zero temperature. In fact, Van Vleck[13] proved that the classical demagnetizing factor can be used even in the microscopic standpoint for an ellipsoidally cut body. He further proved that if spins are nearly all parallel in cubic crystal, macroscopic anisotropy energy are equivalent to the result derived from the microscopic picture with a model having quadrupolar coupling between spins. These explanations can also be applied in the spin wave theory.

It is worth noting that in ferromagnetic substance, all spins move in phase by oscillating field, yet effective Hamiltonian of only one spin is meaningless unlike in paramagnetic substance, because any spin cannot absorb oscillating field quantum independently of one another owing to strong exchange interaction. In other words, if an oscillating field quantum is absorbed, in ferromagnetic crystal any spin does not decrease by a unity but all spins deviate in order of $1/n$ in classical picture, where n is the total number of spins.

In § 4 we showed that it was difficult to interpret that a second peak was caused in the non-alignment state of magnetization. But we can see from Fig. 3, that if lower frequency had been used, a resonance in the non-aligned state would have been possible to observe. In fact under such a condition Weiss-Anderson[14] observed in ferroxdure and Suhl[15] observed in nickel ferrite double resonances.

If a mechanism of double resonances is that discussed in § 4, resonance field will delicately depend on domain structure and line width will be broad. Double resonances in iron single crystal, manganese ferrite[16] and cobalt ferrite are broad indeed.

Recently, Suhl[17] proposed the mechanism of subsidiary absorption peaks in ferromagnetic resonance at high signal levels. He could explain with his theory the observations of Bloembergen and Wang[18] on subsidiary absorption.

In addition, it may be possible that another resonance due to wall displacement occurs. In any case, we hope that the domain structure will be studied in detail whenever the double resonances occur.

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On Solutions of He^3 and He^4

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A theory of He^3 and He^4 solutions has been developed on the basis of the following assumptions:

(i) He^3 mixes with the whole of He^4 (normal as well as superfluid),

(ii) the contribution to the entropy of mixing is due only to the normal part of He^4, and

(iii) the solution of He^3 in He^4 is not an "Ideal Solution" even above the λ-temperature. It obeys laws of a "Strictly Regular Solution".

Agreement between theory and experiment concerning the λ-temperature of the solution, molar specific heat of mixing, vapour pressure in solution, the distribution coefficient and the measurements of vapour pressures at dew points is found to be satisfactory.

§ 1. Introduction

Several theoretical attempts have been made in recent years to explain the properties of solutions of He^3 and He^4. The behaviour of solutions of He^3 and He^4 has been described following two different lines of thought. The thermodynamical description based on the properties of He^4 has been extended to solutions by making an ad hoc assumption about the entropy of mixing, or, starting from a more fundamental theory of the λ-transition in He^4, the influence of He^3 on the properties of He^4 has been computed. The first method has been made use of by de Boer and Gorter[1], Rice et al.,[2] Stout[3] and Koide and Usui[4], using the assumption of Taconis et al.[5] that He^3 dissolves *only* in the normal part of He^4 and that the solution obeys the laws of an "Ideal Solution". The second line has been followed in the theories of Harasima[6], Heer and Daunt[7] later refined by Mikura[8], and of Toda and Ishihara[9], starting from an interpretation of the λ-point in He^4 as a consequence of the Bose-Einstein condensation, and of Pomeranchuk[10] starting from Landau's excitation theory[11].

These theoretical models were proposed mainly to explain the earlier observations of the vapour pressure[12] of dilute solutions of He^3 in He^4 and of the variations of the λ-temperature of the solution[13][14] with the concentration of He^3. It is now an established fact that these earlier observations were largely in error. Thus there exists no completely satisfactory theory for solutions of He^3 and He^4, as none of these theories is in good agreement with the recent experimental data about λ-temperature[15], specific heat[16], effective "normal fluid" density[17] and vapour pressure of solutions of He^3 and He^4 [18],[19].

The theories based on the difference in the statistics are unsatisfactory, though the recent approach made by Mikura[20] leads to good agreement with the experiments. The

parameters involved in this model render it rather easy to arrive at the desired results by their suitable adjustment. In 1946 Franck[21] suggested that a separation of the two isotopes He^3 and He^4 might be based on the expectation that He^3 would not take part in the superfluid phenomena which are characteristic for liquid He^4 below the λ-point. This hypothesis was supported* by experiments of Daunt et al.,[22] which indicated that He^3 is not carried with the superflow in liquid helium films or through narrow slits. Subsequent experiments by Fairbank et al.[23] gave further support to this general idea by demonstrating that, in bulk liquid also, He^3 follows the movements of the normal fluid. When a state of internal convection is set up by local heating, the He^3 atoms are flushed along by the heat current and are thus concentrated where the temperature is lowest. Hence the flow of He^3 atoms seems to be governed by the flow of the normal fluid and not by that of the superfluid. This is in line with the suggestion by Landau and Pomeranchuk[24] that massive foreign bodies, like the rotating disk, in liquid HeII are directly coupled with the normal fluid only. This behaviour, therefore, does not reveal anything about the importance of statistics of the isotopes for superfluidity. The experimental results of Guttman and Arnold[25] indicate clearly that He^6 behaves exactly like He^3 in He^4 in spite of different statistics. Thus the second line of thought (in which the difference of statistics being obeyed by He^4 and He^3 is regarded as important) appears to be misleading.

The phenomenological thermodynamical approach of de Boer and Gorter[1] does not give any clue to the understanding of the molecular origin of the deviations of the properties of He^3 and He^4 solutions from the laws of an "Ideal Solution". A different choice of the Gibbs function and an introduction of a non-ideality parameter, as has been done by Nanda[26], make no appreciable improvement in the original theory. The root cause of the severe disagreement of the original theory of de Boer and Gorter[1] with the recent experimental results, therefore, seems to lie in the basic assumption of the theory. The assumption of Taconis et al.[5] (that He^3 dissolves *only* in the normal part of He^4) on which the theory of de Boer and Gorter[1] is based, appears too artificial. It must be regarded with a due amount of caution, because taken literally, it implies that the normal part of He^4 is separated in space from the superfluid part (See e.g., Rice et al.[2]). This interpretation of the two fluid theory leads us into difficulties with the explanation of the basic properties of He II, such as second sound, where the superposition of the two fluids is always taken granted for. We have, therefore, tried to develop[27,28,29,30] a more consistent theory of He^3 and He^4 solutions on the basis of the following assumptions :

(i) He^3 mixes with the whole of He^4 (normal as well as superfluid).

(ii) The contribution to the entropy of mixing is due only to the normal part of He^4.

* E. F. Hammel and A. F. Schuch [Phys. Rev. 87 (1952), 154] have observed a pressure-independent stationary flow of He^3 through superleaks in the presence of superfluid He^4. As this stationary flow has been found accompanied by a pressure difference, we would be inclined to ascribe it to the irreversible process involved. However, a final interpretation of these results must wait for further investigations.

(iii) The solution of He^3 in He^4 is not an "Ideal Solution" even above the λ-temperature. It obeys laws of a "Strictly Regular Solution."

In the present paper, it will be shown that the agreement between theory and experiment concerning the λ-temperature of the solution, molar specific heat of mixing, vapour pressure, distribution coefficient, and the vapour pressure measurements at the dew points is satisfactory.

In a paper to be published, it will be shown that fairly good agreement for the velocity of second sound and the effective "normal fluid" density in solution is obtained, on the basis of this theory, if we assume that (i) He^3 atoms partake in the motion of the normal part particles, and (ii) the effective mass of the He^3 atoms in solution is twice the actual mass. Theory and experiment are also found to be in agreement as regards the osmotic pressure, the fountain effect and the heat resistance in He^3 and He^4 solutions.

§ 2. The influence of He^3 on the λ-temperature

On the basis of the assumptions of the present theory (Sec. 1), the Gibbs function ($G=U-TS+PV$) for the solution can be written in the form

$$G(T, X_L, x) = (1-X_L)\,G_4(T, x) + X_L \cdot G_3(T) + RT[X_L \ln X_L + x(1-X_L)\ln(1-X_L)] + xX_L \cdot (1-X_L)\,W, \qquad (2\cdot 1)$$

where X_L is the concentration of He^3 in the liquid phase. $G_4(T,x)$ and $G_3(T)$ are the Gibbs functions for pure He^4 and He^3 respectively. $x\,(=\rho_{4n}/\rho)$ is the fraction of the total number of atoms of He^4 which constitute the normal fluid, while $w\,(=W/N)$ is the interchange energy[3], and N is Avogadro's number. The third (mixing) term on the right hand side of Eq. $(2\cdot 1)$ has the same form as in the case of an "Ideal Solution" of two liquids, the only, but very important difference being that only a fraction x of the He^4 atoms contributes to it. The last term on the right hand side of eq. $(2\cdot 1)$ accounts for the assumption that the solution of He^3 and He^4 is not an "Ideal Solution" even above the λ-temperature, but it obeys laws of a "Strictly Regular Solution". When $x=1$, eq. $(2\cdot 1)$ becomes identical with the Gibbs function of a "Strictly Regular Solution".

Following de Boer and Gorter[1], we consider G_4 as a function of the parameter x. The equilibrium values of x are determined by the fact, that "for each temperature and concentration Gibbs function must be a minimum with respect to variations of the parameter x at constant T and X_L". Thus applying the condition of equilibrium

$$[\partial G(T, X_L, x)/\partial x]_{T, X_L} = 0, \qquad (2\cdot 2)$$

to eq. $(2\cdot 1)$, we obtain the following implicit equation for $x=x(T, X_L)$:

$$[\partial G_4(T, x)/\partial x]_{T, X_L} + RT \ln(1-X_L) + X_L W = 0. \qquad (2\cdot 3)$$

In order to make an actual calculation of the λ-temperature of the solution, some functional dependence of Gibbs function, $G_4(T, x)$, for pure He^4 on the normal fraction x has to be chosen. The expression for G_4 is chosen so as to conform the approximate representation of the experimental data for pure He^4 (See Tisza[2]) which is expressed by the relation

$$x_0 = \rho_{n0}/\rho_0 = S_0/S_\lambda \approx (T/T_\lambda)^{5.5}, \tag{2·4}$$

where the subscript 0 refers to pure He^4. S_λ is the entropy of pure He^4 at T_λ, the λ-temperature of pure He^4.

A. Linear expression for G_4.

For G_4, de Boer and Gorter[1] adopted the expression

$$G_4(T, x) = -E_0(1-x^{7/6}) - xS_\lambda T, \tag{2·5}$$

where E_0 is a constant equal to $(\frac{6}{7} \cdot T_\lambda S_\lambda)$. In agreement with eq. (2·4) we choose the "Modified linear expression for G_4" which is expressed as

$$G_4(T, x) = -\tfrac{5.5}{6.5} \cdot S_\lambda T_\lambda (1-x^{6.5/5.5}) - xS_\lambda T, \tag{2·6}$$

where $S_\lambda = 1.622$ cal.deg^{-1}. mole^{-1}., and $T_\lambda = 2.182°$K. Substitution of eq. (2·6) in eq. (2·3) gives the implicit equation for x as

$$S_\lambda T_\lambda x^{1/5.5} - S_\lambda T + RT \ln(1-X_L) + X_L W = 0. \tag{2·7}$$

The λ-temperature of the solution ($T = T_{\lambda X_L}$, when $x = 1$) is thus given by the equation

$$\frac{T_{\lambda X_L}}{T_\lambda} = \frac{\left(S_\lambda + \dfrac{X_L \cdot W}{T_\lambda}\right)}{S_\lambda - R\ln(1-X_L)} = \frac{1 + 0.283 X_L \cdot W}{1 - 2.821 \log_{10}(1-X_L)}. \tag{2·8}$$

Recently, Dash and Taylor[15] have measured the λ-temperatures of solutions of He^3 and He^4. They have proposed an empirical relation

$$\frac{T_{\lambda X_L}}{T_\lambda} = (1-X_L)^{2/3}, \tag{2·9}$$

for X_L up to 10 per cent.

Using the experimentally observed value of $(\partial T_{\lambda X_L}/\partial X_L)_{X_L \to 0} = -1.455$ (from eq. (2·9)), eq. (2·8) leads to a value of $W = 1.976$ cal. mole^{-1}. It is in fairly good agreement with the average value, 1.81 cal. mole^{-1}, of W calculated by Morrow[33] from Sommers'[18] experimental results of vapour pressures for solutions of He^3 and He^4.

B. Quadratic expression for G_4.

The choice of linear expression for G_4 is not satisfactory, as above T_λ this leads to temperature independent expression for entropy implying zero specific heat for He I. Another expression for G_4, also used by de Boer[34], involves a quadratic function for term in T. We have in this case (see also Daunt[35]),

$$G_4(T, x) = -\frac{4.5}{13.0} S_\lambda T_\lambda (1 - x^{6.5/5.5}) - \frac{1}{2}\frac{S_\lambda}{T_\lambda} \cdot T^2 x^{4.5/5.5}. \qquad (2\cdot10)$$

This expression for G_4 (known as "Quadratic Expression" to distinguish it from "Linear Expression" given by eq. (2·6)) gives an entropy for He I which increases linearly with temperature, as observed experimentally[36].

Substitution of eq. (2·10) in eq. (2·3) gives the implicit equation for x as

$$(T_\lambda/T)\, x^{1/5.5} - (T/T_\lambda)\, x^{-1/5.5} + \left(\tfrac{11.0}{4.5} \cdot R/S_\lambda\right)[\ln(1-X_L) + X_L W/RT] = 0. \qquad (2\cdot11)$$

The λ-temperature of the solution ($T = T_{\lambda X_L}$, when $x = 1$) is thus given by

$$(T_{\lambda X_L}/T_\lambda)^2 - 6.896\,(T_{\lambda X_L}/T_\lambda)\log_{10}(1-X_L) - 1 - 0.691 X_L W = 0. \qquad (2\cdot12)$$

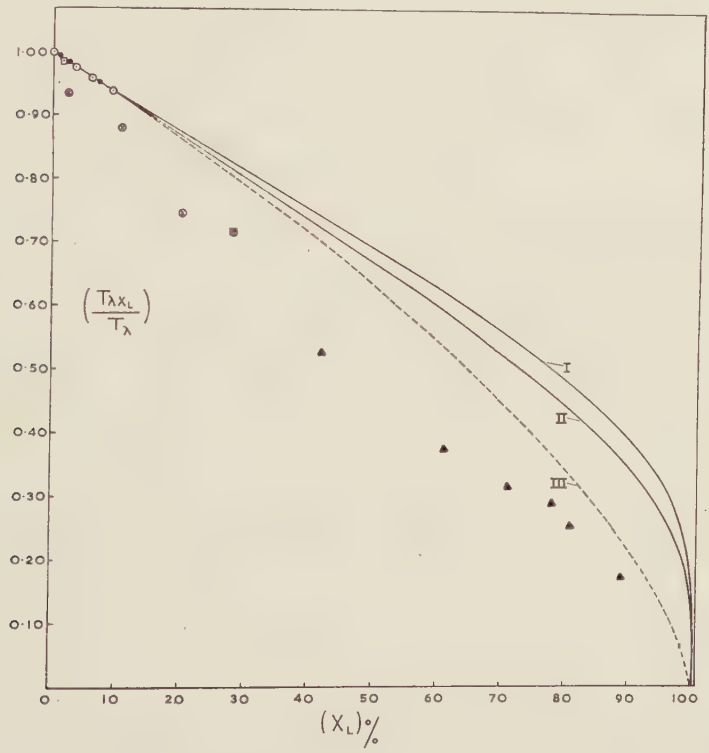

Fig. 1. The ratio of the λ-temperature of He^3 and He^4 solution, $T_{\lambda X_L}$, to that of pure He^4, T_λ, as a function of the He^3 concentration X_L.
Curve I : Present Theory, Linear Expression for G_4.
Curve II : Present Theory, Quadratic Expression for G_4.
Curve III : Empirical relation (2·9), proposed by Dash and Taylor[15] for X_L up to 10 percent.
⊗ : Experimental results of Abraham et al.[13]
⊡ : Experimental result of Eselsohn and Lazarev[37].
▲ : Experimental results of Daunt and Heer[14].
● : Experimental results of Dokoupil et al.[16]
⊙ : Experimental results of Dash and Taylor[15].

Using the experimentally observed value of $(\partial T_{\lambda x_L}/\partial X_L)_{x_L \to 0}$ as before, eq. (2·12) gives a value of $W = 2.406$ cal. mole^{-1}. The λ-temperature of the solution, as a function of the concentration of liquid He^3, calculated by using eqs. (2·8) and (2·12) is compared in Fig. 1 with the empirical relation (2·9). It will be noticed that for X_L up to 20 per cent, the calculated values of the λ-temperature of the solution (using both the expressions for G_L) are in excellent agreement with the empirical relation* (2·9) proposed by Dash and Taylor[15]. For the sake of comparison, we have also included in Fig. 1 the experimental results of various other authors.

The early experimental observations of the lowering of the λ-temperature of He^4 due to an admixture of He^3, reported by Abraham et al.[13] for concentrations 2.4 per cent up to 28.2 per cent He^3 by using a superleak formed around a platinum wire, seem to be largely in error, presumably due to the concentration gradient set up in the liquid solution as a result of the heat flush effect. The value given for the λ-temperature of 1.5 per cent He^3 solution by Eselsohn and Lazarev[17] is uncertain, due to the uncertainty of the concentration of liquid He^3. They used an apparatus consisting of two reservoirs connected by a narrow capillary and made use of an observation of the supra-surface flow as the criterion for the existence of the λ-temperature. Daunt and Heer[14] used, in their experiments for concentrations from 42 to 89 percent He^3, the same criterion and determined the λ-temperatures at which the heat influxes to the reservoir changed abruptly. They expected that their results were not seriously affected by the heat flux effect.

The experimental results of King and Fairbank[3], using the disappearance of second sound at the λ-point as the criterion, for concentrations below 4.2 per cent He^3 revealed a linear dependence of the λ-temperature of the solution on the concentration of He^3, giving a slope of -1.5 degrees per mole concentration. These observations led to the suspicion that the earlier observations were erroneous. The suspicion was confirmed by the recent measurements of the specific heat of solutions of He^3 and He^4 by Dokoupil et al[16], and by the results of the oscillating pendulum experiments performed by Dash and Taylor[15]. In the case of the specific heat measurements for concentrations of He^3 up to 7.13 per cent, the values of the λ-temperature of the solutions were computed from the discontinuities in the slope of the warm-up curves (of temperature versus time). These measurements revealed a linear dependence of the λ-temperature of the solution on the concentration of He^3, giving a slope of -1.48 degrees per mole concentration. Dash and Taylor[15] obtained the λ-temperatures of the individual solutions by locating the discontinuity in the slope of the torsion period, of a torsion pendulum immersed in the solution, versus temperature curves. This is equivalent to finding the temperature at which the normal fluid density becomes equal to the total density. Their experiments also give a linear shift of -1.47 ± 0.3 degrees per mole concentration for the whole concentration range (up to $X_L = 0.092$) they studied.

The principles involved in the methods to determine the lowering of the λ-point of

* We have been informed that the unpublished results of the Los Alamos Group give support to the empirical relation (2·9) even for higher values of X_L.

He^4 due to the presence of He^3 in the solution are entirely different from each other in the experiments of King and Fairbank[38], Dokoupil et al.[16] and of Dash and Taylor[15]. In spite of that, the excellent agreement of the results suggests that the shift of λ-point of He^4 due to the presence of He^3 as given by Dash and Taylor[15] may be treated as an established fact. In view of this fact, the previous theories of solutions of He^3 and He^4 need attention and may be revised because they were proposed mainly to explain the earlier observations, by Abraham et al[13]. and by Daunt and Heer[14], of the λ-temperature of the solutions of He^3 and He^4 versus the concentration of He^3.

§ 3. The heat of mixing of He^3 and He^4 solutions

The mixing term in eq. (2·1) is *not* exactly proportional to T for temperatures below the λ-temperature. Thus there is a *mixing enthalpy*, corresponding to which there is also a specific heat of mixing. This mixing enthalpy, being defined as the excess of the enthalpy of the solution over the sum of the enthalpies of the pure substances at the same temperature (and pressure, which is always supposed to be constant), can be calculated from Gibbs function by using the relation

$$H \equiv G - T\,(\partial G/\partial T)_{X_L} = G - T\,(\partial G/\partial T)_{X_L, x}. \qquad (3\cdot1)$$

In writing eq. (3·1) use has been made of the equilibrium condition for G, given by eq. (2·2). In what follows we shall not distinguish between enthalpy and internal energy, as is usual for liquids not subject to high pressures.

Using eq. (2·1) in eq. (3·1), the expression for enthalpy (or internal energy) becomes

$$H = xX_L(1-X_L)W + X_L[G_3(T) - T\,(\partial G_3(T)/\partial T)] + (1-X_L)[G_4(T, x) - T\,(\partial G_4(T, x)/\partial T)_{X_L, x}]. \qquad (3\cdot2)$$

Thus the expression for the mixing enthalpy $[\Delta H = H - (1-X_L)H_4^0 - X_L H_3^0]$ can be written as

$$\Delta H = xX_L(1-X_L)W + (1-X_L)\{[G_4(T, x) - T\,(\partial G_4(T, x)/\partial T)_{X_L, x}] - [G_4(T, x_0) - T\,(\partial G_4(T, x_0)/\partial T)_{x_0}]\}, \qquad (3\cdot3)$$

where x_0 denotes the fraction of the normal fluid in pure He^4 at temperature T.

A. Linear expression for G_4

Substituting the linear expression for G_4, as given by eq. (2·6), eq. (3·3) leads us to the following expression for the molar heat of mixing:

$$\Delta H = xX_L(1-X_L)W + \tfrac{5\cdot5}{6\cdot5} S_\lambda T_\lambda (1-X_L)\,(x^{6\cdot5/5\cdot5} - x_0^{6\cdot5/5\cdot5}). \qquad (3\cdot4)$$

B. Quadratic expression for G_4

Substitution of the quadratic expression for G_4, as given by eq. (2·10), eq. (3·3) leads us to the following expression for the molar heat of mixing:

$$\Delta H = x X_L (1-X_L) W + S_\lambda T_\lambda (1-X_L) \left[\tfrac{4\cdot5}{13\cdot0}(x^{6\cdot5/5\cdot5} - x_0^{6\cdot5/5\cdot5}) + \tfrac{1}{2}(T/T_\lambda)^2 (x^{4\cdot5/5\cdot5} - x_0^{4\cdot5/5\cdot5})\right] \qquad (3\cdot5)$$

(i) *Above the λ-temperature for pure* He^4 $(T \geq T_\lambda)$, $x = x_0 = 1$. Therefore, the molar heat of mixing is given by the "Strictly Regular Solution" value $X_L(1-X_L)W$.

(ii) *Below the λ-temperature for pure* He^4 $(T \leq T_\lambda)$, x_0 is obtained from the relation

$$x_0 = (T/T_\lambda)^{5\cdot5}, \qquad (2\cdot4')$$

while x can be calculated from eqs. (2·7) and (2·11) respectively when linear and quadratic expressions for G_4 are used.

Sommers[18] calculated the latent heat of evaporation of solutions of He^3 and He^4 for different concentrations of He^3 with the aid of the Clausius-Clapeyron equation for a solution and by using his measured values of the vapour pressure of the solution. As a result he estimated the heat of mixing for dilute solutions of He^3 in He^4 at 1.5°K to be 3 cal per mole of He^3.

Up to now there is available only one directly determined value of the heat of mixing. At a constant temperature of 1.02°K for a 8.6 per cent solution of He^3 in He^4, Sommers et al.[39] measured the heat of mixing to be 1.98 cal per mole of He^3 (=0.17 cal per mole of the 8.6% solution).

We have calculated the molar heat of mixing, as a function of temperature, for 8.6% solution of He^3 in He^4 on the basis of the present theory by using the linear and

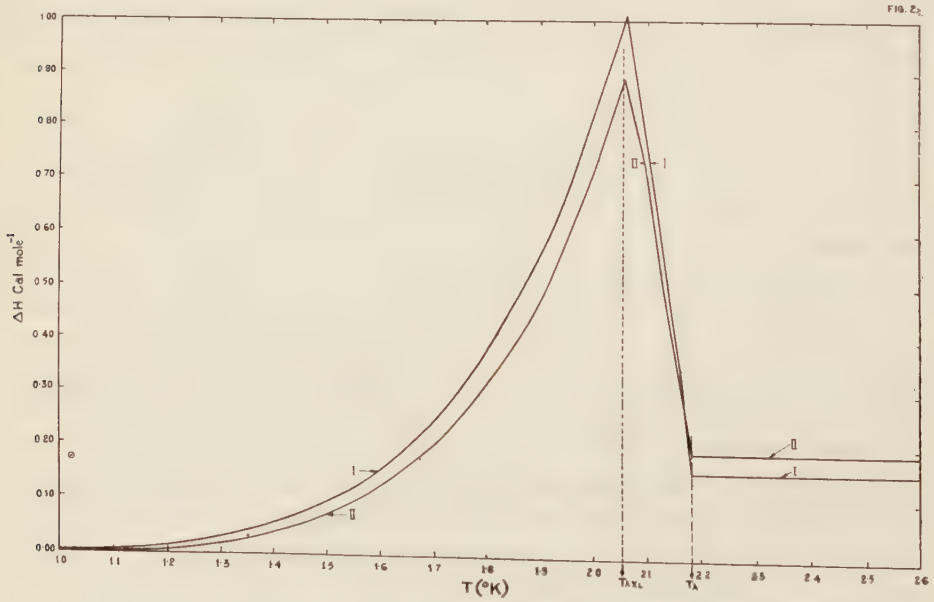

Fig. 2. The Heat of mixing per mole of 8.6 per cent solution of He^3 in He^4 versus T(°K).
Curve I : Present Theory, Linear Expression for G_4.
Curve II : Present Theory, Quadratic Expression for G_4.
⊙ : Experimental result of Sommers et al.[39]

quadratic expressions for G_4. The values so obtained are compared in Fig. 2 with the only one direct experimental result by Sommers et al[39]. It is evident from Fig. 2 that the magnitude of the predicted heat of mixing at 1.02° K is smaller by an order of magnitude in comparison with the direct experimental result. However, we cannot draw any conclusion from it, in view of the fact that there is one and only one direct experimental result reported so far. The method by which the experimental estimation of this single value for the heat of mixing has been carried out by Sommers et al.[39] is not free from doubt. We will discuss this point in more detail in a subsequent publication.

In a paper to be published, we will show that the values of the heats of mixing, derived from the vapour pressure data of Sommers[18] and of Daunt and Tseng[19] for solutions of He^3 in He^4, are in fairly good agreement with the corresponding values calculated on the basis of the present theory.

§ 4. The specific heat of mixing of He^3 and He^4 solutions

The expression for the specific heat of mixing can be obtained by differentiating eq. (3·3) with respect to T.

A. Linear expression for G_4.

Using the linear expression for G_4, eq. (3·3) can be written in the form of eq. (3·4). Differentiating eq. (3·4) we have the following expression for the specific heat of mixing of He^3 and He^4 solutions:

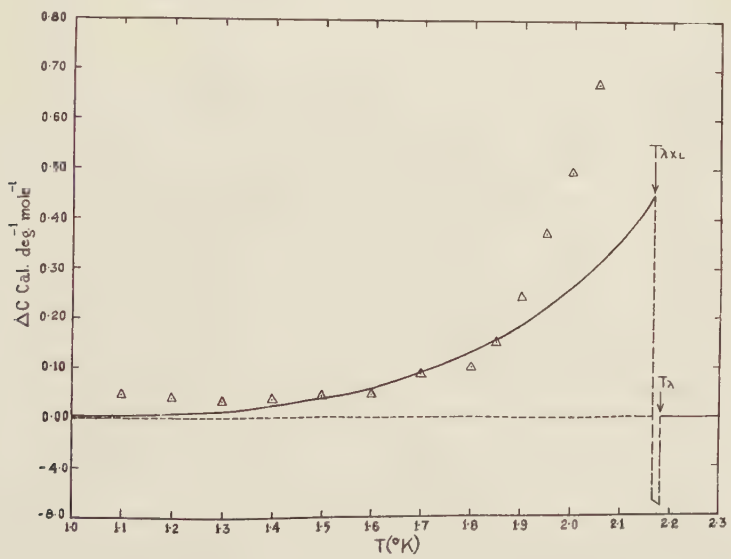

Fig. 3a. The specific heat of mixing per mole of 1.00 per cent solution of He^3 in He^4 versus T (°K).
——: Present Theory, Quadratic Expression for G_4.
△: Experimentally derived points[16), 40)].

(i) *For* $T \geq T_\lambda$, $x = x_0 = 1$, and we have

$$\Delta C = 0. \tag{4·1}$$

(ii) *For* $T_\lambda \geq T \geq T_{\lambda X_L}$, $x = 1, x_0 < 1$, and we have

$$\Delta C = -5.5 x_0 S_\lambda (1 - X_L). \tag{4·2}$$

(iii) *For* $T \leq T_{\lambda X_L}$ we have

$$\Delta C = (1 - X_L) S_\lambda \{T_\lambda (\partial x/\partial T)_{X_L} \cdot [x^{1/5 \cdot 5} + X_L W/S_\lambda T_\lambda] - 5.5 x_0\}. \tag{4·3}$$

B. Quadratic expression for G_4.

The expression obtained for the specific heat of mixing of solutions of He^3 and He^4

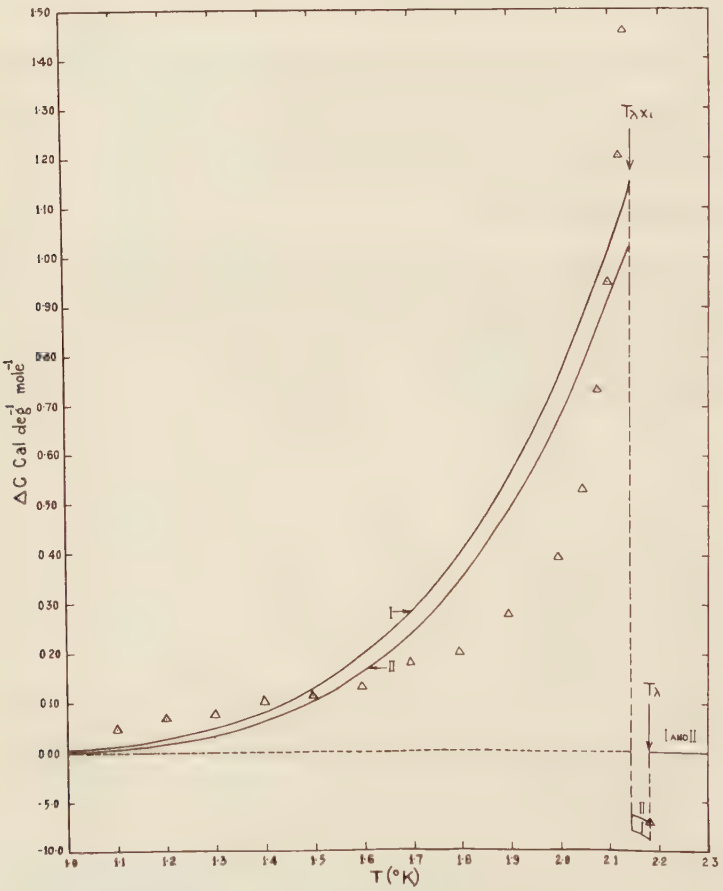

Fig. 3b. The specific heat of mixing per mole of 2.50 per cent solution of He^3 in He^4 versus $T(°K)$.
Curve I : Present Theory, Linear Expression for G_4.
Curve II : Present Theory, Quadratic Expression for G_4.
△ : Experimentally derived points.[16)40)]

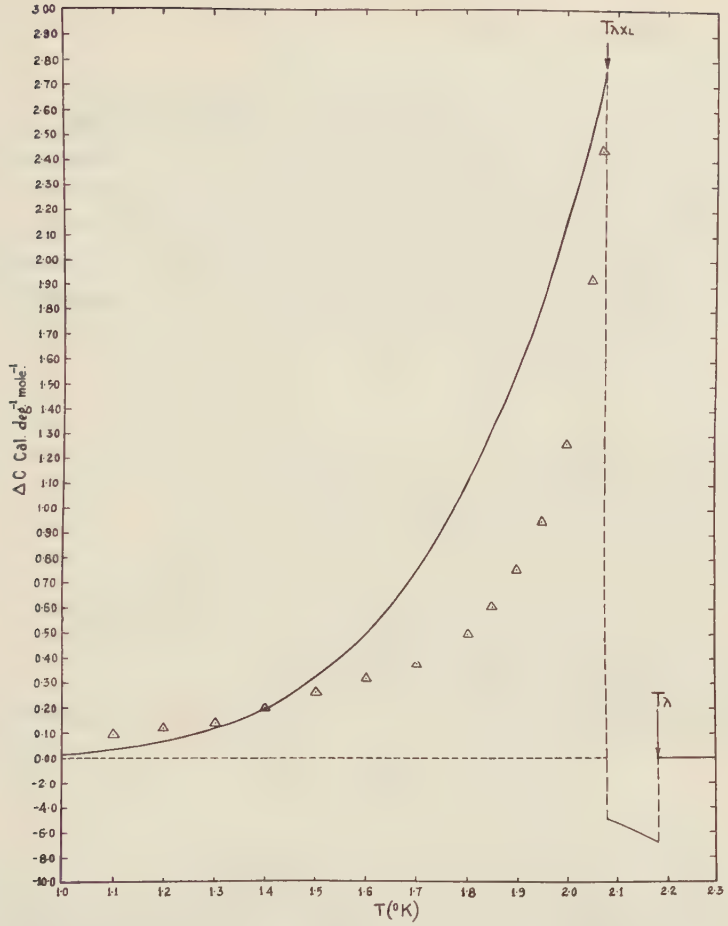

Fig. 3c. The specific heat of mixing per mole of 7.13 per cent solution of He^3 in He^4 versus $T(^\circ K)$.
——: Present Theory, Quadratic Expression for G_4.
△: Experimentally derived points.[16), 40)]

in this case by differentiating eq. (3·3) with respect to T is given by:

(i) *For* $T \geq T_\lambda$, $x = x_0 = 1$, and we have

$$\Delta C = 0. \tag{4·4}$$

(ii) *For* $T_\lambda \geq T \geq T_{\lambda X_L}$, $x = 1$, $x_0 < 1$, and we have

$$\Delta C = -(1 - X_L) S_\lambda [5.5 x_0 - (T/T_\lambda)]. \tag{4·5}$$

(iii) *For* $T \leq T_{\lambda X_L}$, we have

$$\Delta C = (1 - X_L) S_\lambda \{\tfrac{4·5}{11·0} T (\partial x/\partial T)_{X_L} \cdot [(T_\lambda/T) x^{1/5·5} + (T/T_\lambda) x^{-1/5·5} + \tfrac{11·0}{4·5} \cdot X_L W/T S_\lambda] + [(T/T_\lambda) x^{4·5/5·5} - 5.5 x_0]\}. \tag{4·6}$$

In these equations x_0 is given by equation (2·4) while x and $(\partial x/\partial T)_{X_L}$ can be com-

puted from the equations (2·7) and (2·11) respectively for the linear and quadratic expressions for G_4.

We have calculated the molar specific heat of mixing, as a function of temperature, for 1.00, 2.50 and 7.13 per cent solutions of He^3 in He^4 on the basis of the present theory by using the linear and quadratic expressions for G_4. The values so obtained are compared in Figures 3a, 3b and 3c with the correponding experimentally determined values. To calculate the experimental values of the molar specific heat of mixing $[\Delta C = C-(1-X_L)C_4^0-X_L C_3^0]$, we have used the smoothed values of C and C_4^0 from the data given by Dokoupil et al[16]., while for C_3^0 we have used the empirical relation

$$C_3^0 = 0.667 + 0.290T + 0.078T^3 \quad \text{cal. deg}^{-1}\text{. mole}^{-1}, \tag{4·7}$$

for $0.4 < T < 2.4\,(^\circ\text{K})$,

proposed by Roberts and Sydoriak[40]. It is evident from Fig. 3 that the calculated values, of the molar specific heat of mixing (using both the expressions for G_4), are in fairly good agreement with the experimentally determined values.

§ 5. The vapour pressure of He^3 and He^4 solutions

5(a). *Partial Vapour pressures of He^3 and He^4.*

The thermodynamical potentials of He^4 and He^3 in the liquid phase are given by the relations

$$\mu_{4L} = G - X_L (\partial G/\partial X_L)_T, \tag{5·1}$$

and

$$\mu_{3L} = G + (1-X_L)(\partial G/\partial X_L)_T, \tag{5·2}$$

where the subscript L indicates the quantity in the liquid phase. Making use of the expression (2·1) for G and the equilibrium condition (2·2) we thus have

$$\mu_{4L} = G_4(T, x) + RTx \ln(1-X_L) + xX_L^2 W - X_L RT(1-x), \tag{5·3}$$

and

$$\mu_{3L} = G_3(T) + RT \ln X_L + x(1-X_L)^2 W + (1-X_L) RT(1-x). \tag{5·4}$$

Thus we have

$$\mu_{4L} - \mu_{4L}^0 = [G_4(T, x) - G_4(T, x_0)] + RTx \ln(1-X_L) + xX_L^2 W - X_L RT(1-x), \tag{5·5}$$

and

$$\mu_{3L} - \mu_{3L}^0 = RT \ln X_L + x(1-X_L)^2 W + (1-X_L) RT(1-x), \tag{5·6}$$

where μ_{4L}^0 and μ_{3L}^0 are the thermodynamical potentials respectively of liquid He^4 and liquid He^3 in the pure state.

Assuming the vapours of He^3 and He^4 to be slightly imperfect, we have according to Guggenheim[41]

$$\mu_{4v}-\mu_{4v}^0=RT\ln\{(1-X_v)P/p_4^0\}+B_{44}(P-p_4^0)+PX_v^2\cdot\gamma, \qquad (5\cdot7)$$

and

$$\mu_{3v}-\mu_{3v}^0=RT\ln(X_vP/p_3^0)+B_{33}(P-p_3^0)+P(1-X_v)^2\cdot\gamma, \qquad (5\cdot8)$$

where p_4^0 and p_3^0 are the vapour pressures of He^4 and He^3 respectively in the pure phase; P is the total pressure, $\gamma=2B_{34}-B_{33}-B_{44}$, B's being the second virial coefficients. In these equations the subscript v refers to the quantity in the vapour phase.

Equating the thermodynamical potentials of He^4 and He^3 in the liquid phase to the thermodynamical potentials of He^4 and He^3 in the vapour phase respectively, the expressions for the partial vapour pressures of He^4 and He^3 are obtained as $[p_4\equiv P(1-X_v)$ and $p_3\equiv PX_v]$

$$p_4=p_4^0(1-X_L)^x\cdot\exp\{(1/RT)[G_4(T,x)-G_4(T,x_0)+xX_L^2W-(P-p_4^0)B_{44}-PX_v^2\cdot\gamma]-X_L(1-x)\}, \qquad (5\cdot9)$$

and

$$p_3=p_3^0X_L\cdot\exp\{(1/RT)[x(1-X_L)^2W-(P-p_3^0)B_{33}-P(1-X_v)^2\gamma]+(1-X_L)(1-x)\}, \qquad (5\cdot10)$$

where X_L and X_v are the concentrations of He^3 in the liquid and the vapour phases respectively.

The partial vapour pressure of He^4 above $T_{\lambda X_L}$, the λ-temperature of the solution, is still higher than

$$p_4^0(1-X_L)\cdot\exp\{(1/RT)[X_L^2W-(P-p_4^0)B_{44}-PX_v^2\cdot\gamma]\},$$

as long as $T<T_\lambda$, the λ-temperature of pure He^4. This is so, because, in solutions above the λ-temperature, all the He^4-atoms are in the normal state, whereas for pure He^4 at the same temperature part of the atoms may be in the superfluid state and it causes a small difference in vapour pressure.

We have calculated the total vapour pressure $P(=p_3+p_4)$, as a function of temperature, of the solutions He^3 in He^4 for various concentrations of He^3 ranging from 0.58 to 20.3 per cent on the basis of the present theory by using the linear and quadratic expressions for G_4. The values so obtained are compared in Fig. 4 with the experimental results of Sommers[18], Weinstock et al.[42] and of Daunt and Tseng[19]. The vapour imperfection corrections have been applied using the second virial coefficients computed by Kilpatrick et al.[43] For p_4^0 we have used the vapour pressure data given by Van Dijk and Shoenberg[44], while for p_3^0 we have used the smoothed values given by Araham et al.[45] It is evident from Fig. 4 that the calculated values on the basis of the present theory are in fairly good agreement with the experimental results.

It is of interest to note that Sommers[18] as well as Daunt and Tseng[19] measured directly p_4^0 and the difference $\Delta P(=P-p_4^0)$ in vapour pressure between a solution of He^3 in liquid He^4 and pure liquid He^4. Thus it would be more appropriate to compare the theoretically evaluated values of ΔP with those determined experimentally. The values

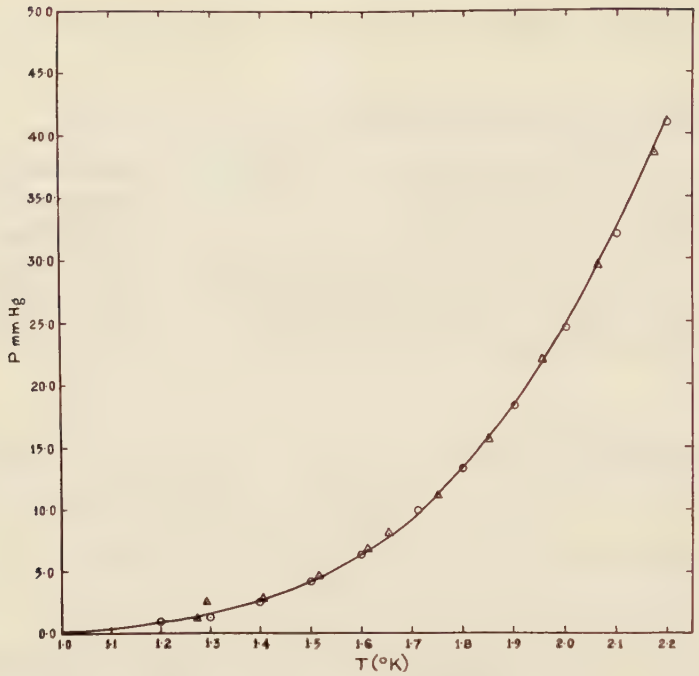

Fig. 4a. The vapour pressure for 0.58 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results for 0.50 per cent solution of He^3 in He^4.
△: Sommers[18] unsmoothed experimental results for 0.58 per cent solution of He^3 and He^4.

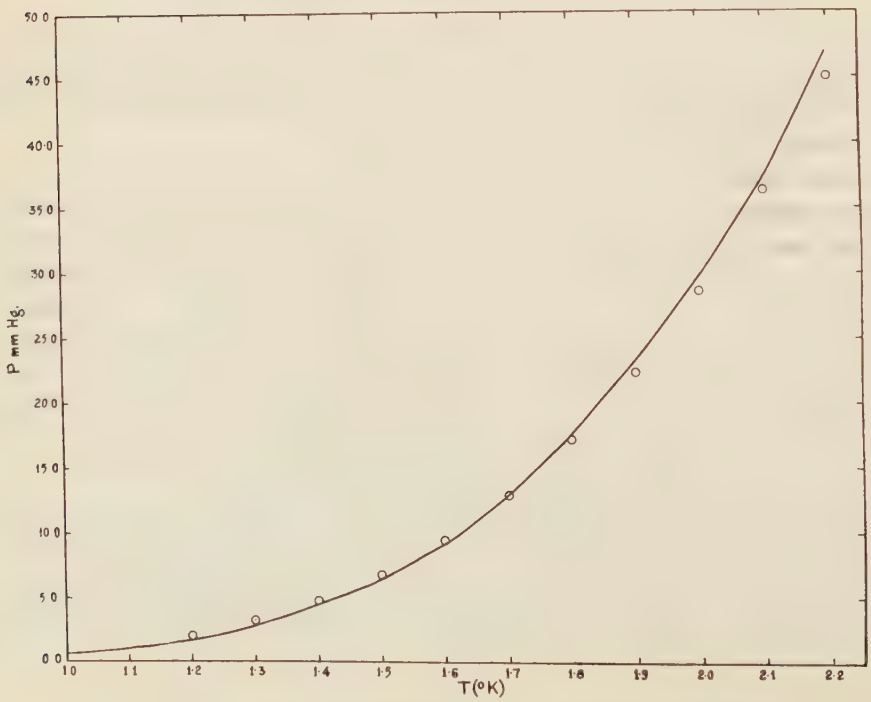

Fig. 4b. The vapour pressure for 2.50 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results

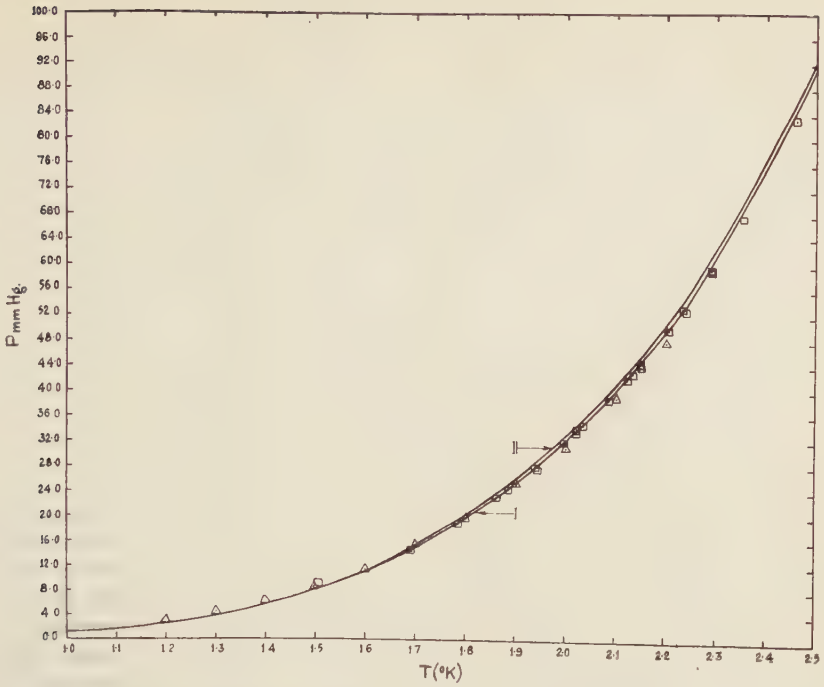

Fig. 4c. The vapour pressure for 4.00 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
Curve I : Present Theory, Linear Expression for G_4.
Curve II: Present Theory, Quadratic Expression for G_4.
◬ : Sommers[18] smoothed experimental results.
⊙ : Experimental results of Daunt and Tseng.[19]

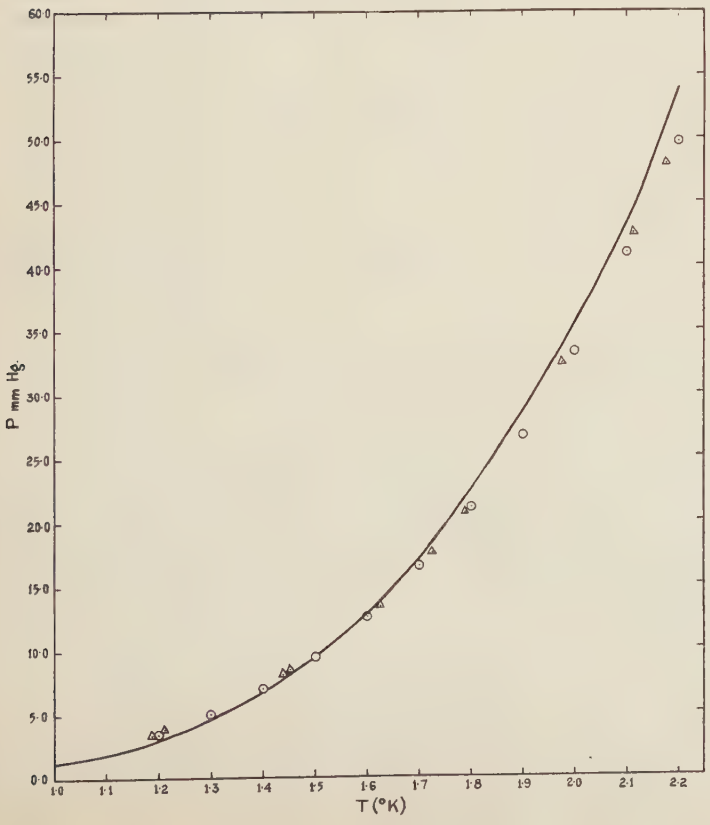

Fig. 4d. The vapour pressure for 5.00 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
——: Present Theory, Quadratic Expression for G_4.
⊙ : Sommers[18] smoothed experimental results for 5.00 per cent solution of He^3 in He^4.
◬ : Sommers[18] unsmoothed experimental results for 5.21 per cent solution of He^3 in He^4.

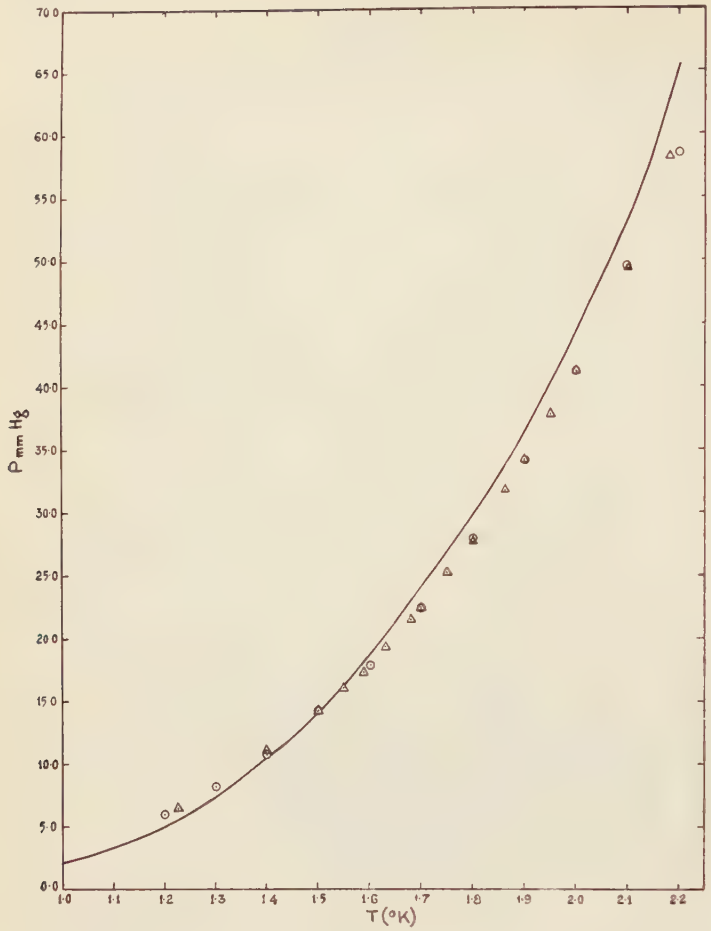

Fig. 4e. The vapour pressure for 9.50 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results for 9.50 per cent solution of He^3 in He^4.
△: Sommers[18] unsmoothed experimental results for 9.49 per cent solution of He^3 in He^4.

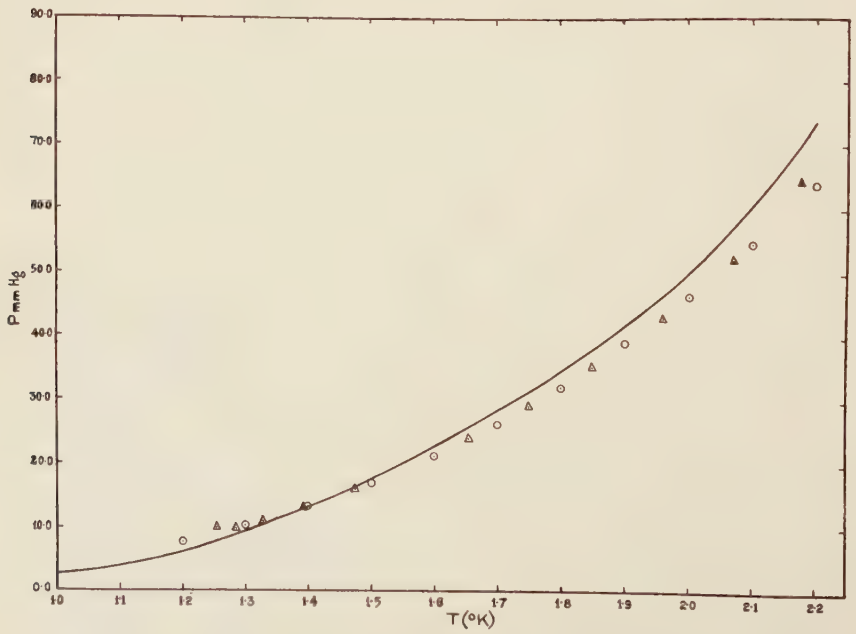

Fig. 4f. The vapour pressure for 13.00 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results.
△: Sommers[18] unsmoothed experimental results.

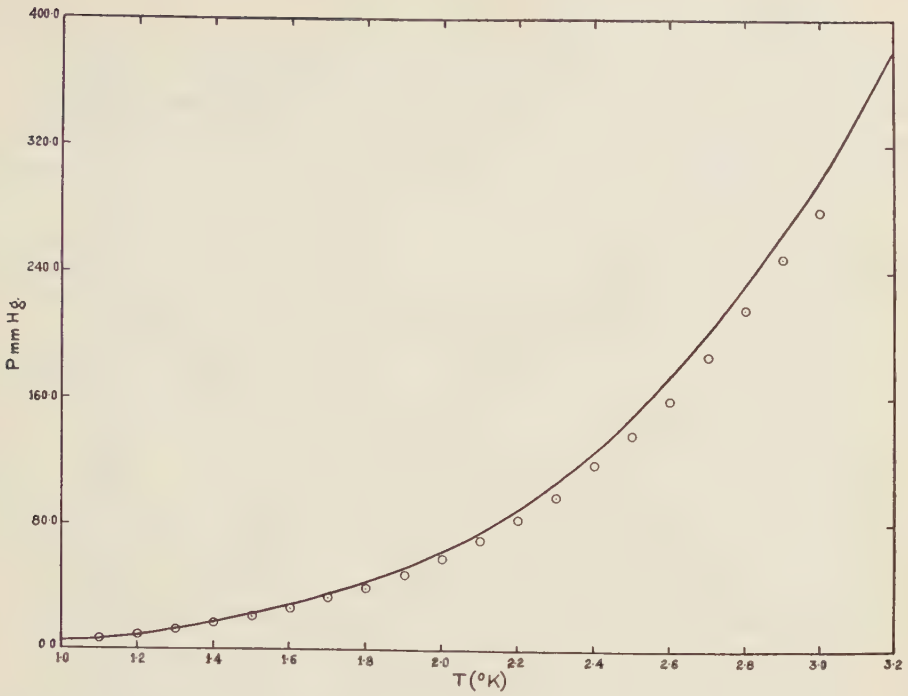

Fig. 4g. The vapour pressure for 20.3 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Smoothed experimental results of Weinstock et al[42].

of ΔP calculated on the basis of the present theory are compared in Fig. 5 with the corresponding experimental results[18),19),42)]. For a 4.00 per cent solution of He^3 in He^4, the results obtained on the basis of the present theory are compared with various theories[1),7),8)] and the experimental results[18),19)]. It is evident from Fig. 5 (especially for $X_L = 0.04$, for which the comparison is shown with other theories*) that the present theory, especially with the linear expression for G_4, describes the observed behaviour better than all the other theories.

Fig. 5 indicates clearly that on the basis of the present theory the theoretical curve of ΔP versus T shows an inflexion point at the λ-temperature of the solution. It is borne out clearly by the experimental data of Daunt and Tseng[19)]. The agreement between theoretically calculated ΔP and the corresponding experimental results is fairly good in the lower temperature region up to 1.6°K. For higher temperatures the predicted values are higher than the experimental results. This discrepancy can be accounted for by the facts that at higher temperatures the vapours of the solution are far from being perfect and thus higher virial coefficients should be included in the vapour pressure equations. Secondly, we have ignored in our treatment the correction on account of the

* This figure is the same as in reference 29).

volume change on mixing. It is found that the agreement in the higher temperature region can be improved by a proper selection of the expression for G_1 which leads to a smaller value of W. The lower value of W, in the case of linear expression for G_4, improves the agreement because the terms involving W are important for higher temperatures where x is appreciable.

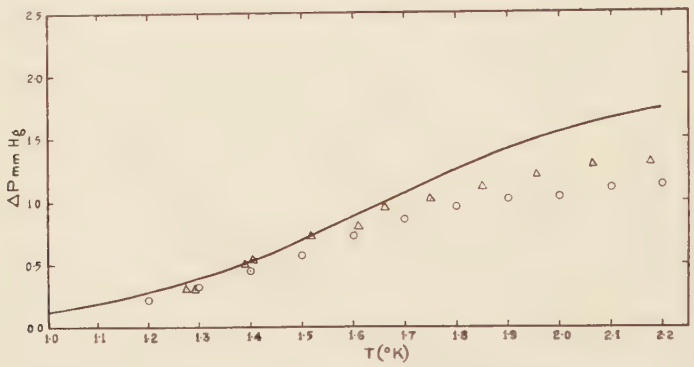

Fig. 5a. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 0.58 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results for 0.50 per cent solution of He^3 in He^4.
△: Sommers[18] unsmoothed experimental results for 0.58 per cent solution of He^3 in He^4.

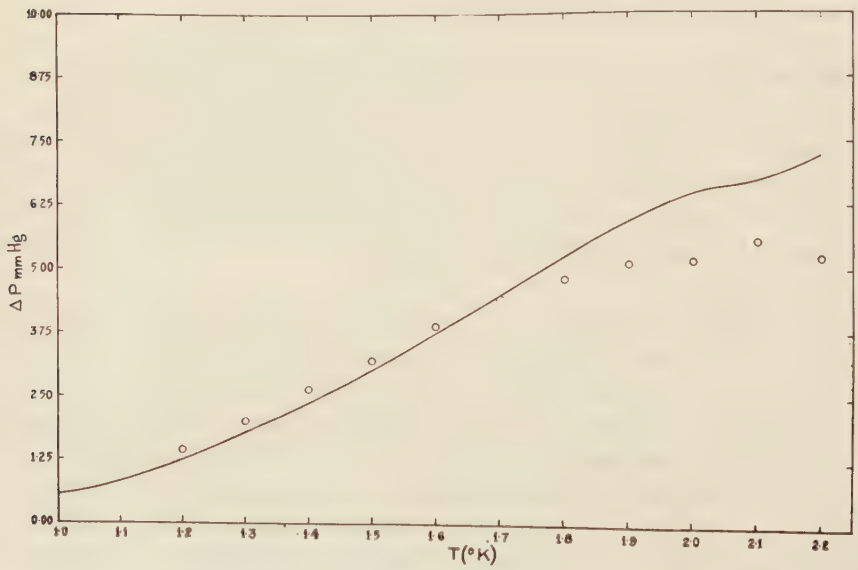

Fig. 5b. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 2.50 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results.

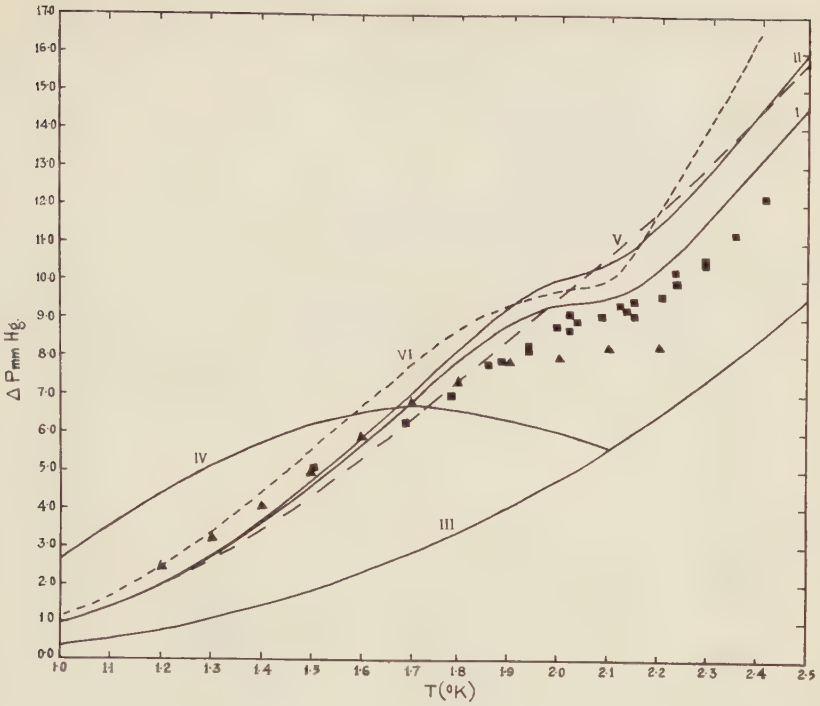

Fig. 5c

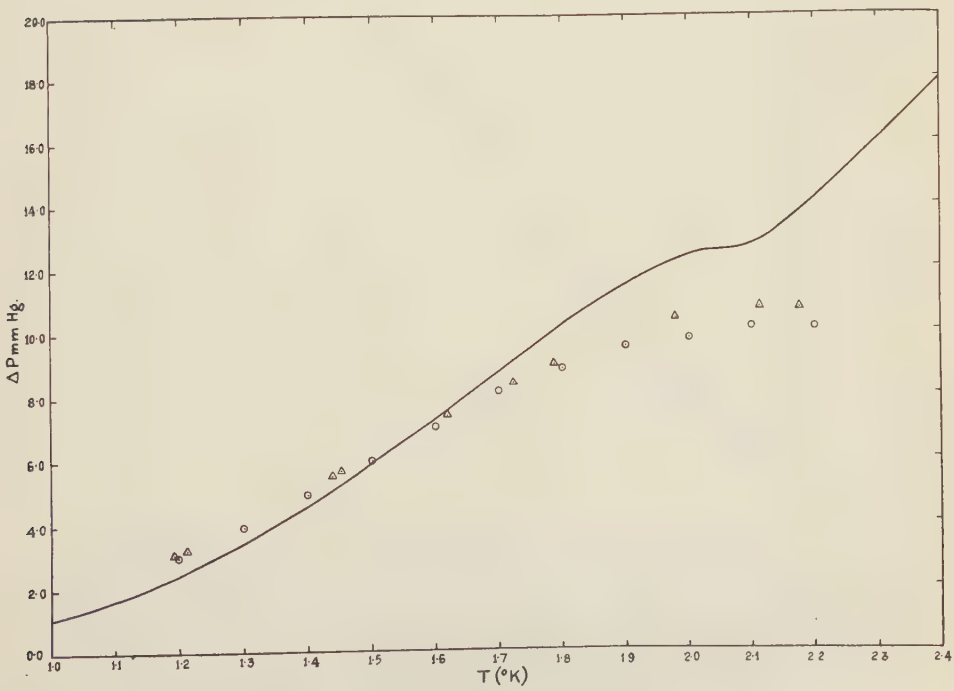

Fig. 5d

Fig. 5c. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 4.00 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
Curve I : Present Theory, Linear Expression for G_4.
Curve II : Present Theory, Quadratic Expression for G_4.
Curve III : Perfect Classical Solution Theory.
Curve IV : de Boer and Gorter's Theory.[1]
Curve V : Heer and Daunt's Theory[7].
Curve VI : Mikura's Theory[8].
▲ : Sommers[18] smoothed experimental results.
■ : Experimental results of Daunt and Tseng[19].

Fig. 5d. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 5.00 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
—— : Present Theory, Quadratic Expression for G_4.
◉ : Sommers[18] smoothed experimental results for 5.00 per cent solution of He^3 in He^4.
△ : Sommers[18] unsmoothed experimental results for 5.21 per cent solution of He^3 in He^4.

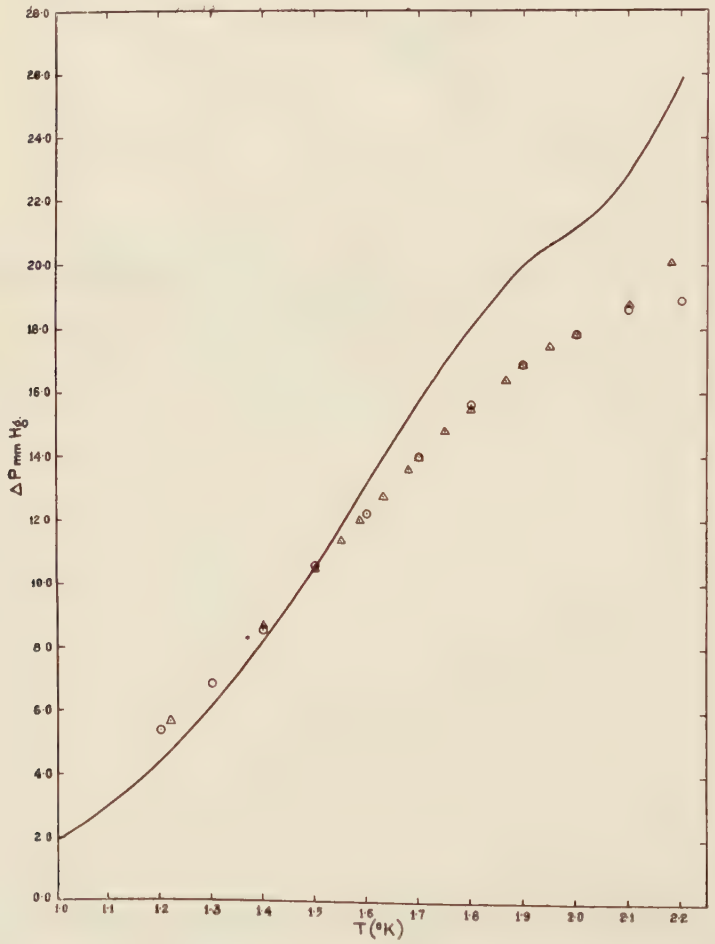

Fig. 5e. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 9.50 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
—— : Present Theory, Quadratic Expession for G_4.
◉ : Sommers[18] smoothed experimental results for 9.50 per cent solution of He^3 in He^4.
△ : Sommers[18] unsmoothed experimental results for 9.49 per cent solution of He^3 in He^4.

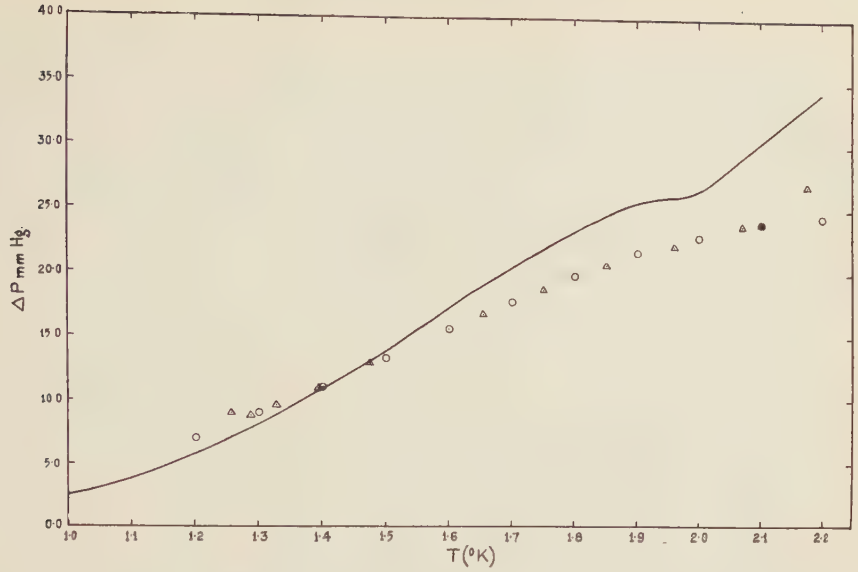

Fig. 5f. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 13.00 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] smoothed experimental results.
△: Sommers[18] unsmoothed experimental results.

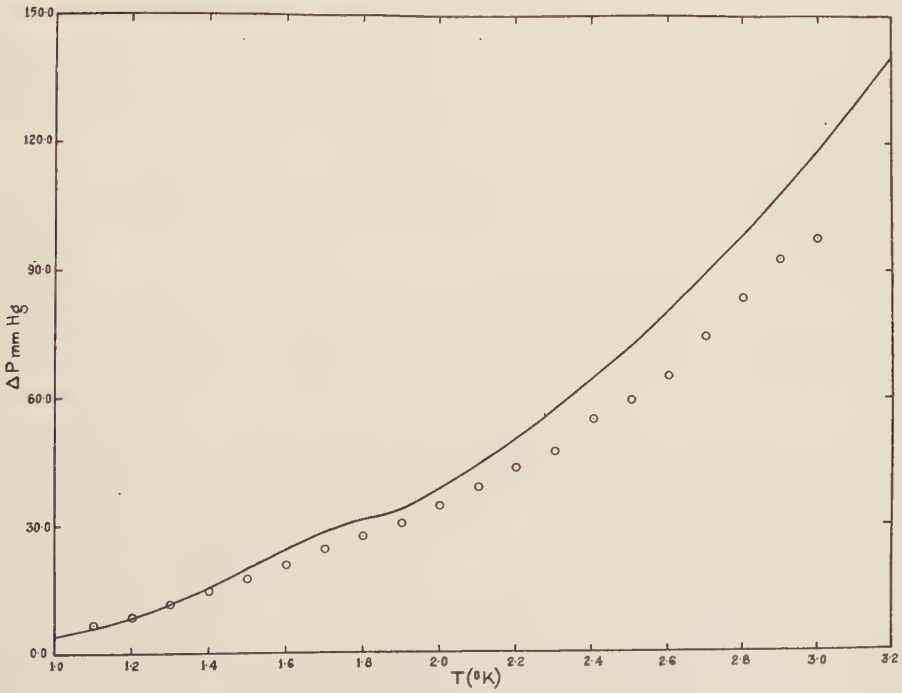

Fig. 5g. The vapour pressure difference $\Delta P_{mmHg}(=P-p_4^0)$ for 20.3 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Smoothed experimental results of Weinstock et al[42].

5(b). *Discontinuities in the temperature derivatives of the vapour pressure of He^3 and He^4 solutions at the λ-temperature of the solution.*

The existence of discontinuities in the temperature derivatives of vapour pressure at the λ-point of the solution is of interest both from the experimental and theoretical standpoints. Assuming a second order transition for the solution, de Boer and Gorter[46] have shown from purely thermodynamical considerations that (dp_3/dT) and (dp_4/dT) should have discontinuities at $T_{\lambda X_L}$. Following the procedure adopted by Nanda[26], we have

$$\delta\{\partial(P-p^0_4)/\partial T\} = -\{(1-X_L)p_3 - X_L p_4\}\,\delta(\Delta C)/RT^2\cdot\partial T_{\lambda X_L}/\partial X_L. \qquad (5\cdot11)$$

In deriving this relation we have dropped the vapour imperfection corrections for the sake of simplicity. Here $\delta\Psi\,(=\Psi_I-\Psi_{II})$ is the change in the value of the function Ψ on crossing the λ-temperature from phase I to phase II. Thus the theoretical value of the discontinuity in the temperature derivative of $\Delta P\,(=P-p_1^0)$ can be computed from equation (5·11), as all the quantities involved are calculable from the formulas of this paper.

It was believed that Sommers[18] measurements of the vapour pressures of He^3 and He^4 solutions gave no evidence of discontinuities in the partial vapour pressures at the λ-temperature of the solution. However, Nanda[26] showed that Sommers[18] data reveals a discontinous change in slope of the ΔP versus temperature curve at the λ-temperature of the solution. It is evident from Fig. 5c that the recent experimental data of vapour pressure of Daunt and Tseng[19] for 4.00 per cent solution of He^3 in He^4, in agreement with the predictions of the present theory, clearly shows a change in slope, of ΔP versus temperature curve, at the λ-temperature of the solution.

5(c). *Dew-point measurements of vapour-pressure of solutions of He^3 in He^4.*

Sommers[18] has also measured the vapour pressure of solutions of He^3 in He^4 at the dew point when the vapour just beings to condense. These measurements should not be seriously affected by heat fluxes which may come into play when the bulk liquid has been actually formed. Accordingly, we have analyzed Sommers' dew-point data in the same way as has already been done by Nanda[26]. From $(\Delta P/P)$ values given in Table I (of reference (18)) we have calculated p_3, which is used to determine X_L on the basis of the present theory, by using equation (5·10) and the quadratic expression for G_4. Knowing X_L, p_1 can be calculated by usiug equation (5·9) and compared with the experimental values. The results of such calculations are presented in Table 1, where we have used only these $(\Delta P/P)$ values for which the experimental error is less than 2 per cent. For comparison we have also included in Table 1 the corresponding values of X_L and p_1 calculated on the basis of C.R.S. (Classical Regular Solution) and de B.G.+R.S.[26] (de Boer and Gorter's theory based on regular solution model). A strange feature of Sommers' dew-point measurements is that at low temperatures and high concentrations of He^3 in the vapour phase, partial pressures for He^4 higher than the pressure

in the pure phase are observed. Such a behaviour cannot be understood on the basis of the "Classical Regular Solution" theory. Unlike the de Boer-Gorter-Taconis approach (based on the regular solution model, as given by Nanda[2(6)]) which offers a natural explanation for observed high values of p_3, this strange feature cannot be explained on the basis of the present theory. Sommers, therefore, may be regarded as right when he

Table 1. Calculation of X_L and P_3 from Sommers' dew-point data according to C.R.S. (Classical Regular Solution), de B.G.+R.S. (de Boer and Gorter's theory based on regular solution model) and present theory (using the quadratic expression for G_4).

X_v	T(°K)	p_3 mmHg	X_L			p_4 mmHg				p_4^0 mmHg
			C.R.S.	de B.G.+ R.S.	Present Theory	C.R.S.	de B.G.+ R.S.	Present Theory	Exptal.	
0.781	2.142	85.91	0.418	0.418	0.382	23.45	23.47	25.34	24.09	34.32
	1.703	27.65	0.270	0.270	0.183	6.88	7.46	7.66	7.75	8.62
	1.702	26.71	0.257	0.257	0.175	6.99	7.58	7.68	7.49	8.60
	1.688	25.62	0.254	0.254	0.170	6.56	7.15	7.25	7.18	8.10
	1.285	3.86	0.107	0.013	0.059	1.02	1.12	1.06	1.08	1.11
	1.268	3.58	0.080	0.0067	0.058	0.93	1.00	0.95	1.00	0.99
0.58	1.978	28.13	0.154	0.154	0.131	19.48	19.80	20.30	20.37	21.97
	1.662	9.87	0.130	0.084	0.059	6.56	7.07	7.05	7.15	7.32
	1.546	6.35	0.072	0.0096	0.048	4.20	4.43	4.30	4.59	4.44
	1.486	4.76	0 061	0.0011	0.041	3.20	3.40	3.29	3.44	3.38
	1.448	3.85	0.053	0.0008	0.036	2.67	2.79	2.72	2.79	2.79
0.405	2.155	22.84	0.092	0.092	0.080	33.14	33.15	33.66	33.56	35.53
	1.697	5.76	0.048	0.016	0.032	8.10	8.38	8.24	8.45	8.40
	1.695	5.72	0.048	0.016	0.031	8.04	8.32	8.18	8.41	8.34
0.13	2.159	5.36	0.027	0.027	0.018	35.75	35.76	35.87	35.84	36.30
	2.052	4.07	0.020	0.016	0.014	27.08	27.17	27.25	27.23	27.48
	1.918	2.76	0.015	0.008	0.011	18.36	18.46	18.45	18.44	18.57

Table 2. Calculation of X_L and p_3 from Sommers' smoothed dew-point data according to H.D. (Heer and Daunt) theory, C.R.S., de B.G.+R.S., and the present theory (using the quadratic expression for G_4) at 1.7°K. (P_{mmHg} is obtained by smoothing out the data graphically).

X_v	P mmHg	p_4 mmHg	X_L				p_3 mmHg				
			H.D.	C.R.S.	de B.G.+ R.S.	Present Theory	H.D.	C.R.S.	de B.G.+ R.S.	Present Theory	Exptal.
0.781	34.49	7.55	0.178	0.172	0.245	0.181	23.55	19.13	25.26	27.33	26.94
0.58	19.58	8.22	0.054	0.053	0.091	0.056	8.63	6.90	14.42	9.98	11.36
0.405	14.38	8.56	0.0042	0.0042	0.0126	—*	0.71	0.58	4.77	—*	5.82

* Absurd, because $p_4 > p_4^0$.

concludes that there must have been some hidden source of error in his measurements at low temperatures. However, the agreement between the values of p_4 calculated on the basis of the present theory, using the quadratic expression for G_4, and the experimentally determined values is in general satisfactory (See Table 1). The agreement on the basis of the present theory is as good as that obtained on the basis of de B.G.+R.S., if not better.

Table 3. Calculation X_L and p_3 from Sommers' smoothed dew-point data according to the present theory (using the quadratic expression for G_4) at 1.7°K (P_{mmHg} is obtained by using the empirical expression for X_v, as given by Sommers.).

X_v	P_{mmHg}	p_4 mmHg	X_L	p_3 mmHg	
			Present Theory	Present Theory	Exptal.
0.58	19.20	8.06	0.0751	13.08	11.14
0.405	13.97	8.31	0.0328	6.00	5.66
0.13	9.72	8.46	0.0060	1.13	1.26

Table 4. Calculation of X_L and p_3 from Sommers' unsmoothed data of vapour pressure at the dew point according to H.D., C.R.S., de B.G.+R.S. and the present theory (using the quadratic expression for G_4).

X_v	T(°K)	p_4 mmHg	X_L				p_3 mmHg				
			H.D.	C.R.S.	de B.G.+R.S.	Present Theory	H.D.	C.R.S.	de B.G.+R.S.	Present Theory	Exptal.
0.781	2.142	24.09	0.447	0.398	0.398	0.423	91.56	82.63	82.63	92.61	85.91
	1.688	7.18	0.163	0.205	0.249	0.182	22.37	21.57	25.12	26.92	25.62
0.58	1.978	20.37	0.133	0.111	0.127	0.127	25.72	20.92	25.79	27.49	28.13
	1.662	7.15	0.035	0.034	0.059	0.045	3.73	2.93	8.52	7.56	9.87
0.405	2.155	33.56	—	—	—	0.083	—	—	—	23.61	22.84
0.13	2.159	35.84	—	—	—	0.019	—	—	—	5.64	5.36

The reverse calculation, i.e., using p_4 for calculating X_L and determining p_3 from it, brings out the difference between various theories more clearly as can be seen from Tables 2, 3 and 4. In Table 2 we have used the graphically smoothed values of pressures, P_{mmHg}, at 1.7°K for different X_L, while in Table 3 use has been made of the empirical relation for X_v given by Sommers to calculate P_{mmHg} at 1.7°K for different X_v. In Table 4 unsmoothed dew-point vapour pressure data have been used.* It will be seen that the deviations of the C.R.S. and the Heer and Daunt[7] theory from the experimental

* Only the data for $T > 1.6$°K has been used for the calculation of X_L from p_4 because of the limited accuracy of the experimental results.

values are more severe for low He^3 concentrations in the vapour; for higher concentrations they become pronounced only at lower temperatures. It appears that the deviations are large whenever the concentration of the normal fluid in the solution is small. The formulas of the present theory, however, always yield results which are (considering the uncertainty in the measured values) in resonable agreement with experiment.

The general agreement of the theoretical results of the present theory with Sommers' dew-point measurements as well as with the direct vapour pressure results for fixed liquid concentrations clearly indicates that Sommers' two sets of data, obtained by different methods, are in mutual agreement. The sign and the order of magnitude of the discontinuity in the temperature derivative of JP at the λ-temperature of the solution, as is indicated by the observations of Sommers[18] and of Daunt and Tseng[19], is also in agreement with the theoretical predictions of the present theory. The root cause of the severe disagreement of the theory of de Boer and Gorter[1] (and of de Boer and Gorter's theory based on regular solution model[26] with the experimental results, therefore, seems to lie in the basic assumption of the theory.

§ 6. The distribution coefficient of He^3 and He^4 solutions

The expression for the vapour-liquid concentration ratio (distribution coefficient)

$$C_V/C_L = p_3/p_4 \cdot (1-X_L)/X_L \qquad (6\cdot 1)$$

can be easily derived from equations (5·9) and (5·10). The calculated values of (C_V/C_L) for various He^3 concentrations ranging from 0.58 to 13.00 per cent are compared in Fig. 6 and Fig. 7 with the experimental data given by Sommers[18].

As is evident from Fig. 6, the agreement between theoretical and experimental values is satisfactory.

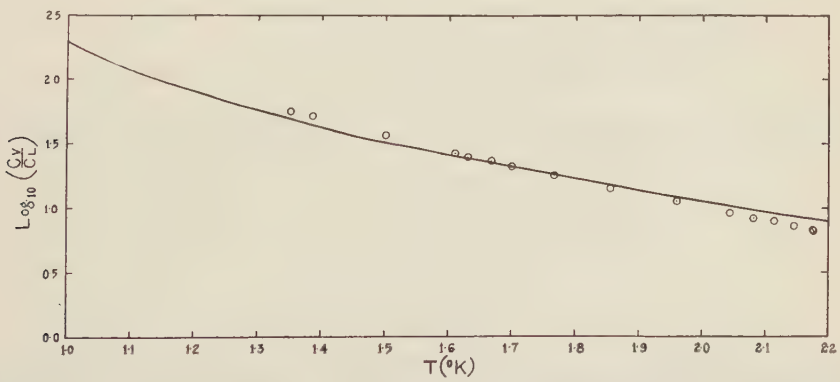

Fig. 6a. The distribution coefficient of 2.50 per cent solution of He^3 in He^4, as a function of absolute temperature, T(°K).
——: Present theory, Quadratic Expression for G_4.
⊙: Sommers[18] experimental results for 1.98 per cent solution of He^3 in He^4.

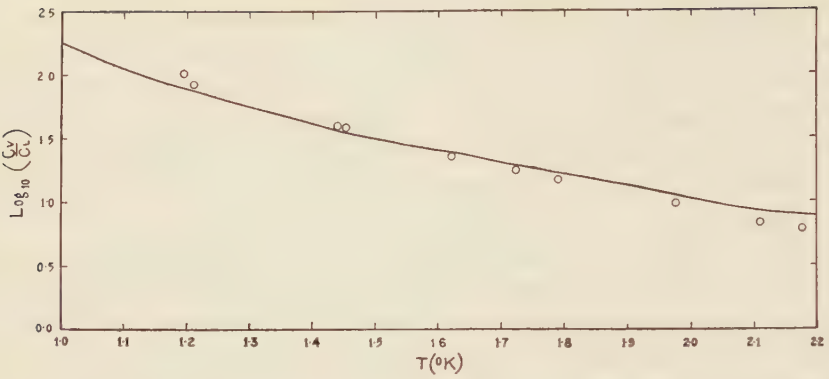

Fig. 6b. The distribution coefficient of 5.00 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] experimental results for 5.21 per cent solution of He^3 in He^4.

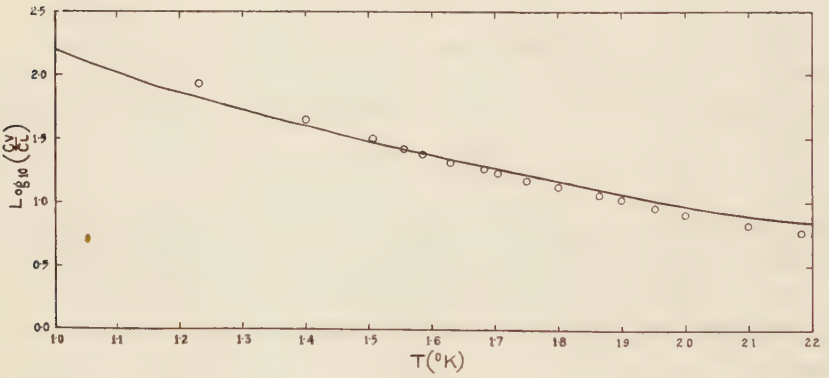

Fig. 6c. The distribution coefficient of 9.50 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
—: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] experimental results for 9.49 per cent solution of He^3 in He^4.

For dilute solutions, the expressions (5·9) and (5·10) for the partial vapour pressures reduce to

$$p_4 = p_4^0, \tag{6·2}$$

and

$$p_3 = p_3^0 X_L \cdot \exp\left[\frac{xW}{RT} + (1-x)\right], \tag{6·3}$$

where the virial correction terms have been dropped for the sake of simplicity. Thus for dilute solutions, the vapour-liquid concentration ratio reduces to

$$C_V/C_L = (p_3^0/p_4^0) \cdot \exp\left[\frac{x_0 W}{RT} + (1-x_0)\right], \tag{6·4}$$

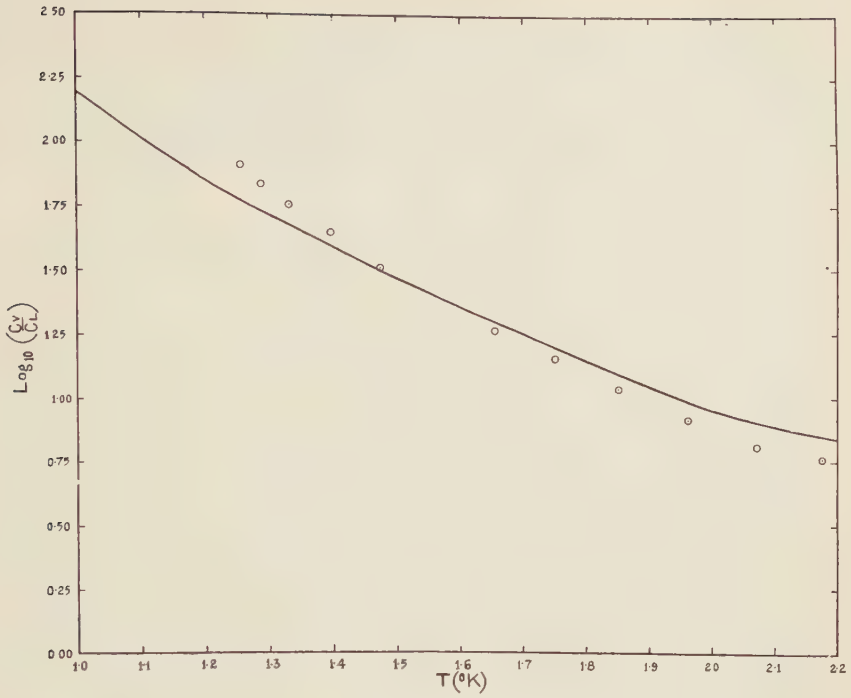

Fig. 6d. The distribution coefficient of 13.00 per cent solution of He^3 in He^4, as a function of absolute temperature, $T(°K)$.
——: Present Theory, Quadratic Expression for G_4.
⊙: Sommers[18] experimental results.

as compared with the corresponding expressions

$$C_V/C_L = p_3^0/x_0 \cdot p_4^0, \tag{6·5}$$

on the basis of de Boer and Gorter's[1] theory, and

$$C_V/C_L = p_3^0/p_4^0, \tag{6·6}$$

on the basis of an "Ideal Solution" theory (Raoult's Law).

The corresponding expression for dilute solutions on the basis of the "Modified B. E. Liquid" theory proposed by Mikura[8] is found to be

$$C_V/C_L = p_3^0/p_4^0 \cdot (V_3^0/V_4^0) \exp\left[1 - \frac{\varkappa_0}{\nu}(V_3^0/V_4^0)\left(1 + \frac{\Delta_0}{T}\right)\right], \tag{6·7}$$

where $\nu = 8.8$, $\Delta_0 = 8.609°K$ and $(V_3^0/V_4^0) = 1.36$.

We have calculated (C_V/C_L) for dilute solutions of He^3 in He^4, as a function of temperature, on the basis of the present theory by making use of the linear and quadratic expressions for G_4. The results thus obtained are compared in Fig. 7 with various theories[1),8)] and the experimental results of Sommers[18] for 0.58% solution of He^3 in He^4 and of Wansink et al.[47] for dilute solutions.

The vapour imperfection corrections have been applied using the second virial coefficients computed by Kilpatrick et al[43]. As is evident from Fig. 7 the experimental data of Sommers appears to be in strong support of the present theory rather than of de Boer and Gorter's[1] or "Ideal Solution" or Mikura's[8] theory. However, the recent experimental data of Wansink et al.[47] appears to be in good agreement with Mikura's[8] theory, especially at low temperatures.

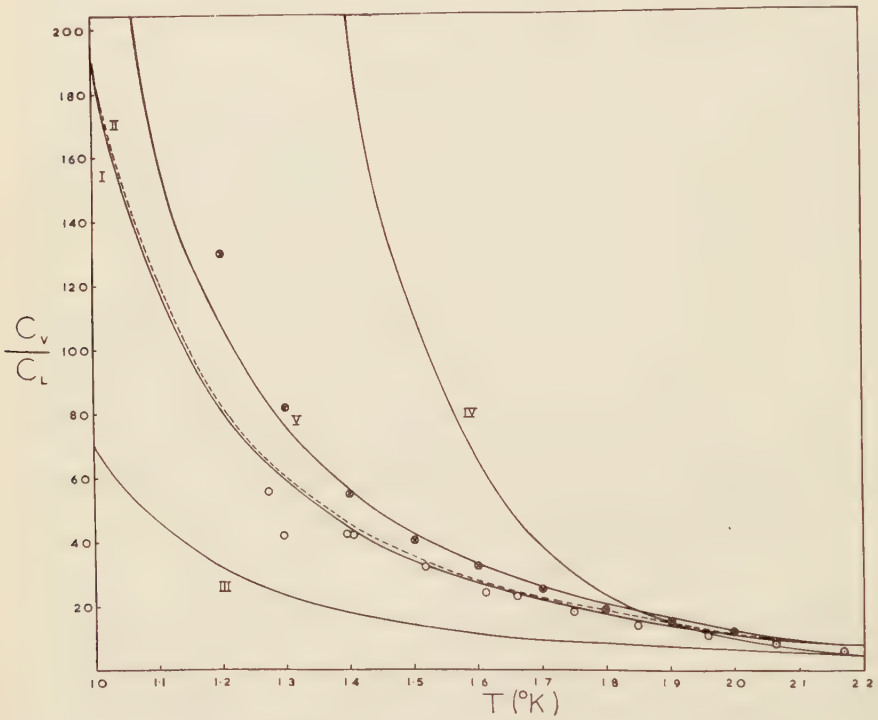

Fig. 7. The distribution coefficient of a dilute solution of He^3 in He^4, as a function of absolute temperature, $T(K°)$.
Curve I : Present Theory, Linear Expression for G_4.
Curve II : Present Theory, Quadratic Expression for G_4.
Curve III : Perfect Classical Solution Theory ("Ideal Solution").
Curve IV : de Boer and Gorter's theory[1].
Curve V : Mikura's theory[8].
⊙ : Experimental results of Sommers[18] for 0.58 per cent solution of He^3 in He^4.
⊗ : Smoothed Experimental results of Wansink et al.[47] for dilute solutions of He^3 in He^4 ($X_L \sim 0$).

§ 7. The Henry's law coefficient

The Henry's law coefficent gives us a nice chance to compare the experimental data for dilute solutions of He^3 in He^4 with the values obtained on the basis of various theories. The "Henry's Law Coefficient", h, is defined by the equation

$$h = \lim_{X_L \to 0} L_t \, (p_3/X_L) . \qquad (7 \cdot 1)$$

Thus according to the present theory, it is given (using equation (6·3)) by the relation

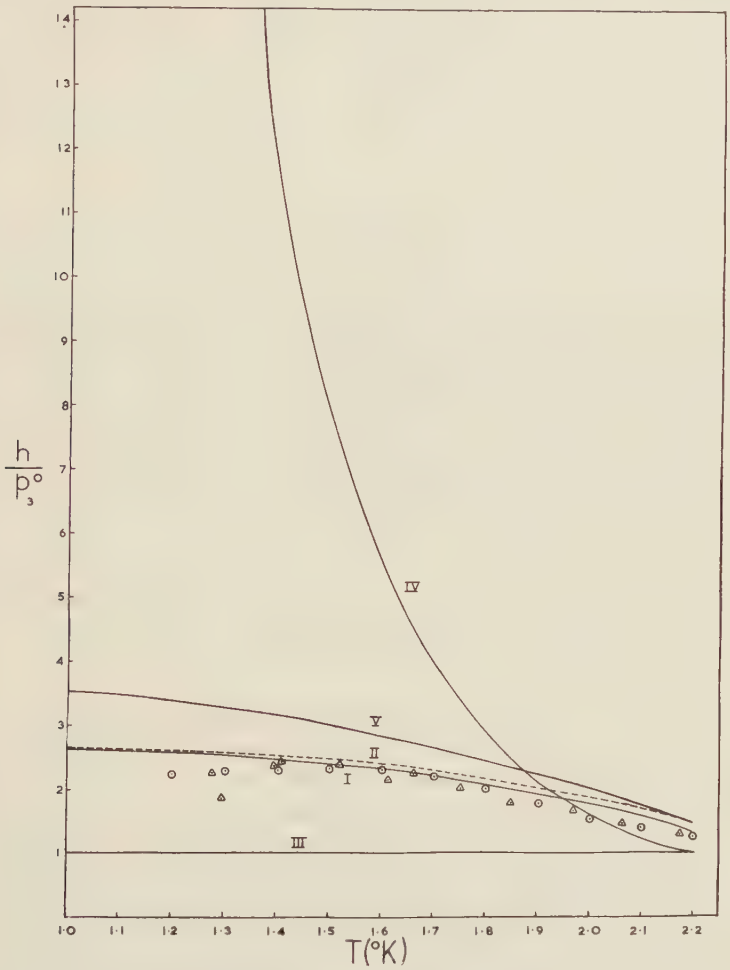

Fig. 8. The ratio of the "Henry's Law Coefficient", for solution of He^3 in He^4, h, to the vapour pressure of liquid He^3 in the pure state, p_3^0, as a function of absolute temperature, T(°K).

Curve I : Present Theory, Linear Expression for G_4.
Curve II : Present Theory, Quadratic Expression for G_4.
Curve III : Perfect Classical Solution Theory ("Ideal Solution").
Curve IV : de Boer and Gorter's theory[17].
Curve V : Mikura's theory[8].
◉ : Experimental points, derived by using Sommers[18] smoothed experimental vapour pressure data for 0.5 per cent solution of He^3 in He^4.
△ : Experimental points, derived by using Sommers[18] unsmoothed experimental vapour pressure data for 0.58 per cent solution of He^3 in He^4.

$$h = p_3^0 \cdot \exp[(1-x_0) + x_0 W/RT], \tag{7·2}$$

whereas, according to de Boer and Gorter's[1] theory

$$h = p_3^0/x_0, \tag{7·3}$$

and according to "Ideal Solution" theory (Raoult's Law)

$$h = p_3^0. \tag{7·4}$$

The corresponding expression on the basis of the "Modified B. E. Liquid" theory proposed by Mikura[8] is

$$h = p_3^0 (V_3^0/V_4^0) \exp\left[1 - \frac{x_0}{\nu}(V_3^0/V_4^0)(1 + \Delta_0/T)\right]. \tag{7·5}$$

We have calculated (h/p_3^0) for solution of He^3 in He^4, as a function of temperature, on the present theory by making use of the linear and quadratic expressions for G_4. The results thus obtained are compared in Fig. 8 with various theories[1,8] and the experimental results. The vapour imperfection corrections have been applied using the second virial coefficients computed by Kilpatrick et al[45]. The observed points plotted in Fig. 8 are calculated from Sommers' vapour pressure data concerning 0.5 and 0.58 per cent solutions of He^3 in He^4 by applying the relation

$$P = hX_L + p_4^0(1 - X_L). \tag{7·6}$$

It is evident from Fig. 8 that the experimental data is in strong support of the present theory rather than of de Boer and Gorter[1] or "Ideal Solution" or Mikura's theory[8].

§ 8. The osmotic pressure of He^3 and He^4 solutions

It is a well-known phenomenon that the liquid level in the solution of He^3 and He^4 is found to be higher than the pure He^4 level when the two liquids at the same temperature are connected by a very narrow slit above both liquid levels and equilibrium is established. This excess pressure in the solution is called "Osmotic Pressure"; the slit plays here the role of semi-permeable membrane in the similar phenomena well-known in usual solutions. The osmotic effect was first observed by Daunt et al[48]., and has subsequently been studied by Taconis et al[49]., who found that the osmotic pressure for very dilute solutions (concentration X_L up to 2.3×10^{-3}) of He^3 in He^4 is roughly given by 'Van't Hoffs Law,' namely

$$\Pi = RTC_3, \tag{8·1}$$

where C_3 is the number of moles of He^3 found in unit volume of the solution. Theoretically, eq. (8·1) has been derived by Gorter et al.[50] by considerations based on de Boer and Gorter's[1] theory. Even when de Boer-Gorter-Taconis point of view is now given up, we can derive eq. (8·1) on the basis of the present theory.

The osmotic pressure Π_0, for the osmotic equilibrium of the solvent species He^4, is related to the partial vapour pressure of the solvent by the thermodynamical relation[51]

$$\Pi_4 = (RT/\bar{V}_4) \cdot \ln (p_4^0/p_4) , \tag{8·2}$$

where p_4, p_4^0 denote respectively the partial vapour pressure of the solvent in the solution and the vapour pressure of the solvent in the pure state (at the same external pressure). $\bar{V}_4$ denotes the partial molar volume of the solvent in the solution, which for dilute solutions is indistinguishable from the molar volume V_4^0 of the pure solvent.

Ignoring the vapour imperfection corrections for dilute solutions of He^3 in He^4, the expression (5·9) reduces to

$$p_4 = p_4^0 (1-X_L)^x \cdot \exp [-X_L (1-x)] , \tag{8·3}$$

which reduces to (6·2) for $X_L \approx 0$.
Thus for dilute solution of He^3 in He^4 ($X \lesssim 0.01$),

$$\ln (p_4/p_4^0) = x \ln (1-X_L) - X_L (1-x) \approx -X_L , \tag{8·4}$$

where terms of higher powers in X_L have been ignored.

Thus the osmotic pressure for dilute solutions of He^3 in He^4 is given by

$$\Pi_4 = RT \cdot X_L / V_4^0 , \tag{8·5}$$

which is identical with eq. (8·1) proposed empirically by Taconis et al[49].

Measurements of osmotic pressure for solutions of higher He^3 concentrations are highly desirable, in view of the fact that they will provide a clue to the correct model of the He^3 and He^4 solutions.

Acknowledgements

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On Collective and Internal Motions in a Bose System

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By introducing a number of auxiliary variables, quantum-mechanical collective and internal motions in a Bose system are described. In a collective representation obtained here, the wave functions are expressed as functions of the auxiliary variables and the internal coordinates.

The energy for the ground state and the excitation energies of collective motion and internal motion are obtained for the case of a long-range interaction potential. The lowest energy obtained proves to be smaller than that afforded by the Hartree-Fock approximation. The excitations of the collective motion with smaller wave numbers than a cut-off wave number k_c correspond to Landau's phonons. The excitations of the internal motion resemble Landau's rotons; but they are forbidden for the smaller wave numbers $K \leqq k_c$.

When one applies this method to the problem of liquid helium, it becomes necessary to take account of the two-body correlation of individual particles. It is pointed out that the phonon energy depends upon the zero-point motion of the individual particles.

§ 1. Introduction

As pointed out by Landau[1], in order to clarify the superfluidity of liquid helium and its equilibrium properties at very low temperatures, it becomes necessary to introduce the organized wavy motion characterized by phonons and the hydrodynamical vortex motion characterized by rotons. The low energy part of the energy spectrum is given by phonons and the high energy part is afforded by rotons. Landau's conception of phonon and roton involves some *a priori* assumptions, although his theory has been supported by a number of experimental results. Bogolubov[2] presented an approximation method of obtaining the energy spectrum of weakly interacting Bose particles, which behaves in the same manner as Landau's spectrum if the Fourier coefficients of the interaction potential are properly adjusted. Recently, by utilizing the hydrodynamical approximation, almost the same result has been given by several authors[3]. Even in the case of a long-range weak interaction potential these results are valid only for the excitations of low energies and the excitations of the individual particles corresponding to the excitations of Landu's rotons have been quite overlooked.

On the other hand, Bohm and Pines[4] presented the collective description of metallic electrons, in which the total Hamiltonian splits into the energy of the plasma oscillation and that of the individual particles. In their theory, the collective motion of the electrons appearing as the plasma oscillation is described by a set of auxiliary variables. The superfluous

degrees of freedom of these variables are canceled by imposing a set of subsidiary conditions upon the wave functions. It has been pointed out by several authors[5] that the neglect of these subsidiary conditions can lead to absurd results.

We have shown in a foregoing paper[5] that we can modify Bohm's method so that we may obtain the correct eigenvalues, even if the subsidiary conditions are neglected. We shall apply this modified method to a Bose system to find the excitations of the internal motion as well as those of the collective oscillation. In so doing, we shall notice that the so-called random phase approximation utilized in Bohm's theory becomes invalid in the case of a short-range potential.

In Section 2, we shall consider a set of curvilinear coordinates to illustrate the present method. There we shall clarify the role of the subsidiary condition and make a remark on Adams'[5] criticism of Bohm's theory. In Section 3, we shall consider a formal problem of a long-range interaction potential taking the Fourier coefficients of the density operator as the collective variables. When the interaction potential is of such a long-range that its Fourier coefficients with larger wave numbers than a cut-off wave number k_c are negligible, the energy spectrum becomes the same with that obtained from the hydrodynamical approximation for the smaller wave numbers than k_c and it resembles the energy of Landau's roton for the larger wave numbers. The energy for the ground state is in accord with the previous result. It will be shown that this lowest energy is certainly smaller than the lowest energy obtained by the Hartree-Fock approximation. In Section 4, we shall point out that the occupation number of the Bose particles is given by that of collective quanta or phonons for the small wave numbers and that of internal quanta or rotons for the large wave numbers. Finally, in Section 5, we shall examine the applicability of the random phase approximation. The result obtained in Section 3 cannot be applied to the problem of liquid helium, because the frequency of the collective oscillation becomes divergent. By taking account of the non-orthogonality of the Fourier coefficients, it will be shown that no divergent term appears in the expression of the frequency for the sake of the correlation effect of the individual particles.

§ 2. A set of curvilinear coordinates

In order to illustrate our method of the collective description we consider a set of curvilinear coordinates in the $3N$-dimensional configuration space. We select out a coordinate $\xi(x_1,\ldots x_N)$ regarded as a simple generalization of the center-of-mass coordinate. The momentum operator associated with ξ is defined by

$$\pi=\frac{i}{\hbar}[T,\xi]=\frac{\hbar}{2mi}\sum_{j=1}^{N}\left(\frac{\partial}{\partial x_j}\cdot\frac{\partial\xi}{\partial x_j}+\frac{\partial\xi}{\partial x_j}\cdot\frac{\partial}{\partial x_j}\right), \tag{2·1}$$

where T implies the kinetic energy of particles and m is the particle mass. Geometrically speaking, the above momentum operator represents a differential operator in the direction normal to the surface given by $\xi=const.$ Here we impose a restriction on the functional form of ξ such as

$$\sum_{j=1}^{N}(\partial\hat{\xi}/\partial x_j)^2=m, \qquad \sum_{j=1}^{N}\partial^2\hat{\xi}/\partial x_j^2=0. \tag{2·2}$$

Then the momentum operator becomes

$$\pi=\frac{\hbar}{mi}\sum_{j=1}^{N}\frac{\partial\hat{\xi}}{\partial x_j}\cdot\frac{\partial}{\partial x_j}, \tag{2·3}$$

and we find

$$[\hat{\xi},\ \pi]=i\hbar, \tag{2·4}$$

which shows that π becomes the canonical conjugate of $\hat{\xi}$. When we compose $3N-1$ curvilinear coordinates $\zeta_1,\ldots,\zeta_{3N-1}$ orthogonal to $\hat{\xi}$ and also orthogonal to one another, the momentum operator turns out to be

$$\pi=\frac{\hbar}{2mi}\left\{g^{-1/2}\frac{\partial}{\partial\hat{\xi}}(g^{1/2}g_\xi^{-1})+2g_\xi^{-1}\frac{\partial}{\partial\hat{\xi}}\right\}=\frac{\hbar}{i}\frac{\partial}{\partial\hat{\xi}}, \tag{2·5}$$

remembering that the determinant of the metric tensor denoted by g is independent of $\hat{\xi}$ and the diagonal element of the tensor with respect to $\hat{\xi}$ denoted by g_ξ becomes m^{-1}. The total Hamiltonian can be expressed in terms of the internal coordinates as

$$\begin{aligned}H&=T+V\\&=-\frac{\hbar^2}{2m}\sum_{j=1}^{3N-1}g^{-1/2}\frac{\partial}{\partial\zeta_j}(g^{1/2}g_j^{-1})\frac{\partial}{\partial\zeta_j}+\frac{\pi^2}{2}+V(\zeta_1,\ldots,\zeta_{3N-1},\hat{\xi}),\end{aligned} \tag{2·6}$$

where g_j stands for a diagonal element of the metric tensor. When H is expanded in a power series of $\hat{\xi}$ the expansion coefficients become independent of both $\hat{\xi}$ and π, hence they are commutative with both $\hat{\xi}$ and π. Inversely, as shown by Tomonaga[6], for a given set of canonical variables $\hat{\xi}$ and π the total Hamiltonian can be expanded in such a way that the expansion coefficients of $\hat{\xi}$ become commutative with both $\hat{\xi}$ and π.

We shall apply Bohm's theory[4] of the collective description to separate the variable $\hat{\xi}$ from the total Hamiltonian in the way mentioned above, and it will be pointed out that there appears an important difference between Bohm's theory and the Tomonaga's theory.

Considering $\hat{\xi}$ as the collective variable to describe the collective oscillation of this boson system, we transform the total Hamiltonian by a contact transformation of the form

$$U_1=\exp(i/\hbar\cdot\hat{\xi}_1\beta). \tag{2·7}$$

In the original representation we impose the subsidiary condition of the form

$$\alpha\phi_0=0, \tag{2·8}$$

where α is the auxiliary variable and β is its canonical conjugate.

Noting (2·1), the transformed Hamiltonian turns out into

$$H_1=U_1^{-1}HU_1=T+V+\pi\beta+\beta^2/2, \tag{2·9}$$

and in this representation the subsidiary condition takes the form

$$(\alpha-\hat{\xi})\phi_1=0. \tag{2·10}$$

In our case, a contact transformation called the transformation of Bohm and Pines that cancels the interaction term $\pi\beta$ is given by

$$U_B=\exp(-i/\hbar\cdot\pi\alpha). \tag{2·11}$$

The Hamiltonian in Bohm's representation is now expressed by

$$\begin{aligned} H_B&=U_B^{-1}H_1U_B \\ &=T-\pi^2/2+V+(i/\hbar)\,\alpha[\pi,\ (T+V)] \\ &\quad+\tfrac{1}{2}(i/\hbar)^2\alpha^2[\pi,\ [\pi,\ (T+V)]]+\cdots, \end{aligned} \tag{2·12}$$

and the wave function ϕ_B is imposed by the subsidiary condition such as

$$\hat{\xi}\phi_B=0 \quad \text{or} \quad \partial\phi_B/\partial\pi=0. \tag{2·13}$$

The Hamiltonian H_B commutes with $\hat{\xi}$ but does not commute with π. This fact implies that the coefficients of α^n in H_B depend upon the collective variable. On the other hand, the subsidiary condition means that the eigenfunctions obtained in the momentum space should be independent of the momentum operator π. But the eigenvalue problem in the momentum space is much more difficult than that in the ordinary space, so that it is difficult for us to select the wave functions satisfying (2·13) from the set of eigenfunctions of H_B. If we solve the eigenvalue problem regardless of the subsidiary condition there appear, of course, some fictitious states giving rise to the incorrect eigenvalues as pointed out in a foregoing paper[5]. This difficulty in taking account of the subsidiary condition does not seem to be a small drawback in Bohm's theory, and it is connected with the incomplete separation of the collective variable and the internal variables. By taking account of the orthogonality between the collective coordinate and the internal coordinates, we can modify Bohm's theory so that the subsidiary condition may be treated in a systematic way. It can be shown that by applying the transformation U_1 again, the transformed Hamiltonian, as well as other transformed quantities, becomes commutative with both $\hat{\xi}$ and π. To show this we have only to notice that

$$\hat{\xi}U=U\alpha, \quad \pi U=U\beta, \tag{2·14}$$

$$U=U_1U_BU_1. \tag{2·15}$$

Indeed, the transformed Hamiltonian H_c is given by

$$\begin{aligned} H_c&=T+V+(i/\hbar)\,(\alpha-\hat{\xi})[\pi,\ (T+V)] \\ &\quad+\beta^2/2+\tfrac{1}{2}(i/\hbar)^2(\alpha-\hat{\xi})^2[\pi,[\pi,\ (T+V)]]+\cdots, \end{aligned} \tag{2·16}$$

where all the expansion coefficients with respect to α are precisely in accord with Tomonaga's result.[6] As the collective variable has been completely separated in this representation, the eigenfunctions are expressed by the internal coordinates and the auxiliary variable. In principle, the eigenvalue problem can be solved in the $3N-1$ dimensional restricted configuration space. But, in practice, we inquire about the eigenfunctions of

H_c in the $3N$ dimensional total configuration space. In this case the eigenfunctions obtained may depend upon the collective coordinate $\tilde{\xi}$. If these eigenfunctions are correct, each of them must be of the form $\phi_c(x_1, x_2, \cdots, x_N, \alpha) = \phi(\zeta_1, \cdots, \zeta_{3N-1}, \alpha) F(\tilde{\xi})$, where ϕ is independent of $\tilde{\xi}$ and $F(\tilde{\xi})$ is an arbitrary function of $\tilde{\xi}$. The eigenvalue corresponding to ϕ_c is independent of the form of $F(\tilde{\xi})$. In fact, we get

$$H_c\phi_c = F(\tilde{\xi}) H_c\phi = E_c\phi_c . \tag{2·17}$$

Example: In order to clarify this point, we consider a Hamitonian of a one-dimensional two-particle system which is independent of the center-of-mass coordinate and momentum. As H_c, we assume

$$H_c = \frac{(p_1-p_2)^2}{4} + \frac{(x_1-x_2)^2}{4} + \beta^2/2 + \alpha^2/2 = H_0 - (p_1+p_2)^2/4 - (x_1+x_2)^2/4. \tag{E·1}$$

In the restricted configuration space of the relative coordinate, the eigenfunction for the lowest state is

$$\phi_0 \propto \exp[-(x_1-x_2)^2/4] \exp(-\alpha^2/2), \quad \hbar = 1. \tag{E·2}$$

When we solve the problem in the representation that H_0 is diagonal, the eigenfunction is given by

$$\phi_{c,0} \propto \exp[-(x_1^2+x_2^2)/2] \exp(-\alpha^2/2) = \exp[-(x_1-x_2)^2/4] \exp(-\alpha^2/2) \exp[-(x_1+x_2)^2/4], \tag{E·3}$$

which depends upon the center-of-mass coordinate and $\exp[-(x_1+x_2)^2/4]$ corresponds to the above arbitrary function $F(\xi)$.

Here we can see that the role of the subsidiary condition is to determine the functional form of $F(\tilde{\xi})$. For instance, if we put

$$\{(\omega/2\hbar)^{1/2}\tilde{\xi} + i(1/2\hbar\omega)^{1/2}\pi\}\phi_c = 0, \tag{2·18}$$

$F(\tilde{\xi})$ becomes the Gaussian such as

$$F(\tilde{\xi}) = \exp(-\omega\tilde{\xi}^2/2\hbar). \tag{2·19}$$

In the original representation the subsidiary condition corresponding to (2·18) is expressed as

$$\{(\omega/2\hbar)^{1/2}\alpha + i(1/2\hbar\omega)^{1/2}\beta\}\phi_0 = 0, \tag{2·20}$$

which takes the place of Bohm's condition (2·8).

The present representation has the advantage of Bohm's representation, because all the eigenvalues are correct, even if we neglect the subsidiary condition to obtain them. As pointed out in a foregoing paper,[5] the eigenvalues obtained in Bohm's representation are not always correct, unless we take account of the subsidiary condition in each step of the calculation, because the Hamiltonian H_B, is not independent of the collective variable.

Among the eigenfunctions of H_c, those satisfying the subsidiary condition are called proper functions. The other eigenfunctions incompatible with the subsidiary condition are called improper functions which should be excluded from the family of the eigenfunctions. The eigenfunctions independent of the collective variable are called true functions.

The form of the subsidiary condition is not unique. As another form, we may assume

$$\pi\phi_c = \hbar\partial\phi_c/i\partial\tilde{\xi} = 0, \tag{2·21}$$

instead of (2·18). Inasmuch as the true functions are unaffected by the subsidiary condition, the form of the condition is quite at our disposal.

The eigenfunctions of H_c are obtained in the following manner. When the higher order commutators of π and H such as $[\pi, [\pi, [\ldots[\pi, H]]\ldots]$ can be negligible, the total Hamiltonian H_c splits into three parts: the internal Hamiltonian H_{in} being independent of α and β, the collective Hamiltonian H_{coll} being quadratic in α and β and the interaction Hamiltonian H' being linear in α. First, we try to diagonalize the Hamiltonian $H_0 = H_{in} + H_{coll}$ regarding H' as the perturbation energy. We assume the trial function for H_0 in a product form such as

$$\phi_0 = \psi'(x_1, \ldots, x_N, \alpha)\,\chi(\alpha), \tag{2·22}$$

where ψ' is a function of the N particle coordinates and α and χ is a function of the auxiliary variable α. This assumption corresponds to the well-known Born-Oppenheimer approximation[7]. If the coefficient of α^2 is constant or almost commutative with H_{in}, we can take advantage of an eigenfunction of H_{in} as ψ' and the average of the coefficient of $\alpha^2/2$ with respect to ψ' can be recognized as the square frequency of the collective oscillation. This frequency denoted by ω corresponds to the plasma frequency in Bohm's theory. The eigenfunctions of H_0 obtained in this way may depend upon the collective coordinate $\tilde{\xi}$ through the arbitrary functions $F(\tilde{\xi})$. Among them we select out the proper functions. This adiabatic approximation can be justified, when the differences of the energy eigenvalues given by ψ' are much larger than that afforded by χ, namely $\hbar\omega$, and when the derivatives of ψ' with respect to α such as $\partial\psi'/\partial\alpha$ and $\partial^2\psi'/\partial\alpha^2$ can be ignored.

In the end of this section we would like to make a remark on Adams' criticism[5] pointing out that the conception of the number of the oscillation quanta is meaningless. Also in the above problem, his statement is true, so far as the original representation concerns. In the original representation where the subsidiary condition is given by (2·10) the expectation value of β^2 is given by

$$\langle\beta^2\rangle_1 = (\phi_1, \beta^2\phi_1) = (\phi_0, \beta^2\phi_0) = \langle\beta^2\rangle_0, \tag{2·23}$$

since β commutes with U_1. Considering that ϕ_0 belongs to the sharp value of α, this expectation value becomes divergent. This is the essential point of Adams' criticism. On the other hand in the present collective representation, we get

$$\langle\beta^2\rangle_c = (\phi_c, \beta^2\phi_c) = (\phi_0, \pi^2\phi_0) = \langle\pi^2\rangle_0, \tag{2·24}$$

which is not divergent in general. Hence the number of the oscillation quanta defined by

$$M = (2\hbar\omega)^{-1}(\beta^2 + \omega^2\alpha^2 - \hbar\omega) \tag{2·25}$$

involves no divergent term. Therefore, in this representation, the occupation number of the quanta proves to be a useful conception.

The condition (2·2) is satisfied by a linear combination of the particle coordinates such as the center-of-mass coordinate. In Tomonaga's theory[6], there appear collective variables satisfying the second condition of (2·2) which satisfy the first condition only approximately. When we introduce the canonical momentum of the collective variable by the Lagrangian, the above transformation (2·15) can be applied to a more generic collective coordinate of an arbitrary functional form; however, the Hamiltonian H_c will become very much complicated in such a case.

§ 3. Density waves

Here we consider the Fourier coefficients of the density operator* $\rho_k=\sum_j^N \exp(-ikx_j)$ or $\xi_k=(m/N)^{1/2}|k|\rho_k$ as the collective variables. Being different from the case of the set of curvilinear coordinates, these variables are not orthogonal to one another and do not satisfy the condition: $\sum_j^N \partial^2\xi_k/\partial x_j^2=0$ which corresponds to (2·2). But, within the limit of the following approximation, these variables may be considered as orthogonal to one another.

Introducing the momentum operators defined by

$$\pi_k=\frac{i}{\hbar}[T,\hat{\xi}_{-k}]=\frac{\hbar}{2mi}\sum_j^N\left(2\frac{\partial\xi_{-k}}{\partial x_j}\cdot\frac{\partial}{\partial x_j}+\frac{\partial^2\xi_{-k}}{\partial x_j^2}\right)$$

$$=i\hbar(mN)^{-1/2}|k|^{-1}\sum_j^N\exp(ik\cdot x_j)\left\{(k\cdot p_j)+\frac{\hbar k^2}{2}\right\}, \tag{3·1}$$

we get

$$[\pi_k,\hat{\xi}_l]=\frac{\hbar}{mi}\sum_j^N\frac{\partial\hat{\xi}_{-k}}{\partial x_j}\cdot\frac{\partial\hat{\xi}_l}{\partial x_j}=-\frac{i\hbar}{N}\frac{(k\cdot l)}{|k||l|}\rho_{l-k},\quad \rho_0=N. \tag{3·2}$$

For $k=l$, (3·2) becomes

$$[\pi_k,\hat{\xi}_k]=\hbar/i. \tag{3·3}$$

Tomonaga[6] considered that these variables specified by small wave numbers such as $|k|\leq k_c$, are almost orthogonal to one another, where $\lambda_c=2\pi/k_c$ is smaller than the mean atomic spacing. Then the commutator (3·2) is replaced by

$$[\pi_k,\hat{\xi}_l]=\hbar/i\cdot\delta_{kl},\quad |k|,\ |l|\leq k_c. \tag{3·3}$$

The error arising from this assumption can be estimated by a statistical reasoning to take account of the mean deviation of the commutator (3·2) with respect to the ground state, which is expressed by

$$D=(\hbar^2/N\cdot(k\cdot l)^2/k^2l^2\cdot F_{k-l})^{1/2} \tag{3·4}$$

* We consider a cubic box of unit volume. The Fourier coefficent of a function $f(x)$ is defined by

$$f_k=\int f(x)\exp(-ik\cdot x)\,d^3x.$$

where F_k given by the average with respect to the ground state: $N^{-1}\langle \rho_k \rho_{-k} \rangle_0$ implies the form factor observed by the scattering of x-rays. As will be mentioned in Section 5, the form factor increases linearly with the wave number for small wave numbers and tends to unity as the wave number increases in the case of liquid helium as well as in the formal case of a long-range potential discussed here. Therefore we can see that the mean deviation (3·4) is at most of order $N^{-1/2}$ and the approximation (3·3) is useful as far as the mean deviation is negligible for smaller wave numbers than $2k_c$. In Section 4, in the case of a long-range potential, it will be shown that the form factor obtained by using the exact commutation relations (3·2) proves to be equal to that derived by using (3·3). Even in the case of a short-range potential, (3·3) will be useful for sufficiently small wave numbers.

Next, the commutator between π_k and π_l is given by

$$[\pi_k, \pi_l] = \frac{\hbar}{mN} \frac{(k \cdot l)}{|k||l|} \sum_j^N \exp\{i(k+l) \cdot x_j\} \, (k-l) \cdot \{p_j + \hbar (k+l)/2\}, \tag{3·5}$$

which turns out for $k = -l$ into the form

$$[\pi_k, \pi_{-k}] = -2 \frac{\hbar}{mN} \sum_j^N (k \cdot p_j), \tag{3·6}$$

whose mean deviation becomes negligible, if the center-of-mass motion can be ignored. According to a similar statistical reasoning applied to (3·2), the commutator (3·5) is replaced by

$$[\pi_k, \pi_l] = 0, \quad |k|, |l| \leq k_c. \tag{3·7}$$

Now, in terms of ξ_k and π_k for $|k| \leq k_c$, the contact transformation corresponding to (2·15) takes the form

$$\left.\begin{aligned} &U = U_1 U_2 U_1, \\ &U_1 = \exp(i/\hbar \cdot \sum_k^{k_c} \xi_k \beta_k), \quad U_2 = \exp(-i/\hbar \cdot \sum_k^{k_c} \pi_k \alpha_k), \end{aligned}\right\} \tag{3·8}$$

$$[\alpha_k, \beta_l] = i\hbar \delta_{kl}, \tag{3·9}$$

where α_k and β_k represent the set of auxiliary variables.

Here we restrict ourselves to such a formal case that the interaction potential is so long-range that the Fourier coefficients of the potential specified by the larger wave numbers than k_c are negligible and the interaction potential can be written in terms of the collective variables ξ_k as

$$V = \frac{N}{2m} \sum_{k \neq 0}^{k_c} k^2 V_k \xi_k \xi_{-k} + \frac{N^2}{2} V_0 - \frac{N}{2} V(0), \quad V(0) = \sum_k V_k. \tag{3·10}$$

Next, in order to cancel the redundant degrees of freedom of the auxiliary variables, we impose a set of subsidiary conditions of the form

$$\Omega_k^{(0)} \psi_0 = 0, \quad \Omega_k^{(0)} = \left(i \frac{m^{1/2}}{\hbar |k|} \beta_{-k} + \frac{|k|}{2m^{1/2}} \alpha_k \right), \tag{3·11}$$

which corresponds to (2·19) and turns out in the collective representation to be

$$Q_k^{(c)}\psi_c=0, \quad Q_k^{(c)}=\left(i\frac{m^{1/2}}{\hbar|k|}\pi_{-k}+\frac{|k|}{2m^{1/2}}\xi_k\right). \tag{3·12}$$

Under these conditions we consider the eigenvalue problem of the transformed Hamiltonian $H_c=U^{-1}HU$ expressed by

$$\begin{aligned}H_c=&T-\tfrac{1}{2}\sum_k^{k_c}\pi_k\pi_{-k}+\frac{i}{\hbar}\sum_k^{k_c}(\alpha_k-\tilde{\xi}_k)[\pi_k, T]\\&+\tfrac{1}{2}(i/\hbar)^2\sum_k^{k_c}(\alpha_k-\tilde{\xi}_k)(\alpha_{-k}-\tilde{\xi}_{-k})[\pi_{-k}, [\pi_k, T]]+\frac{N}{2m}\sum_k^{k_c}k^2V_k\alpha_k\alpha_{-k}\\&+\tfrac{1}{2}\sum_k^{k_c}\beta_k\beta_{-k}+\frac{N^2}{2}V_0-\frac{N}{2}V(0),\end{aligned} \tag{3·13}$$

where we have used the approximation akin to (3·3) such that $[\pi_l, [\pi_k, T]]=[\pi_{-k}[\pi_k, T]]\delta_{-lk}$. As mentioned in Section 2 those terms independent of the auxiliary variables are called the internal energy which becomes

$$H_{\text{in}}=T-\tfrac{1}{2}\sum_k^{k_c}\pi_k\pi_{-k}-\frac{i}{\hbar}\sum_k^{k_c}\tilde{\xi}_k[\pi_k, T]+\tfrac{1}{2}\left(\frac{i}{\hbar}\right)^2\sum_k^{k_c}\tilde{\xi}_k\tilde{\xi}_{-k}[\pi_{-k}[\pi_k, T]]. \tag{3·14}$$

The second order terms in α_k and β_k are called the collective energy which is given by

$$H_{\text{coll}}=\tfrac{1}{2}\sum_k^{k_c}\beta_k\beta_{-k}+\tfrac{1}{2}\sum_k^{k_c}\left\{(i/\hbar)^2[\pi_{-k}, [\pi_k, T]]+\frac{N}{m}k^2V_k\right\}\alpha_k\alpha_{-k}, \tag{3·15}$$

and the energy composed of the linear terms of α_k given by

$$H'=\frac{i}{\hbar}\sum_k^{k_c}[\pi_k, T]\alpha_k-\left(\frac{i}{\hbar}\right)^2\sum_k^{k_c}\tilde{\xi}_{-k}[\pi_{-k}, [\pi_k, T]]\alpha_k \tag{3·16}$$

is considered as the interaction energy between the internal motion and the collective oscillation. We ignore the other higher order terms appearing in H_c which come from the higher order commutators such as $[\pi_l, [\ldots, [\pi_{-k}, [\pi_k, T]]\ldots]]$. These higher order terms give rise to the anharmonic terms* of higher order than the third order in α_k.

Before we proceed to the eigenvalue problem of H_c, we give here the explicit forms of the commutators as follows:

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* Utilizing the hydrodynamical approximation, we have shown for the one-dimensional case that the contribution from the third order anharmonic term corresponding to the above one is negligible compared with that coming from another anharmonic term such as $\sum_{k+l\neq 0}k-l||k|^{-1}|l|^{-1}(k\cdot l)\alpha_{k-l}\beta_k\beta_{-l}$ which would appear, if we applied the exact commutation relation (3·2):(T. Nishiyama and K. Kamiya, Proc. the International Conference of Theor. Phys. Kyoto, September 1953.). As far as the approximate commutation relations are utilized, the detailed discussion about the above anharmonic terms is less interesting.

$$i/\hbar\cdot[\pi_k,\ T]=(m/N)^{1/2}|k|^{-1}(kk:K_{-k})\,, \tag{3·17}$$

$$(i/\hbar)^2[\pi_{-k},[\pi_k,T]]=(m^2N)^{-1}\sum_j^N\{3(k\cdot p_j)^2+\hbar^2k^4/4\}\,, \tag{3·18}$$

where K_k is the Fourier coefficient of the kinetic stress tensor defined by

$$K_k^{(12)}=\int K^{(12)}(x)\exp(-ik\cdot x)\,dx=m^{-2}\sum_j^N\exp(-ik\cdot x_j)\Big\{p_j^{(1)}p_j^{(2)}-\frac{\hbar k^{(1)}}{2}p_j^{(2)}-\frac{\hbar k^{(2)}}{2}p_j^{(1)}+\frac{\hbar^2}{4}k^{(1)}k^{(2)}\Big\}\,, \tag{3·19}$$

and the notation $A:B$ implies the inner product of tensors: $\sum_{i,j}A_{ij}B_{ji}$, $i,j=1,2,3$. These expressions are in accord with those given by Tomonaga for the one-dimensional fermion problem.

For the many Bose particle system in question, first we consider the eigenvalue problem of $H_{\rm in}$ and we select the proper functions from the set of the eigenfunctions. In so doing, we restrict ourselves to the zero-particle state and the one-particle states. In the adiabatic approximation the eigenfunction of H_c for the ground state is given by the eigenfunction of $H_{\rm in}$ for the lowest zero-particle state.

When all the particles are degenerate to the single-particle state of the zero momentum, the wave function normalized in the unit volume is of the simple form such as

$$\psi_0=1 \tag{3·20}$$

which corresponds to the wave function (2·22). According to the adiabatic approximation[7] we determine the Hamiltonian of the oscillation quanta by obtaining the eigenvalue of H_c for ψ_0, regarding the auxiliary variables α_k as the external parameters. Noting that

$$H_{\rm in}\psi_0=E_{\rm in,0}\psi_0=\frac{\hbar^2}{8mN}\sum_k^{k_c}k^2\Big[\sum_{j\neq j'}^N\exp\{ik\cdot(x_j-x_{j'})\}-\sum_{j,j'}^N\exp\{ik\cdot(x_j-x_{j'})\}-N\Big]\psi_0$$

$$=-\sum_k^{k_c}\frac{\hbar^2k^2}{4m}\psi_0, \tag{3·21}$$

$$H_{\rm coll}\psi_0=\tfrac{1}{2}\sum_k^{k_c}(\beta_k\beta_{-k}+\omega_k^2\alpha_k\alpha_{-k})\,\psi_0,\quad \omega_k=|k|\Big(\frac{\hbar^2k^2}{4m^2}+\frac{N}{m}V_k\Big)^{1/2*}\,, \tag{3·22}$$

$$H'\psi_0=0, \tag{3·23}$$

$$H_c\psi_0=E_\alpha\psi_0\,, \tag{3·24}$$

the eigenvalue becomes

$$E_\alpha=\sum_k^{k_c}\Big(\tfrac{1}{2}\beta_k\beta_{-k}+\frac{\omega_k^2}{2}\alpha_k\alpha_{-k}-\frac{\hbar^2k^2}{4m}\Big)+\frac{N^2}{2}V_0-\frac{N}{2}V(0)\,, \tag{3·25}$$

* This expression is slightly different from that given by Bogolubov[2] by the total number of particles N which is replaced by n_0, the occupation number of the zero-momentum state, in his paper.

and we get the wave functions of the oscillation quanta from the eigenvalue equation of the form

$$\frac{1}{2}\sum_{k}^{k_c}(\beta_k\beta_{-k}+\omega_k^2\alpha_k\alpha_{-k})\chi=\epsilon_{\rm coll}\chi\,. \tag{3·26}$$

The wave function χ_0 for the zero-point oscillation is given by

$$\chi_0=\prod_{|k|<k_c}(\omega_k/\pi\hbar)^{1/2}\exp(-\omega_k\alpha_k\alpha_{-k}/2\hbar)\,. \tag{3·27}$$

Accordingly, the eigenfunction for the lowest state becomes

$$\phi^{(0)}=\psi_0\chi_0\,, \tag{3·28}$$

for which the lowest eigenvalue turns out to be

$$E_0=\frac{\hbar}{2}\sum_{k}^{k_c}\omega_k-\frac{\hbar^2}{4m}\sum_{k}^{k_c}k^2+\frac{N^2}{2}V_0-\frac{N}{2}V(0)\,. \tag{3·29}$$

This result is precisely in accord with the former one obtained by means of the hydrodynamical approximation[3]. Recently the same result has been obtained by Bogolubov* and Zubarev[3], independently.

It can easily be shown that (3·28) satisfies the subsidiary conditions. As far as we deal with the approximate commutation relations, the subsidiary conditions can also be satisfied by the wave function of the form

$$\phi^{(0)\prime}=\prod_{|k|\leq k_c}F(|\xi_k|)\,\phi^{(0)},\quad F(|\xi_k|)=\exp(-k^2\xi_k\xi_{-k}/4m)\,.$$

Therefore the wave function is determined except for this arbitrariness arising from the approximate commutation relation.

The adiabatic approximation can be justified when the spacing of the energy levels of the collective oscillation, namely $\Delta\hbar\omega_k$ is much smaller than that of the internal motion. In the following we shall show that the lowest excitation energy of the internal motion is much larger than the energy difference $\Delta\hbar\omega_k$.

The lowest excited states for the wave numbers smaller than k_c are represented by the one-phonon wave functions of the form

$$\phi_k^{(1)}=\psi_0\chi_k^{(1)}\,, \tag{3·30}$$

$$\chi_k^{(1)}=\frac{\alpha_{-k}}{(\hbar/2\omega_k)^{1/2}}\chi_0\,,$$

and the excitation energy is given by

$$(H_c-E_0)\,\phi_k^{(1)}=\epsilon_k\phi_k^{(1)}=\hbar\omega_k\phi_k^{(1)}\,, \tag{3·31}$$

where the wave functions (3·30) are the proper functions.

* In the original paper written in English,[2] he spells his name "Bogolubov". According to the variant translations of cyrillic characters, it may be spelled "Bogoliubov" or "Bogolyubov".

For the larger wave numbers than k_c we consider the one-particle wave functions of internal quanta expressed by

$$\psi_k^{(1)} = N^{-1/2} \sum_j^N \exp(iK \cdot x_j), \quad |K| > k_c, \tag{3·32}$$

corresponding to the configuration in the momentum space that the K-th single-particle state is occupied by one quantum and the occupation number of the lowest single-particle state is $N-1$. Being different from (3·28) and (3·30), these wave functions are not the eigenfunctions of H_c, but they serve as the trial functions for the excited states (Appendix 1). The excitation energy corresponding to (3·31) amounts to

$$\epsilon_k = (\psi_k^{(1)}\chi_0,\ (H_c - E_0)\psi_k^{(1)}\chi_0) = \frac{\hbar^2 K^2}{2m^*},\quad \frac{1}{m^*} = \frac{1}{m} + \frac{2\pi\hbar}{Nm^2}\int_0^{k_c} \frac{k^4 d|k|}{\omega_k (2\pi)^3}. \tag{3·33}$$

If we neglect the interaction energy H', the energy spectrum for the small wave numbers is solely determined by the phonons, while for the large wave numbers it is given by the internal quanta and we obtain

$$\epsilon_k = \begin{cases} \hbar\omega_k, & |k| \leq k_c \\ \hbar^2 k^2/2m^*, & |k| > k_c. \end{cases} \tag{3·34}$$ *

Utilizing the second quantization the above results will be confirmed in Appendix 1. The expression (3·32) may be considered as a special case of the trial function proposed by Feynman.[9]

Now, let us consider the interaction energy H', which causes the scattering of an internal quantum accompanied with emission of a phonon. The matrix element corresponding to such a scattering is denoted by

$$\langle K-k\,;\,k|H'|K\,;\,O\rangle, \quad |K| > k_c, \tag{3·35}$$

where K and $K-k$ stand for the wave numbers of the internal quanta, k means the wave number of the emitted phonon and the state of the zero-point oscillation represented by χ_0 is indicated here by O. The matrix elements between the ground state and the one-particle state of the internal quanta disappear. Utilizing the time-dependent perturbation theory (Appendix 2), we obtain the self-energy Δ_v and the level width $\hbar\Gamma_v$ such as

$$\Delta_v = \left(\frac{1}{3} - \frac{\gamma}{2}\right)\alpha^2 \frac{N'}{N}\frac{m^* v^2}{2} + \left(1 + \frac{1}{2}\log\frac{|1-\beta|}{1+\beta}\right)\frac{\hbar\Gamma_v}{\pi\beta}, \tag{3·36}$$

* Brenig[10] obtained almost the same result with (3·34). He assumed the effective mass to be $3m/2$. From the classical view-point, when a particle moves, the surrounding particles set in a flow motion which increases the effective mass. This is Feynman's reasoning[11] to derive the large effective mass $3m/2$. On the other hand, in case of a system of particles interacting with one another through a potential of finite range, the spacings of the energy levels of the particles are extended, because every particle is considered as moving with a hard core which makes the extent of the space, in which the particles move, effectively small. This effect can be taken into account by introducing a small effective mass in contrast with the classical consideration. The present result (3·33) shows that such a quantum-mechanical effect predominates over the classical effect.

$$\Gamma_v=\left(1-\frac{3}{2}\gamma+\frac{3}{5}\gamma^2\right)\alpha^3\frac{\pi\beta}{2\hbar}m^*s^2, \tag{3·37}$$

$$N'=k_c^3/6\pi^2,\ v=\hbar|K|/m^*,\ \alpha=m^*/m,\ \beta=s/v,\ \gamma=\hbar k_c/m^*s, \tag{3·38}$$

where we have put $\omega_k+\hbar k^2/2m^*=|k|s$. If the sound speed s is smaller than $\hbar k_c/m^*$, no divergent term appears in (3·36). But if s becomes larger than $\hbar k_c/m^*$, (3·36) diverges at the wave number $|K|=m^*s/\hbar$. Here, restricting ourselves to the former case, we get the modified spectrum for the large wave numbers in the form*

$$\epsilon_k=\left\{1+\left(\frac{1}{3}-\frac{\gamma}{2}\right)\frac{\alpha^3N'}{N}\right\}\frac{\hbar^2K^2}{2m^*}+\left(1-\frac{3}{2}\gamma+\frac{3}{5}\gamma^2\right)\frac{\alpha^3m^*s^2}{2}+O(1/|K|). \tag{3·39}$$

This expression is similar to the energy of Landau's roton exhibiting an "energy gap". Therefore we can conclude that there appears no excitation quantum except for phonons at very low temperatures.

Finally, we would like to show that the lowest energy given by (3·29) is smaller than the lowest energy of the Hartree-Fock approximation. In the present formal problem of a long-range potential the mean energy is expressed by

$$E=\left\langle\frac{\hbar^2}{2m}\sum_k k^2n_k+\frac{1}{2}\sum_{k\neq 0}\sum_l V_kN_lN_{l+k}+\frac{N^2-N}{2}V_0\right\rangle_{AV}, \tag{3·40}$$

where n_k stands for the number of the particles occupying the single-particle state of the momentum $\hbar k$. As far as the Fourier coefficients V_k are positive, the lowest state is represented by the configuration that all the particles occupy the zero-momentum state**. Hence, in the Hartree-Fock approximation, the energy for the ground state is given by

$$E_0=\tfrac{1}{2}N(N-1)V_0\simeq\frac{N^2}{2}V_0, \tag{3·41}$$

which is equal to the third term in the right-hand side of (3·29). We can readily show that the other three terms in (3·29) become the negative correction energy improving the Hartree-Fock approximation, noting that

$$\left(\frac{\hbar^2k^2}{2m}+NV_k\right)^2-(\hbar\omega_k)^2=N^2V_k^2\geq 0. \tag{3·42}$$

In the above we have assumed that no Fourier coefficients inclusive of V_0 are divergent. This assumption excludes the infinitely long-range potentials such as the Coulombian, for which the positively or negatively charged particles tend to condense at the surface of the vessel containing them, unless they are embedded in a uniform negative or positive

* The terms independent of γ have been given in a foregoing paper: T. Nishiyama and S. Lee, Busseiron Kenkyu, **93** (1956), p. 185 (in Japanese).

** This is a consequence of the Heisenberg uncertainty relation concerning the canonically conjugate pair: relative momentum and relative distance of two particles. As the particles tend to stay apart from one another in the ordinary space, they tend to condense into the same momentum state.

charge distribution. We have restricted ourselves to such well-behaving potentials as their ranges are finite though sufficiently long.

§ 4. Occupation number (Second quantization)

In this section we shall deal with the quantized wave functions in the momentum space subjected to the following commutation relations:

$$[a_k,\ a_l^+]=\delta_{kl}\,,\quad [a_k,\ a_l]=[a_k^+,\ a_l^+]=0. \tag{4·1}$$

In terms of these wave functions, the creation operators ξ_k and π_k are expressed by

$$\xi_k=(m/N)^{1/2}|k|^{-1}\sum_l{}' a^+_{l-(k/2)}a_{l+(k/2)}\,,\quad \pi_k=i\hbar\,(m/N)^{-1/2}|k|^{-1}\sum_l{}'(k\cdot l)\,a^+_{l+(k/2)}a_{l-(k/2)}\,. \tag{4·2}$$

Now we try to obtain the expectation value of the occupation number $n_k=a_k^+a_k$ of the particles which turns out in the collective representation to be

$$\begin{aligned}U^{-1}n_kU=n_k&-\frac{i}{\hbar}\sum_l^{k_c}(\beta_l-\pi_l)\,[\xi_l,\ n_k]+\frac{i}{\hbar}\sum_l^{k_c}(\alpha_l-\xi_l)\,[\pi_l,\ n_k]\\&+\tfrac{1}{2}\Big(\frac{i}{\hbar}\Big)^2\sum_l^{k_c}(\beta_l-\pi_l)\,(\beta_{-l}-\pi_{-l})\,[\xi_{-l},[\xi_l,\ n_k]]\\&+\tfrac{1}{2}\Big(\frac{i}{\hbar}\Big)^2\sum_l^{k_c}(\alpha_l-\xi_l)\,(\alpha_{-l}-\xi_{-l})\,[\pi_{-l},[\pi_l,\ n_k]]+\cdots,\end{aligned} \tag{4·3}$$

where the non-diagonal bilinear terms such as $\alpha_k\alpha_l$ and ${}_i\beta_k{}_j\beta_l$ for $k\neq -l$ and the higher order terms than the third order in α_k and ${}_i\beta_k$ have been neglected. The commutators in (4·3) are given explicitly in Appendix 1. For the lowest state represented by $\Phi^{(0)}=|N_0=N,\ldots;0\rangle$, we get

$$a_K^+a_0\Phi^{(0)}=N^{1/2}\Phi_K^{(1)},\quad a_0^+a_K\Phi_K^{(1)}=N^{1/2}\Phi^{(0)}, \tag{4·4}$$

and the expectation value becomes

$$\left.\begin{aligned}&\langle n_K\rangle_0=0,\quad |K|>k_c,\\&\langle n_k\rangle_0=\frac{\dfrac{\hbar^2k^2}{2m}+NV_k}{2\hbar\omega_k}-\tfrac{1}{2}\,,\quad |k|\leqq k_c\\&\langle n_0\rangle_0=N-\sum_{k\neq 0}^{k_c}{}'\langle n_k\rangle_0\end{aligned}\right\}, \tag{4·5}$$

the second of which is exactly the same as Bogolubov's result[2]. The expression (4·5) shows that a number of the Bose particles occupy the single-particle states of non-zero momenta even in the ground state, the vacuum state. Since the total number of these Bose particles cannot exceed N, we get

$$\sum_{k\neq 0}^{k_c}{}'\langle n_k\rangle<N\,, \tag{4·6}$$

which gives an upper limit for the cut-off wave number k_c. As k_c tends to this limit, the excitations of both phonons and internal quanta become impossible, because any excitations of a small number of phonons or internal quanta make the occupation number n_0 negative. Therefore the interaction potential should be of such a long-range that the magnitude of k_c remains within the above limit.

Noting that the total number of particles N commutes with the transformation operator U, we can see that the total number of internal quanta is given by

$$\sum_K \langle N_K \rangle = \langle N_0 \rangle + \sum_{|K|>k_c} \langle N_K \rangle = N. \tag{4·7}$$

For an n-phonon state represented by $|N_0=N, \cdots ; M_k=n\rangle$; $|k| \leq k_c$, the expectation values become

$$\left.\begin{aligned} &\langle n_K \rangle_{M_k} = 0, \quad |K| > k_c \,; \ \langle n_l \rangle_{M_k} = \langle n_l \rangle_0, \quad |l| \leq k_c,\ l \neq k,\ -k,\ 0 \,; \\ &\langle n_k \rangle_{M_k} = \langle n_{-k} \rangle_{M_k} = \langle n_k \rangle_0 + (\langle n_k \rangle_0 + \tfrac{1}{2})\, n, \end{aligned}\right\} \tag{4·8}$$

where $M_k = b_k^* b_k$ and

$$b_k^* = (\omega_k/2\hbar)^{1/2} \alpha_{-k} - i\,(1/2\hbar\omega_k)^{1/2} \beta_k, \quad b_k = (\omega_k/2\hbar)^{1/2} \alpha_k + i\,(1/2\hbar\omega_k)^{1/2} \beta_{-k}. \tag{4·9}$$

For these small wave numbers the one-to-one correspondence between the number of particles and that of excitation quanta can be clearly shown, but for the large wave numbers our approximation method does not work without limit.

For a state of one internal quanta, which may be called one roton state, represented by $|N_0=N-1,\ N_K=1,\ N_l=0,\ l \neq 0,\ K\,;\,0\rangle$, the expectation values turn out to be

$$\left.\begin{aligned} &\langle n_M \rangle_{N_k} = 0, \ M \neq K, \\ &\langle n_K \rangle_{N_k} = 1 + \tfrac{1}{2}\frac{N'}{N} - \left(\frac{K}{k_c}\right)^2 \frac{N'}{N}, \quad N' = k_c^3/6\pi^2, \end{aligned}\right\} \tag{4·10}$$

except for terms of order N^{-1}. For smaller wave numbers than $2k_c$ this expression seems to be reasonable, but for much larger wave numbers than $2k_c$ it becomes meaningless as the expectation value becomes negative. This situation shows that our approximation becomes worse for the larger wave numbers than k_c, than for the smaller wave numbers.

For very large wave numbers it can be shown that the average value of the occupation number taken in the momentum space remains positive even if each expectation value of the occupation number is negative. This is because of the fact that the large correction terms are smeared to cancel out on taking the average in the momentum space. Indeed, we can show that

$$\langle\!\langle n_K \rangle\!\rangle_{N_k} = (6\pi^2/k_c^3) \sum_k \langle n_k \rangle_{N_k} = (6\pi^2/k_c^3) \sum_M \langle N_M \rangle_{N_k} = \langle\!\langle N_K \rangle\!\rangle_{N_k}, \tag{4·11}$$

where the sum is taken within a sphere of radius k_c whose center is at a phase point of a large wave number K such as $|K| > 2k_c$ (see Appendix 1).

§ 5. Discussions

The above method can be applied only to the case of a long-range potential. In the real problem of liquid helium the potential is of the Lennard-Jones type[11] such as

$$V(r)=4\varepsilon\left|(\sigma/r)^{12}-(\sigma/r)^{6}\right|, \tag{5·1}$$

$$\varepsilon=1.40\times10^{-15}\ \text{erg},\quad \sigma=2.56\times10^{-8}\ \text{cm},$$

whose Fourier coefficients become divergent, hence the frequency of the collective oscillation given by (3·22) diverges. In relation to this difficulty involved in our theory, we shall examine more closely the approximate commutation relations applied in the above and the orthogonality among the Fourier coefficients of the density operator. In the case of a long-range potential the Fourier coefficients with large wave numbers do not appear in the Hamiltonian, while in the case of a short-range potential there appear the Fourier coefficients with the large wave numbers which cannot be considered as orthogonal to those with the small wave numbers. In this case, in order to take account of the non-orthogonality of the Fourier coefficients we deal with the exact commutation relation (3·2). Utilizing these exact formulas we try to examine to what extent the orthogonality relation or the approximate commutation relation can be justified.

In the collective representation the form factor is expressed by

$$\begin{aligned}F_k=\frac{1}{N}\Big\langle &\rho_k\rho_{-k}-\frac{1}{N}\sum_{l}^{k_c}l^{-2}(k\cdot l)\,(\rho_{-k}\rho_{k-l}-\rho_k\rho_{-k-l})\,\rho_l+\frac{1}{2N^2}\sum_{l}^{k_c}l^{-2}f(k,l)\,\rho_l\rho_{-l}\\&+\Big(\frac{1}{mN}\Big)^{1/2}\sum_{l}^{k_c}|l|^{-1}(k\cdot l)\,(\rho_{-k}\rho_{k-l}-\rho_k\rho_{-k-l})\,\alpha_l+\Big(\frac{1}{mN^3}\Big)^{1/2}\sum_{l}^{k_c}|l|^{-1}f(k,l)\,\rho_{-l}\alpha_l\\&+\frac{1}{2mN}\sum_{l}^{k_c}f(k,l)\,\alpha_l\alpha_{-l}\Big\rangle_{AV},\quad f(k,l)=l^{-2}(k\cdot l)^2(\rho_{k-l}\rho_{-k+l}+\rho_{k+l}\rho_{-k-l}-2\rho_k\rho_{-k}),\end{aligned} \tag{5·2}$$

where the bilinear terms such as $\alpha_k\alpha_l$ for $k\neq -l$ and the higher order terms than the third order in α_k have been neglected. With respect to the lowest state represented by $\phi^{(0)}$, we obtain

$$\begin{aligned}\langle(k\cdot l)\,(\rho_{-k}\rho_{k-l}-\rho_k\rho_{-k-l})\,\rho_l\rangle_0&=\begin{cases}k^2N^2(\delta_{kl}+\delta_{-kl}), & |k|\leq k_c\\ 0, & |k|>k_c\end{cases}\\ \langle f(k,l)\,\rho_l\rho_{-l}\rangle_0&=\begin{cases}k^2(N^3-3N^2)\,(\delta_{kl}+\delta_{-kl}), & |k|\leq k_c\\ 0, & |k|>k_c\end{cases}\\ \langle f(k,l)\,\alpha_l\alpha_{-l}\rangle_0&=\begin{cases}(\hbar k^2/2\omega_k)(N^2-N)(\delta_{kl}+\delta_{-kl}), & |k|\leq k_c\\ 0, & |k|>k_c\end{cases}\end{aligned} \tag{5·3}$$

hence it follows that

$$F_k = \begin{cases} \hbar k^2/2m\omega_k, & |k| \leq k_c \\ f_k = 1, & |k| > k_c, \end{cases} \tag{5·4}$$

where f_k is the form factor of the individual particles. This result is of the same form as that derived by assuming the approximate commutation relation (3·3). Therefore in the case of a long-range potential, for which $\phi^{(0)}$ becomes the eigenfunction of the lowest state, the orthogonality relation is useful. On the other hand, in the case of a short-range potential, the wave function for the lowest state becomes dependent of the coordinates of the individual particles so that the form factor is not so simple as (5·4). Although the expression (5·4) is still valid for the small wave numbers, for the large wave numbers, we must take account of the following correction terms:

$$F_k' = \frac{1}{2N^3}\langle \sum_l^{k_c} l^{-2} f(k, l) \rho_l \rho_{-l} \rangle_g + \frac{\hbar}{2mN} \sum_l^{k_c} l^{-2} (k \cdot l)^2 \omega_l^{-1} (f_{k-l} - f_k), \quad |k| > k_c, \tag{5·5}$$

where the three-particle correlation function has been neglected. As the wave number increases these correction terms tend to zero. By expressing the four-particle correlation function in (5·5) in terms of the form factors, we can estimate its order of magnitude from

$$\frac{1}{N} \sum_l^{k_c} l^{-4} (k \cdot l)^2 (f_{k-l} - f_k) f_l . \tag{5·6}$$

Instead of evaluating the form factor of the individual particles by obtaining the wave function, we derive it from the observed form factor[12] F_k by solving the non-linear equation given by

$$F_k = f_k + \frac{\hbar}{2mN} \sum_l^{k_c} l^{-2} (k \cdot l)^2 \omega_l^{-1} (f_{k-l} - f_k) + \frac{1}{N} \sum_l^{k_c} l^{-4} (k \cdot l)^2 (f_{k-l} - f_k) f_l, \quad |k| > k_c. \tag{5·7}$$

This result can be obtained by assuming the approximate commutation relation for the Fourier coefficients $\tilde{\xi}_k$ and π_k and by using the exact commutation relation (3·2) between $\tilde{\xi}_k$ with a large wave number and π_k with a small wave number. On solving (5·7) in a rough way, we may take account of only the linear terms such as

$$F_k = f_k + \frac{\hbar}{2mN} \sum_l^{k_c} l^{-2} (k \cdot l)^2 \omega_l^{-1} (f_{k-l} - f_k) . \tag{5·8}$$

Next we consider the expression for the frequency of the collective oscillation. Utilizing again the approximate commutation relation for the small wave numbers, as the coefficient of $\alpha_k \alpha_{-k}/2$ in the Hamiltonian we get

$$\Omega^2(k,-k)=\frac{1}{m^2N}\sum_j\left\{3(k\cdot p_j)^2+\frac{\hbar^2k^4}{4}\right\}+\frac{Nk^2}{m}V_k$$
$$+\frac{k^{-2}}{mN}\sum_{|K|>k_c}(k\cdot K)^2V_K(\rho_{K-k}\rho_{-K+k}-\rho_{-K}\rho_K), \tag{5·9}$$

where the first two terms are of the same form as Tomonaga's result. The expectation value of $\Omega^2(k,-k)$ affords the square frequency expressed as

$$\omega_k^2=k^2\left(\frac{2T_0}{m}+\frac{\hbar^2k^2}{4m^2}\right)+\frac{N}{m}k^2V_k+\frac{k^{-2}}{m}\sum_{|K|>k_c}(k\cdot K)^2V_K(f_{K-k}-f_K), \tag{5·10}$$

where T_0 implies mean kinetic energy of the individual particles for the lowest state. The correction term expressed by the form factors is similar to (5·8). In order to express (5·10) in the ordinary coordinate space, we add the correction terms for the small wave numbers which are neglected in our theory by using the approximate commutation relation. As far as these correction terms are really negligible, the result is not much changed by this modification. The modified expression turns out to be

$$\omega_k^2=k^2\left(\frac{2T_0}{m}+\frac{\hbar^2k^2}{4m^2}\right)+\frac{k^{-2}}{m}\int P(r)\{1-\exp(-ik\cdot r)\}\left(k\cdot\frac{\partial}{\partial r}\right)^2V(r)\,dr, \tag{5·11}$$

where $P(r)$ is the correlation function of the individual particles defined by

$$P(r)=\sum_k f_k\exp(ik\cdot r), \tag{5·12}$$

which is evaluated from the observed form factor by using the relation (5·8). If the second derivative of the interaction potential is given and if the correlation function $P(r)$ is determined, we can evaluate ω_k^2. It can be expected that the contribution from the hard core of the interaction potential will be much reduced by the correlation function so as to remain finite. The mean kinetic energy T_0 is obtained by the well-known virial theorem :

$$2T_0=-\frac{1}{2}\left\langle\sum_{i\neq j}^{N}\left(r_{ij}\cdot\frac{\partial V(r_{ij})}{\partial r_{ij}}\right)\right\rangle_g,\quad r_{ij}=x_i-x_j, \tag{5·13}$$

of which the right-hand side is given by the correlation function in the form*

$$-\frac{1}{2}\int r\cdot\frac{\partial V(r)}{\partial r}\{P(r)-\delta(r)\}\,dr. \tag{5·14}$$

From our point of view the application to the theory of liquid helium seems to be promising, but we do not tread into details of this problem.

* The correlation function given by (5·12) involves the δ-function, and $P(r)-\delta(r)$ becomes zero at the origin.

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Appendix 1. The expectation values in the representation of the occupation number

Utilizing the creation operators and the destruction operators introduced in Section 4, we can express a number of commutators appeared in Sections 3 and 4 as follows:

$$\frac{i}{\hbar}[\pi_k, T]=(m/N)^{1/2}|k|^{-1}(kk:K_{-k}),\quad K_k^{(ij)}=\left(\frac{\hbar}{m}\right)^2\sum_l l^{(i)}l^{(j)}a^*_{l-(k/2)}a_{l+(k/2)},$$

$$i,\ j=1,\ 2,\ 3,\qquad \text{(stress tensor)}\qquad (A\cdot1\cdot1)$$

$$\left(\frac{i}{\hbar}\right)^2[\pi_{-k},[\pi_k, T]]=\frac{\hbar^2}{Nm^2}\sum_l\left\{3(k\cdot l)^2+\frac{k^4}{4}\right\}a_l^*a_l \qquad (A\cdot1\cdot2)$$

$$\begin{aligned}
&[\xi_l, n_k]=(m/N)^{1/2}|l|^{-1}(a^*_{k-l}a_k-a^*_ka_{k+l}),\\
&[\pi_l, n_k]=i(mN)^{-1/2}|l|^{-1}\{l\cdot(k+\tfrac{l}{2})a^*_{k+l}a_k-l\cdot(k-\tfrac{l}{2})a^*_ka_{k-l}\},\\
&[\xi_{-l},[\xi_l, n_k]]=(m/N)l^{-2}(2a^*_ka_k-a^*_{k-l}a_{k-l}-a^*_{k+l}a_{k+l}), \qquad (A\cdot1\cdot3)\\
&[\pi_{-l},[\pi_l, n_k]]=(\hbar^2/Nm)l^{-2}[2\{(k\cdot l)^2+\tfrac{l^2}{4}\}a^*_ka_k+\{l\cdot(k+\tfrac{l}{2})\}^2a^*_{k+l}a_{k+l}\\
&\qquad\qquad +\{l\cdot(k-\tfrac{l}{2})\}^2a^*_{k-l}a_{k-l}].
\end{aligned}$$

First we consider the eigenvalue of the internal Hamiltonian H_{in} or $H'_{\text{in}}=H_{\text{in}}-T$ expressed by

$$H'_{\text{in}}=\frac{\hbar^2}{2mN}\sum_k^a\sum_l\sum_{l'}k^{-2}a^*_{l'+(k/2)}a_{l'-(k/2)}a^*_{l-(k/2)}a_{l+(k/2)}\left[N^{-1}\sum_K\left\{3(k\cdot K)^2+\frac{k^4}{4}\right\}N_K\right.$$
$$\left.-2(k\cdot l)^2-(k\cdot l)(k\cdot l')\right]. \qquad (A\cdot1\cdot4)$$

For the lowest state, $\Phi^{(0)}=|N_0=N,\ldots;0>$, we get

$$a^*_{l'+k}a_{l'}a^*_{l-k}a_l\Phi^{(0)}=N\delta_{l0}\delta_{ll'+k}\Phi^{(0)}+N^{1/2}(N-1)^{1/2}\delta_{l0}\delta_{l'0}\Phi^{(2)}_{-k,k}, \qquad (A\cdot1\cdot5)$$

where $\Phi^{(2)}_{-k,k}=|N_0=N-2,\ N_{-k}=1,\ N_k=1;0\rangle$ is a two-particle state which disappears under the operation of H'_{in}. The eigenvalue for this state is exactly given by (3·21). For a state of one internal quanta, one roton state represented by $\Phi^{(1)}_K=|N_0=N-1,\ N_K=1,\ N_M=0\ M\neq K,\ 0;0\rangle$, we get

$$\begin{aligned}
a^*_{l'+k}a_{l'}a^*_{l-k}a_l\Phi^{(1)}_K&=\{(N-1)\delta_{l0}+\delta_{lK}\}\delta_{ll'+k}\Phi^{(1)}_K\\
&+(N-1)^{1/2}\delta_{l0}\delta_{l'K}\Phi^{(2)}_{-k,K+k}+(N-1)^{1/2}\delta_{l'0}\delta_{lK}\Phi^{(2)}_{k,K-k}\\
&+(N-1)^{1/2}(N-2)^{1/2}\delta_{l0}\delta_{l'0}\Phi^{(3)}_{-k,k,K}, \qquad (A\cdot1\cdot6)
\end{aligned}$$

where $\Phi^{(3)}_{-k,k,K}=|N_0=N-3,\ N_{-k}=1,\ N_k=1,\ N_K=1,\ N_M=0\ M\neq -k,\ k,\ K,\ 0\ ;0\rangle$ is a three-particle state which vanishes under the operation of $H'_{\rm in}$. Since the two-particle states do not disappear under the operation of $H_{\rm in}'$, $\Phi^{(1)}_K$ cannot represent the true eigenstate. The diagonal terms in $H'_{\rm in}$ are expressed as

$$-\frac{\hbar^2}{2mN}\sum_k^{k_c}\sum_l k^{-2}\Big[3\,(k\cdot l)^2-N^{-1}\sum_K\Big\{3\,(k\cdot K)^2+\frac{k^4}{4}\Big\}N_k\Big]N_{l-(k/2)}N_{l-(k/2)}\,, \qquad (A\cdot1\cdot7)$$

and the expection value for the one-particle state represented by $\Phi^{(1)}_K$ turns out into

$$\langle H'_{\rm in}\rangle_K=-\frac{\hbar^2}{4m}\sum_k^{k_c}k^2,\ |K|>k_c. \qquad (A\cdot1\cdot8)$$

If the wave number is smaller than k_c the expectation value becomes

$$\langle H'_{\rm in}\rangle_k=-\frac{\hbar^2k^2}{2m}-\frac{\hbar^2}{4m}\sum_l^{k_c}l^2,\quad |k|\leq k_c, \qquad (A\cdot1\cdot9)$$

of which the first term is equal to the expectation value of the kinetic energy T, hence the state vector $\Phi^{(1)}_k$ or the wave function (3·31) gives rise to the same eigenvalue of $H_{\rm in}$ as the lowest state. Such a wave function for one of the small wave numbers does not satisfy the subsidiary condition (3·12) and should be excluded from the family of the eigenfunctions. Taking account of the expectation value of $H_{\rm (in)}$ for $\Phi^{(1)}_K$ we get the excitation energy (3·32).

Next, we consider the expectation value of the occupation numbers for the lowest state. Utilizing the following identities:

$$\begin{aligned}
&\langle\pi_l[\tilde\xi_l,n_k]\rangle_0=\frac{i\hbar}{2}\delta_{-lk},\quad \langle\tilde\xi_l[\pi_l,n_k]\rangle_0=\frac{\hbar}{2i}\delta_{lk}\,,\\
&\langle[\tilde\xi_{-l},[\tilde\xi_l,n_k]\rangle_0=ml^{-2}(2\delta_{k0}-\delta_{kl}-\delta_{-kl})\,,\\
&\langle[\pi_{-l},[\pi_l,n_k]\rangle_0=\frac{\hbar^2l^2}{4m}(2\delta_{k0}-\delta_{kl}-\delta_{-kl})\,,
\end{aligned} \qquad (A\cdot1\cdot10)$$

we can readily obtain the expectation value (4·5). In a similar way we can find the expectation value for the single roton states.

Taking the average value of the occupation number n_K for $|K|\cdot 2k_c$ within a sphere of radius k_c whose center is at the point K, all the commutators of (A·1·3) cancel out. In this way we obtain the one-to-one correspondence between the two average values of the occupation numbers as shown by (4·11).

Appendix 2. The self-energy and the level width

Here we compute the self-energy and the level width at half-maximum of an internal quantum. The transformation function $U(t)$ is given by the Tomonaga-Schwinger equation of the form

$$i\hbar\cdot\partial U/\partial t=H'(t)\,U, \tag{A·2·1}$$

where

$$H'(t)=\exp\Big(\frac{i}{\hbar}H_0\,t\Big)H'(0)\exp\Big(-\frac{i}{\hbar}H_0\,t\Big)$$

$$=(m/N)^{1/2}m^{-2}\sum_{k}^{k_c}\sum_{j}|k|^{-1}(\hbar/2\omega_k)^{1/2}\exp[i\{(\hbar k^2/2)-(k\cdot p_j)\}t/m^*]$$

$$\times\{b_k^*\exp(i\omega_k t)+b_{-k}\exp(-i\omega_k t)\}\exp(-ik\cdot x_j)\{(k\cdot p_j)^2-\hbar k^2(k\cdot p_j)\},$$

$$H_0=H_{\rm in}+H_{\rm coll}. \tag{A·2·2}$$

Utilizing the asymptotic expression for the long-time interval such as

$$\lim_{T\to\infty}\int_0^T\exp(i\omega t)\,dt=\pi\delta(\omega)+i\,(1/\omega)_{\text{principal value}}, \tag{A·2·3}$$

the diagonal element of the second order term in the expansion of $U(t)$ takes the form

$$\langle U^{(2)}(t)\rangle_K=-(i/\hbar)^2\int_0^t dt'\int_0^{t'}\langle H'(t')\;H'(t'')\rangle_K dt''$$

$$=-(i/\hbar)^2\int_0^{t/\sqrt{2}}d\tau\int_\tau^{\sqrt{2}t-\tau}\Big\langle H'\Big(\frac{T+\tau}{\sqrt{2}}\Big)H'\Big(\frac{T-\tau}{\sqrt{2}}\Big)\Big\rangle_K dT$$

$$=-(tm/8\pi^2N\hbar)\,\alpha^3\int_0^{k_0}d|k|/\omega_k\int_{-1}^{1}\{x-(\hbar k/m^*v)\}$$

$$\times\Big\{\pi\delta\Big(\omega_k+\frac{\hbar k^2}{2m^*}-|k|vx\Big)+i\left(\frac{1}{|k|vx-\omega_k-\dfrac{\hbar k^2}{2m^*}}\right)_P\Big\}k^4v^4x^2dx$$

$$=-t\Big(\frac{i}{\hbar}\Delta_v+\frac{\Gamma_v}{2}\Big),\quad v=\hbar|K|/m^*,\quad \alpha=m^*/m, \tag{A·2·4}$$

where the self-energy and the level width are denoted by Δ_v and $\hbar\Gamma_v$, respectively. Putting $\omega_k+(\hbar k^2/2m^*)=|k|s$, we obtain the formulas (3·36) and (3·37). We must assume the cut-off wave number k_c to be larger than $m^*s/\hbar$, lest (3·36) should be divergent for all the wave numbers larger than k_c. Further, the effective mass given by (3·33) becomes divergent at the wave number $k=2m^*s/\hbar$, therefore k_c should be taken such as

$$s<\hbar k_c/m^*<2s\,.$$

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Letters to the Editor

On the Systematization of the Strongly Interacting Particles

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March 23, 1957

Recently Salam and Polkinghorne[1] tried to systematize the strongly interacting particles in the four-dimensional charge space. It is attempted here by modifying their scheme to forbid particles with multiple charge and/or high strangeness. We take the view-point that there exist higher degress of symmetry than the charge independence in a system of the strongly interacting particles if there were no interactions between them.

We assume that, if we put all of their interactions out of consideration, the theory is invariant under some continuous transformation group G, which has as its generators of infinitesimal transformations three mutually commuting vectors, $\boldsymbol{M}^{(a)}$, $\boldsymbol{M}^{(b)}$, and $\boldsymbol{M}^{(c)}$, each vector $\boldsymbol{M}$ satisfying the following relations,

$$M_\alpha M_\beta = -M_\beta M_\alpha = \frac{i}{2} M_\gamma,$$

$$(\alpha, \beta, \gamma) = \text{cycl. } (1, 2, 3).$$

From the above relations we easily get

$$M_\alpha{}^3 = \tfrac{1}{4} M_\alpha, \quad \alpha = 1, 2, 3.$$

That is, $\boldsymbol{M}$'s are angular momenta with eigenvalues of 0 and $\frac{1}{2}$ only.

Particles are classified according to the irreducible representations of G, their electric charge Q being given in units of $+e$ as

$$Q = M_3^{(a)} + M_3^{(b)} + M_3^{(c)}.$$

Q must have integer eigenvalues from physical requirements. This restricts them to 0 and ± 1 only. Then, four types of representations are possible (Table I). Moreover, baryon number n is assigned to each type of particles. Particles with $n = 1$, -1 and 0 are called baryons, antibaryons and mesons, respectively.

When strong interactions are taken into account, we assume following Nakano,

Table I Possible representations of the symmetry group G.

Type	$M^{(a)}$	$M^{(b)}$	$M^{(c)}$
1	$\frac{1}{2}$	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	0	$\frac{1}{2}$	$\frac{1}{2}$
4	0	0	0

Nishijima and Gell-Mann[2] that only

$$\boldsymbol{I} = \boldsymbol{M}^{(a)} + \boldsymbol{M}^{(b)}$$

is conserved. $\boldsymbol{I}$ can be interpreted as the usual isq-spin operator as is evident from Table II below, which shows the correspondence with the observed particles.

Table II Correspondence of our classifications with the observed particles.

Type	I	$M_3^{(c)}$	Baryon	Anti-Baryon	Meson
1	0	0	Λ	$\bar{\Lambda}$	(π')
	1		Σ	$\bar{\Sigma}$	π
2 or 3	$\frac{1}{2}$	$+\frac{1}{2}$	N	$\bar{\Xi}$	K
		$-\frac{1}{2}$	Ξ	$\bar{N}$	$\bar{K}$
4	0	0			

$M_3^{(c)}$ is connected with the strengeness quantum number S as follows,

$$M_3^{(c)} = Q - I_z = \tfrac{1}{2}(n + S).$$

According to the above scheme, Λ and Σ should be considered as the particles of the same type, the mass difference of which is due to strong interactions. The same goes for N and Ξ. Such an idea has been expressed by Schwinger[3] on a somewhat different point of view. Emphasizing analogies between the photon field and the π field, he thought that K-mesons can assume both intrinsic parities. To explain the mass identity of τ and θ he assumed parity degeneracy also for Λ and Σ. Such a hypothesis is rather undesirable in our scheme because Λ and Σ belong to the different type from that of K-mesons.* We notice that the large mass difference between N and Ξ does not conflict with the identity of masses of K the $\bar{K}$, so far as strong interactions have such a character that the masses of a particle and its anti-particle are kept the same.

In our scheme particles of the same type are required to have the same spin-parity. Then, if Λ and Σ do not have the same spin-parity we must consider Λ as a baryon of the type 4, and suppose that a possible neutral baryon of the type 1 with $I=0$ (named Σ' for brevity) is difficult of experimental observation as was the case for Σ^0. Our scheme requires the existence of a neutral pseudo-scalar meson with $S=0$ and $I=0$ (π'-meson in Table II). If this hypothetical meson has an opposite charge parity to that of π^0-meson, which is a reasonable assumption in our scheme, it cannot have strong interaccions under the assumption that they are invariant for space reflection and charge conjugation. And so in the case that π'-meson is much heavier than π-mesons, its production may be too scanty to be detected experimentally, because then it would be produced only by collision through weak interactions, not by decay of copiously produced particles.

The writer expresses his thanks to Dr. T. Nakano for helpful discussions.

* Recent experiments originally proposed by Lee and Yang seem to have made the parity doublet theory unnecessary.

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ERRATA

Remarks on the Elastic Scattering of Deuterons by Heavy Nuclei

Yoshihiko Nishida Prog. Theor. Phys. **17** (1957), 506 (L)

The next sentence (in italics) is to be inserted between 8 line and 7 line from the bottom of the right column on page 506.

In addition, the period of the internal motion of the deuteron *is fairly longer than the time which the deuteron* requires to graze the nucleus.

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Volume 12, Number 3, March 1957

CONTENTS

Page

Senzo TOKUNAGA, Kiyoshi NISHIKAWA and Taka ISHII: A Cascade Shower in Nuclear Emulsion 219
Shūichi IIDA: Layer Structures of Magnetic Oxides 222
Masakazu MARUTAKE and Takuro IKEDA: Anisotropy in Polarized Barium Titanate Ceramics 233
Maurice L. HUGGINS and Yosio SAKAMOTO: Lattice Energies and Other Properties of Crystals of Alkaline Earth Chalcogenides 241
Kisaburo SHOGENJI and Susumu UCHIYAMA: On Electrical Resistivity and Hall Coefficient of PbTe Crystals 252
Makoto HIRABAYASHI and Shiro OGAWA: On the Superstructure of the Ordered Alloy Cu_3Pd II. X-ray Diffraction Study 259
Akira ODAJIMA, Junkichi SOHMA and Mitsuru KOIKE: Proton Magnetic Resonance in Chain Polymers 272
Atsuo NISHIOKA, Hachiro KOMATSU and Yoshinobu KAKIUCHI: Nuclear Magnetic Resonance in Some Crystalline Polymers 283
Masataka ARIYAMA, Kunihusa KAYAMA and Masayoshi SUGIMOTO: Self-consistent Field without Exchange for Doubly Ionized Cobalt 285
Shinji KUWABARA: The Forces Experienced by Two Circular Cylinders in a Uniform Flow at Small Raynolds Numbers 291
Otohiko NOMOTO: Theory of the Ultrasonic Absorption in Aqueous Solutions (II) Aqueous Solutions of Some Alchols 300

SHORT NOTES

Takeshi MIYAUCHI: An Empirical Regularity of Energy Gap in Semiconductors 308
Tadashi SUGAWARA: Nuclear Magnetic Resonance in Solid Solutions of Transition Metals in Copper and Aluminum 309
Chiyoe YAMANAKA and Tokuo SUITA: Multiplication of Charge Carriers and Breakdown in Ge p–n Junction I 310
Otohiko NOMOTO and Tadashi KISHIMOTO: Molecular Sound Velocity of Aqueous Sugar Solutions 311
Hisao MIYAZAWA and Suezo SUGAIKE: Phase Transition of In_2Se_3 312
Kiyoshi AZUMA and Hiroshi KAMETANI: The Generation of EMF During Melting and Freezing of Ionic Crystalline Substances 313
Kiyokata MATSUURA, Yoshikazu SUGIURA and G. M. HATOYAMA: Frequency Shift in Ammonia Absorption due to Self-broadening 314

Volume 12, Number 4, April 1957

CONTENTS

Sadao HOSHINO: Crystal Structure and Phase Transition of Some Metallic Halides IV 315
Kazuo HIROIKE: Radial Distribution Function of Fluids I 326
Takao SATO and Seiichiro NOGUCHI: Note on Some Properties of Gamma Brass 335
Tatsumi KUROSAWA: On the Melting of Ionic Crystals 338
Sumitada ASANO and Yasuo TOMISHIMA: Rotational Lattice Vibration in Complex Crystals Part 1, Vibrational Modes and Special Heat 346

Yoshihiko MIZUSHIMA, Zenjiro ODA and Osamu OCHI: The Structure of Evaporated Barium Films 355
Shintaro SUGAI and Jiro FURUICHI: Studies of High Polymers by Radioactive Method II Self-diffusion of Polystyrene in Its Dilute Solution 369
Keiji YAMAMOTO and Yasaku WADA: An Investigation of the Dynamic Mechanical Properties of Glassy Polymers by the Composite Oscillator Method 374
Yuichi IKEDA: On the Effective Diffusion Tensor of a Segment in a Chain Molecule and its Application to the Non-Newtonian Viscosity of Polymer Solutions 378
Motoyoshi YUZURI, Tokutaro HIRONE, Hiroshi WATANABE, Seizo NAGASAKI and Seijiro MAEDA: On the Magnetic Properties of Chromium Sulfides 385
Jun OKANO: Preliminary Report on an Electromagnetic Isotope Separator 390
Ken-ichi KUSUKAWA: On the Transonic Flow of a Compressible Fluid past an Axisymmetric Slender Body at Zero Incidence 401
Ken-ichi KUSUKAWA: On the Transonic Flow of a Compressible Fluid past a nearly Axisymmetric Slender Body 411
Sadatoshi TANEDA: Experimental Studies of the Lift on Two Equal Circular Cylinders placed side by side in a Uniform Stream at Low Reynolds Numbers 419
Hiroomi FUJIKAWA: Expansion Formulae for the Forces acting on Two Equal Circular Cylinders placed in a Uniform Stream at Low Values of Reynolds Number 423

SHORT NOTES

Susumu UCHIYAMA and Kisaburo SHOGENJI: Anomalies of Electrical Properties of Lead Telluride Crystal 431
Zeniti KIYASU, Kazuo FUSIMI and Keisuke KATAOKA: Quasi-static Hysteresis in Barium-titanate Single Crystals 432
Hideo FUTAMA and Hidejirō TANAKA: The Thermal Decomposition of Polyvinyl-Alcohol Using Mass Spectrometer 433
Kisaburo SHOGENJI and Susumu UCHIYAMA: Resistivity and Hall Coefficient of Semiconductors 434
Hidetaro ABE: Manganese Impurities in Silver Halides Studied by the Paramagnetic Resonance Absorption 435
Makoto KIKUCHI: Observation of the Drift of Germanium Surface by Field Effect Experiment 436
Kenji MITANI and Hiroshi KUBO: Randomness of Microwave Gas Breakdown Voltage 437
Goro MIYAMOTO, Giiti IWATA, Sigeru MORI and Kenji INOUE: A Possible Fusion Reactor 438
Yoshio INUISHI: Noise from Ge PN Junction and γ-ray Irradiation Effect 439
Ali Abdel Kerim IBRAHIM: A Graphical Method for Determining the Coefficient of Viscosity of Newtonian Liquids using an Oscillating Cylinder Viscometer III 440
Yasuo KANAI and Riro NII: Anomalous Electrical Properties of Lead Telluride 441
Takeshi MORIMOTO: Cross Section for Displacement of Atoms by Heavy Particle Irradiation 442

Errata

Makoto HIRABAYASHI and Shiro OGAWA: On the Superstructure of the Ordered Alloy Cu_3Pd. II. X-ray Diffraction Study 442
Noriaki ITOH and Tokuo SUITA: Effect of Plastic Deformation on the Dielectric Breakdown of Silver Chloride Crystals 442

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Giant Stars with Shell Sources of C-N and p-p Reactions

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Structures of giant stars of population II producing energy in the C-N shell source are studied taking account of the contribution of the ion pressure in the partially degenerate isothermal core and also of the exact fitting of the radiative region to the convective one. Further, it is found that pure hydrogen stars producing energy in the p-p shell source can also explain the giant branch of population II if the effect of the ion pressure is included in the core.

§ 1. Introduction

It is known that giant stars have such a structure as composed of an isothermal core with dehydrogenized matter, a shell producing energy by C-N reactions and a radiative envelope together with an outermost convective region.[1),2)] Calculations made by Hoyle and Schwarzschild[2)] on the stars of the globular clusters are of an approximate nature in that fitting of the radiative region to the convective one is not exact, and that they used the solutions of isothermal core in which the contribution of the ion pressure is neglected. The first purpose of this paper is to correct the above defects.

There are two alternative theories on the origin of the chemical elements, which have intimate connections with the evolutional schemes of stars and galaxies, that is, the elements were formed in the pre-stellar stage[3)] or in the stellar stage[1)] of our universe. If we adopt the latter one and assume that the oldest stars were condensed from pure hydrogen gas, it will be an important task to find the evolutional tracks of these stars whose interiors are initially composed entirely of hydrogen and whose energy generation depends on the p-p chain reactions only. The heavy elements observed on their surfaces might be explained as the result of accretion of matter during their motions through the interstellar space. The second purpose is to find whether the giant branch of the globular clusters can be explained by the p-p reactions or not.

To carry out the above programs we shall make a simplification of the composite stellar models by assuming that the energy production occurs only in a thin shell and by neglecting a region with the opacity of free-free transitions or a region of outermost convection, the validity of these approximations being checked afterwards.

§ 2. Composite models with isothermal cores

Following two models will be studied. A model (a) consists of three regions :

isothermal core with partially degenerate electrons, radiative envelope with electron-scattering opacity and outermost convection. The validity of this model, which neglects a region with free-free opacity, in the case of C-N reactions is anticipated from the work by Hoyle and Schwarzschild. We shall use the same surface condition as adopted by Osterbrock[5] and Hoyle-Schwarzschild.

Another model (b) consists of three regions also : isothermal core, radiative envelope with electron-scattering opacity and that with free-free opacity. This model will give a better approximation than the model (a) for the stars with a thin convective region.

If the variables in the envelopes are expressed as

$$P=p(GM^2/4\pi R^4),\ T=t(\mu HGM/kR),\ M(r)=qM,\ r=xR, \tag{1}$$

the equations for the hydrodynamical equilibrium and the continuity of mass are given by

$$dp/dx=-pq/tx^2,\quad dq/dx=px^2/t, \tag{2}$$

and the equations for the temperature gradient are

$$dt/dx=-\tfrac{2}{5}\,q/x^2,\ -C_K p^2/x^2t^{6.5},\ -C_F p/x^2t^4, \tag{3a, b, c}$$

in the convective region with $p=Et^{2.5}$, in the radiative region with free-free opacity, and in the radiative region with electron scattering opacity, respectively. Here C_K and C_F are given by

$$C_K=(3/4ac)\,\{3.68\times10^{22}(1+X)/(4\pi)^3\}\,(k/\mu HG)^{7.5}(LR^{0.5}/M^{5.5}), \tag{4}$$

$$C_F=(3/4ac)\,\{0.19\,(1+X)/(4\pi)^2\}\,(k/\mu HG)^4(L/M^3). \tag{5}$$

Fitting of the solutions of the envelopes to those of the isothermal cores is done in the U-V plane, U and V being the following functions invariant under the homology transformations.

$$U=d\log M(r)/d\log r,\quad V=-d\log P/d\log r. \tag{6}$$

The solutions of the isothermal cores used by Hoyle-Schwarzschild are corrected as follows. Since we should consider the regions of strong degeneracy as well as of perfect gas state, the relation between the gas pressure P and the density ρ must be given by[1]

$$P=P_e+P_n,$$

$$P_e=(8\pi/3h^3)\,(2mkT)^{3/2}kTF_{3/2}(\psi),\ P_n=\rho kT/\mu_n H, \tag{7}$$

$$\rho/\mu_e H=(4\pi/h^3)\,(2mkT)^{3/2}F_{1/2}(\psi), \tag{8}$$

where $F_n(\psi)$ are the Fermi-Dirac functions, and μ_e and μ_n are the mean molecular weights of electrons and ions, respectively. A calculation of the effect of the pressure ionization shows that the variation of μ_e for the temperature and density is negligible for $T\lesssim 3\times10^7$ K in the case under consideration where matter of cores is composed entirely of helium. We have $\mu_e=2$ and $\mu_n=4$ in this case.

By putting

$$r=(h^{3/2}/4\pi\mu_e H)\,(2mkT)^{-1/4}(2mG)^{-1/2}\xi, \tag{9}$$

$$M(r) = (h^{3/2}/4\pi\mu_e^2 H^2)\,(2mkT)^{3/4}(2mG)^{-3/2}\varphi, \tag{10}$$

the equations in the isothermal cores are expressed as

$$d\psi/d\xi = -\varphi/(1+\delta)\xi^2, \quad d\varphi/d\xi = \xi^2 F_{1/2}, \tag{11}$$

with

$$\delta = \frac{\mu_e}{\mu_n}\,\frac{1}{F_{1/2}}\,\frac{dF_{1/2}}{d\psi} = \begin{cases} 0 & (\psi \to +\infty) \\ 1/2 & (\psi \to -\infty)\,. \end{cases} \tag{12}$$

The special case $\delta = 0$ $(\mu_n \gg \mu_e)$ corresponds to the solutions calculated by Wares[6),7)] and used by Hoyle-Schwarzschild. One degree of freedom of the homology transformation remains also for a finite constant value of μ_e/μ_n. The U and V functions (6) are given by

$$U = F_{1/2}\frac{\xi^3}{\varphi}, \quad V = \frac{F_{1/2}}{(2/3)F_{3/2} + (\mu_e/\mu_n)F_{1/2}}\,\frac{\varphi}{\xi}. \tag{13}$$

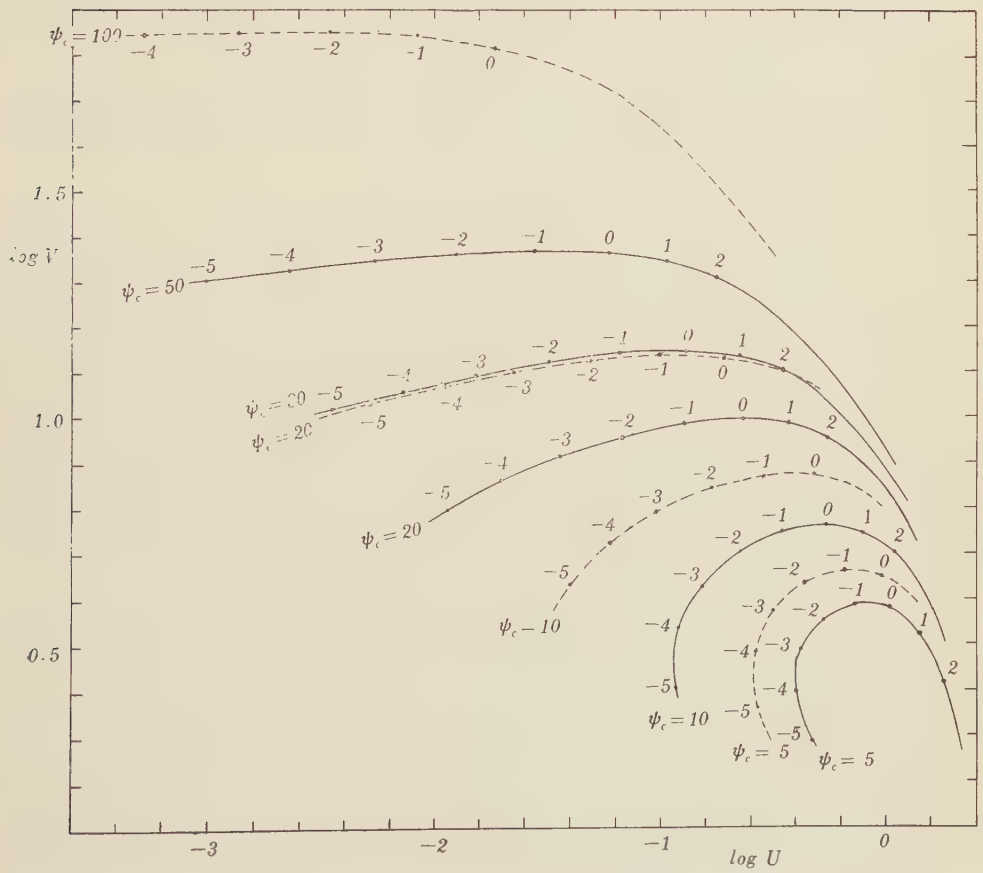

Fig. 1. *U–V* curves of the solutions of the partially degenerate isothermal cores. Full and dotted curves show the cases $\mu_n/\mu_e = 2$ and $\mu_n/\mu_e = \infty$, respectively. Values of ψ are shown on the points on these curves.

Numerical integrations of (11) have been carried out for $\mu_n/\mu_e=2$ and for ψ_c (the central value of ψ) $=50$, 30, 20, 10 and 5. The corresponding U and V are shown in Fig. 1 with labels of the values of ψ, together with those without correction ($\partial=0$). This correction brings a rather serious modification of the results on the stellar structure as shown in the next section.

§ 3. Results on models (a) and (b)

The model (a) is calculated by taking $\mu_{env}=0.533$ (90 percent hydrogen and 10 percent helium) and $M=1.2\,M_\odot$ in the two cases, C-N and p-p reactions. One fixed value $\log L/L_\odot=2.05$ is chosen. For each of five Osterbrock's solutions of the convective region with $E=28$, 24, 20, 16 and 8[5),7)], an interface $x=x_2$ with the radiative region is determined by the continuity of (3a) and (3c), $C_E=q t^4/2.5\,p$. Starting from this point, equations (2) and (3c) are numerically integrated inwards.

Then, several points, $x_1\equiv r_1/R$, are chosen as the interfaces with the isothermal core. For each of them, fitting of the U and V values multiplied by $\mu_{\rm core}/\mu_{env}=1.333/0.533$ to those of the core solutions shown in Fig. 1 determines the value of ψ at the interface. Then, $\tilde{\xi}$ and φ are found from (13) and all the physical values such as T, ρ, $M(r)$ and R are known from (1), (9) and (10). The total energy generations are calculated by using the formulae*

$$\varepsilon_{CN}=4.4\times10^{6}\rho X X_{CN}(T/3\times10^{7})^{15.1}, \tag{14}$$

$$\varepsilon_{pp}=30\rho X^2(T/6\times10^{7})^{2.2}. \tag{15}$$

By interpolation, we can find the solution having the same luminosity as chosen above.

In the next step, the relation between E and $\log R/R_\odot$ given by these solutions is compared with that determined by the surface condition and again by interpolation we find the actual solution. The results thus obtained in the cases with and without the correction of core's solutions are shown in Table 1. The ratio of opacities of free-free and electron scattering, κ_{ff}/κ, at $x=x_2$ shows that this model is good in the C-N case but not in the p-p case. It is inferred, however, from the results that the convective region will be rather thin in the p-p case if a radiative region with free-free opacity is taking into account. This leads us to the study of the model (b).

The model (b) is calculated in the p-p case with $\mu_{env}=0.50$ (100 percent hydrogen) by using the tables of the envelope's solutions given by Sandage and Schwarzschild.[7),8)] One fixed value $\log C_K=-3.5$ is chosen instead of choosing $L/L_\odot$ as in the case of model (a). Fitting to the core's solution and the interpolation are made in a similar way as the model (a).

The results are given in Table 2. We can see from these results and Fig. 2 (the H-R diagram) that stars are still near the main sequence if the effect of the ion pressure

* This formula for ε_{CN} corresponds to the case where the reaction $N^{14}(p,\gamma)O^{16}$ is the slowest in the C-N cycle. The C-N abundance X_{CN} is taken as 10^{-3} corresponding to the population II stars.

Table 1. Model (a) with C-N and p-p reactions for $M=1.2\,M_\odot$

	C-N reactions		p-p reactions	
	without correction	with correction	without correction	with correction
$\log L/L_\odot$	2.05	2.05	2.05	2.05
$\log R/R_\odot$	1.15	1.16	1.11	1.13
q_1	0.33	0.22	0.46	0.39
$\log r_1/R_\odot$	$\bar{2}.64$	$\bar{2}.49$	$\bar{2}.53$	$\bar{2}.32$
$T_1/10^7$	2.8	3.3	5.8	7.3
$\log \rho_{1e}$	1.80	1.74	2.79	3.02
ψ_{1i}	−2.2	−3.1	−0.9	−1.2
ψ_c	14.4	20.6	13	19
$\log x_2$	$\bar{1}.55$	$\bar{2}.98$	$\bar{1}.85$	$\bar{1}.80$
$\log q_2$	$\bar{1}.81$	$\bar{1}.41$	0.00	$\bar{1}.98$
κ_f/κ_s at x_2	1.7	0.43	9.0	4.6
E	16.7	25.7	1.3	5.5
$\Delta r/r_1$	0.04	0.04	0.18	0.16

ρ_{1e} and ψ_{1i} are the values at x_1 on the sides of the envelope and the core, respectively. Δr is the thickness of the shell producing 50 percent of the total energy.

in the core is neglected, but they are far in the giant region if the correction is made. If the surface condition is taken into account in the latter case, they will certainly be brought back to the region occupied by the stars of the globular clusters. In this case the solution should be modified by adding a convective region, but a change in the structure as a whole will not be serious since the convective region will be rather thin as inferred from the results of the model (a).

Table 2. Model (b) with p-p reactions for $M=1.2\,M_\odot$

	without correction	with correction
$\log L/L_\odot$	2.14	1.71
$\log R/R^\odot$	0.80	1.64
q_1	0.53	0.34
$\log r_1/R$	$\bar{2}.51$	$\bar{2}.33$
$T_1/10^7$	6.2	5.7
$\log \rho_{1e}$	2.85	2.95
ψ_{1i}	−0.9	−1.0
ψ_c	13	20
$\log x_s$	$\bar{1}.63$	$\bar{2}.49$
$\Delta r/r_1$	0.19	0.17

x_s is the fraction of radius at the interface between the regions with opacities of free-free and electron scattering.

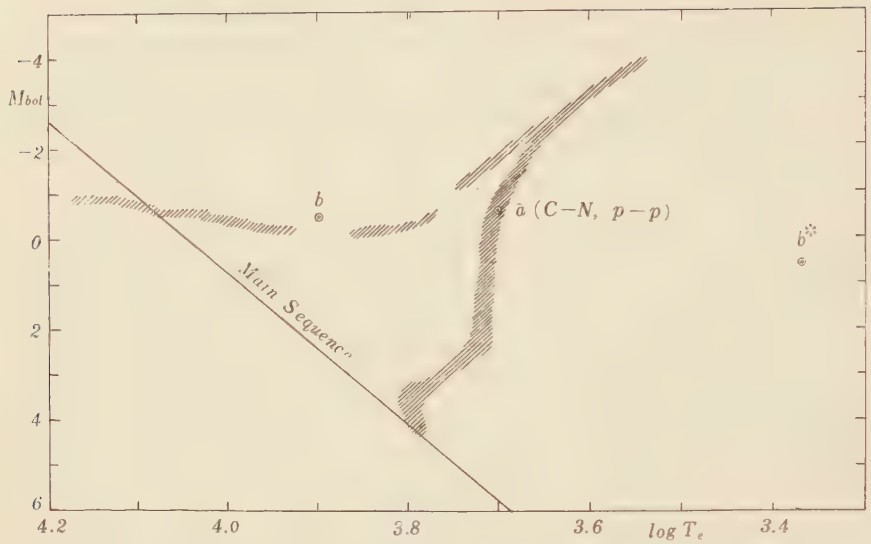

Fig. 2. The positions of the stars in the H-R diagram. The results of the model (a) in the C-N and p-p cases are denoted by *a* and those of the model (b) with and without the correction of cores are denoted by *b** and *b*, respectively. Also, the regions occupied by the stars in the globular clusters are shown.

We can say from the above results that hydrogen stars without mixing of matter will evolve along the giant branch of the globular cluster in the same way as the stars containing carbon and nitrogen initially. The differences between the cases of p-p and C-N reactions lie mainly in the central temperature, the degree of mass condensation, and the thickness of the outermost convective zone, if we set aside the problem about the metal contents observed on the stellar surface. It seems that the rate of accretion of matter containing metals by these stars is not sufficient in the present condition of the interstellar space in our galaxy. However, we can not deny a conjecture that the condition of interstellar matter around these stars has not been constant throughout the age of our galaxy.

In conclusion, the author wishes to express his appreciation to Professors S. Hayakawa and M. Taketani for valuable discussions.

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Selection Rule Imposed by Mass Reversal*

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The author has investigated the structure of the interactions among the fundamental particles by means of a selection rule based on the new invariance principle, the *mass reversal*. A unified model of interactions for the baryon-meson system is presented, and its consistency is proved using this selection rule. This selection rule will be useful for exploring the dynamical structure of the interaction.

§ 1. Introduction

The rapidly increasing experimental evidences concerning hyperons and heavy mesons have led many physicists to propose various methods of classifying these new unstable particles in a unified manner by some principles. For this purpose, it is very important to make clear the structure of interactions existing among these particles. Among several attempts in this direction, Nishijima and Gell-Mann have succeeded in constructing a scheme for new particles based on the concept of "isotopic spin space"[1]. They have made the classification of interactions between baryon and meson under the charge independence hypothesis according to their coupling strengths as: (1) strong interaction for which I and I_3 are conserved, (2) electromagnetic interaction for which only I_3 is conserved and (3) weak interaction for which neither I nor I_3 is conserved. Though it is not yet clear that the charge independence hypothesis can really give the complete characterization of the weak interactions,[2] the N-G theory for new particles may be taken as the one which characterizes the features of the strong interactions among fundamental particles.

Now, the success of the isotopic spin model of the baryon-meson system raised some questions. Has the concept of isotopic spin space any intrinsic significance to the internal structure of fundamental particles? And if there is any significance, what relationship has it to lepton family, namely, to what extent will the straight application of isotopic spin model to lepton family be promising?[3] Though, up to the present, no definite answers to these questions have been given, being influenced by the success of N-G theory the interest of some physicists seems to have been concentrated mainly on the mathematical formulation of this model,[4] and the inclusion of the lepton processes in this model. We stress here another important aspect of the study of the interactions among the fundamental particles. It is to determine the general features of the dynamical

* This is the detailed and revised report which appeared in Soryushiron-Kenkyu (mimeographed circular in Japanese) **9** (1955) 228.

structure of the various kinds of the interaction, whose kinematical characterizations have been partly given in other works. For the purpose of exploring the dynamical feature of the interaction, it is necessary to take into account first of all the existence of the universality of the interactions which are found mainly for the weak interactions through the phenomenological study,[5] and then derive the "interaction dependent" selection rules such as Pais-Jost's[6] or Utiyama's[7] instead of the usual selection rules which are independent of the choice of the interaction Lagrangian.*

In this paper we give an interaction-dependent selection rule based on the invariance principle called *mass reversal*.**[8] To show how this selection rule works, we present a unified model of the interactions between baryons and mesons taking into account the phenomenological study and test its internal consistency by means of this selection rule. The selection rule imposed by mass reversal is almost identical in its effectiveness with the even-odd rule, therefore, not as powerful as the N-G theory. The intention is not to present a model which replaces the isotopic spin model but to get some information about the dynamical character of the interactions existing among the fundamental particles by means of the selection rule obtained by mass reversal. As we are interested in the interactions between baryons and mesons, it is not necessary to touch deeply upon the lepton process, but only a few comments will be given.

The approach taken in this paper has some similarity with that of Utiyama and Tobocman[7] in which the concept of intrinsic space parity was used and the selection rule for the process relating to the baryon-meson system was obtained. Namely, under the hypothesis of mass reversibility, the concept of "intrinsic mass parity" is introduced for the fundamental particles and the structure of interactions among these particles is investigated from this point of view.

§ 2. Transformation properties of field under mass reversal

Before entering into the discussions about the selection rule, we shall summarize the transformation properties of field variables for the later use. If we require the invariance of the free field equation of the spinor field under mass reversion $\kappa \to -\kappa$, ψ' must transform as

$$M\psi'(x, \kappa)M^{-1} = \omega\gamma_5\psi'(x, -\kappa), \tag{1}$$

where M is the time-independent unitary operator of mass reversal and ω is an arbitrary constant. We shall limit this constant to the fourth roots of unity, ± 1, $\pm i$, by the additional requirement

$$Det|\omega\gamma_5| = 1. \tag{2}$$

Thus the spinor field has four different types of transformation under mass reversal with the different values of ω. These four types of spinor field will be denoted as A, B, C and D according as ω takes the value $+1$, -1, $+i$ and $-i$ in an analogous manner

* We mean here, for example, the conservation of space-parity.

** In this paper the mass reversal is considered in the standpoint M_1 of the previous paper.[8]

as Yang and Tiomno[9] did in the case of space-parity.

Defining the charge conjugate field ψ^c by means of the equations $\psi^c = C\overline{\psi}^T$ and $\overline{\psi}^c = (C^{-1}\psi)^T$, the charge conjugate fields of the *A*-, *B*-, *C*- and *D*-type fields become as shown in Table I.

Table I.

Type of spinor field (particle)	*A*	*B*	*C*	*D*
Type of charge conjugate field (antiparticle)	*B*	*A*	*C*	*D*

The details of the transformation properties of spinor field will not be discussed further as the discussion goes analogously to the case of space inversion.

Under mass reversal the boson field transforms as

$$M\phi(x, \kappa)M^{-1} = \omega'\phi(x, -\kappa) \tag{3}$$

where ω' takes the values ± 1, so there exist two types of transformation for the boson field.

§ 3. Selection rule

In this section we show how the requirement of mass reversibility can lead to the selection rule. If the mass reversibility of the system is required, the following equation must hold in the interaction representation,

$$M\int d^4x L_{\text{int}}(x, \kappa)M^{-1} = \int d^4x L_{\text{int}}(x, -\kappa), \tag{4}$$

where L_{int} is the interaction Lagrangian density. Equation (4) can produce the selection rule. This selection rule, as it may be expected, depends on the structure of interactions. Thus, before deriving the selection rule, it is necessary to introduce the explicit forms of the interaction.

Now let us consider the baryon-meson system which is known to interact each other through "strong" and "weak" interactions. In N-G theory, we are likely to understand the difference between the strong and weak interactions as the difference in characters in the isotopic spin space only. But the great difference and universality in the coupling strength seem to reflect the regularity of the dynamical structure of the interactions in Lorentz space. Therefore we present a tentative model of the elementary interaction between baryon and meson based on the following assumptions:

(I) All the bosons have spin 0, and all the fermions have spin 1/2. Any decisive evidence against this assumption has not yet been found.

(II) All the interactions between baryon- and meson-families are of the same type and are expected to be the generalization of the pseudoscalar meson theory. This is suggested from the fact that the interaction between the nucleon and pion, the typical ones of each family, has been well described by the Yukawa-type interaction.

(III) L_{int} is invariant under mass reversal.

It will be natural to take as the most general interactions satisfying these assumptions

$$L_{\text{int}}^{S}=\sum_{abc} G_{abc}\bar{\psi}_a\gamma_5' T^{\alpha}_{(a,b)}\psi_b\phi_c^{\alpha} \tag{5}$$

and

$$L_{\text{int}}^{W}=\sum_{abb'c} g_{ab'c}\bar{\psi}_a\gamma_5'\gamma_\mu T'^{\alpha}_{(a,b')}\psi_{b'}\partial_\mu\phi_c^{\alpha} \tag{6}$$

where ϕ_c denotes c-meson wave function, ψ_a a-baryon spinor, and $T^{\alpha}_{(a,b)}$'s are rectangular matrices in baryon space (a and b are not tensor indices, but express the kinds of fermions in interaction), and γ_5' is equal to γ_5 or I.

Here, it is necessary to emphasize the following two points for later discussions. (a) According to the previous paper[5], under the assumption (III) of this section, the derivative coupling and non-derivative coupling cannot exist simultaneously in L_{int}, when the Yukawa-type interactions are taken. It is therefore considered that in each vertex of the diagrams corresponding to the processes either the interaction (5) or (6) can contribute. (b) G_{abc} and $g_{ab'c}$ will differ in general with the suffices a, b and c but the order of magnitude of G_{abc} or $g_{ab'c}$ are the same irrespective of the suffix, so it will be natural in the model to replace them by the universal couplings G and g between baryon and meson families, just as e in the electromagnetic interactions. The electromagnetic interactions between the baryon and meson families are restricted to some extent by the requirement of gauge invariance. For example the most general interactions between electromagnetic field and baryon family are of the types:

$$L_{1\,\text{int}}^{E}=\sum_{b} e_b\bar{\psi}_b\gamma_\mu\psi_b A_\mu\,, \tag{7}$$

$$L_{2\,\text{int}}^{E}=\sum_{ab} f_{ab}\bar{\psi}_a\sigma_{\mu\nu}T^{E}_{(a,b)}\psi_b F_{\mu\nu}\,, \tag{8}$$

$$L_{3\,\text{int}}^{E}=\sum_{ab'} f_{ab'}\bar{\psi}_a\gamma_5\sigma_{\mu\nu}T'^{E}_{(a,b')}\psi_{b'} F_{\mu\nu}\,, \tag{9}$$

where e_b is equal to $\pm e$, $\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2i$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.

It is also easy to get the interaction Lagrangian for the electromagnetic interactions of meson fields, which we do not write down here explicitly, because in the following discussions these terms are not required. In any case it is evident that $J_\mu A_\mu$-type interactions always occur between same particles, while the Pauli-type ones occur between the different particles,* from the requirement of mass reversibility.

Now requiring the invariance of interaction under the mass reversal, the following conditions are derived from equation (4),

* So far there is no contradiction in this statement. The presence of meson fields causes the Pauli-type interaction as the higher order effect, so the interaction (8) or (9) are not considered as the elementary ones. Then the interaction of type (8) due to the anomalous magnetic moment does not contradict with this statement.

$$\begin{aligned} &\omega_a^* \omega_b \omega_c (-1) = 1 && \text{for (5)} \\ &\omega_a^* \omega_b \omega_c = 1 && \text{for (6)} \\ &\omega_A = 1 && \text{for the electromagnetic field} \end{aligned} \tag{10}$$

and $\omega_a^* \omega_b \omega_A (-1) = 1$ for (8) or (9),

where ω_k is the "intrinsic mass parity"* of k-particle. Thus all the processes occuring through the interactions (5) - (9) are allowed only if the following conditions are satisfied :

$$\left.\begin{aligned} &\omega_{(i)} (-1)^{n_i} = \omega_{(f)} (-1)^{n_f} \\ &\omega_{(i)} = \omega_{(f)} \end{aligned}\right\} \text{for any process} \begin{cases} \text{due to only (5), (8) or (9)} \\ \text{due to only (6) or (7)} \end{cases}$$

and for the process due to (5) + (6) + (7), which is practically important,

$$\omega_{(i)} (-1)^{n_i} = \omega_{(f)} (-1)^{n_f} (-1)^{n.w.}, \tag{11}$$

where the subscript i and f denote the initial and final state respectively and $\omega_{(i)}$ ($\omega_{(f)}$), the total intrinsic parity, is the product of the intrinsic parities of all the particles present in the initial (final) state and n_i is the number of mesons and photons present in the initial state. *n.w.* is the number of vertex (6) of the diagrams under consideration.

§ 4. Parity assignment

The validity of the model is temporarily subjected to the test by the selection rules (11) whether the consistent parity assignment for the particles is possible or not. Therefore it is shown in this section that by applying the selection rule to the well-established decay and production processes, we can assign the mass parities to the fundamental particles on the following assumptions (*A*), (*B*) and (*C*) and classify the elementary interactions between baryon and meson into strong and weak interactions consistently with the experimental evidences.

For the sake of definiteness we assume that (*A*) the baryon interacts strongly with meson by the interaction (5), (*B*) the weak interaction between baryon and meson which is mainly effective for the slow decay processes has the form (6)** and (*C*) the interaction between baryon and electromagnetic field is A_μ-type. The parity assigned for each particle is given in Table II, assuming the type of nucleon to be *A*.

* The term of mass parity may be not adequate, because of the nonexistence of parity conservation in general, but it is used by analogy with space parity.

** The reason why the strong and weak interactions are assigned to (5) and (6) respectively is due to the fact that the strong interaction would be renormalizable and the Yukawa-type interaction with the derivative coupling, though unrenormalized, are proved to be successful for the explanation of the decay processes from the phenomenological analysis of weak interactions.

The magnitude of $G^2/4\pi$ and $g^2/4\pi$ are of the order $1 \sim 10$ and $10^{-13} \sim 10^{-14}$ respectively in unit of $\hbar = c = \mu$(meson mass) $= 1$.

Table II.*

particle	N	Λ	Σ	Ξ	π	θ	τ
mass parity	$A(+1)$	$B(-1)$	$B(-1)$	$A(+1)$	-1	$+1$	$+1$

For the mass parity assignment it is necessary to assure that the lifetimes for the usual observed decays may be explained by means of the diagrams where the weak interaction with the coupling constant g contributes once for all.[5] One remarkable consequence of the mass parity assignment is that θ and τ are shown to have the same mass parity. This fact will be a suggestive hint for the problem concerning the $\theta-\tau$ complexity.[10]

§ 5. Concluding remarks

Although the main purpose of this paper was to present a selection rule by mass reversal, this enabled us to give a model for the baryon-meson interaction. The validity of this model was checked using the selection rule obtained and its consistency with the observed phenomena was proved. The model of the interaction for the baryon-meson system presented here is based on the observed regularity of the coupling strength. The reason why the different types of interaction have been taken in accordance with strong and weak interactions comes from the belief that the observed regularity of coupling constants must originate in the internal structure of fundamental particles. Thus it is very important to study how these interactions are workable for the dynamical description of the phenomena observed in nature. At present the use of mass reversibility as a selection rule is confined to those given in this paper. This is partly due to the fact that no way is found to express the conservation laws of mass parity independently of the interaction. However this way seems to be very useful to grasp the gross structure of the interaction.

Here, we must comment on our model in connection with Utiyama's model. In his model the strong interactions are assumed always to have γ_5 as γ_5' in (5), which seems to require the introduction of two kinds of Λ, Σ, etc., having the opposite parity and makes it impossible for the weak interactions always to have γ_5 or I as γ_5' in (6). If it were accepted that the conservation law of the space parity is violated for the weak interactions, all the strong interactions can have γ_5 as γ_5' in (5) without introduction of two kinds of Λ, Σ, etc., and all the weak interactions can have γ_5 or I as γ_5' in (6). In any case the selection rule imposed by mass reversal remains unaltered. It should be pointed out that the weak interactions in the model proposed here differ from the ones in Utiyama's model in having the derivative coupling instead of the non-derivative coupling.

When we try to describe the fundamental particles in a unified manner by the concept of family, it is interesting to examine the lepton processes. Regarding the leptons,

* The mass parities assigned here are not altered even if the interactions (5) and (6) contain the interaction which are not considered to be elementary.

the weak Fermi interactions and weak boson-fermion interactions are known to exist, but it is indeterminate which is elementary, the weak Fermi interaction or the weak boson-fermion one. This point and the nonexistence of strong interaction between leptons make one hesitate to apply the mass reversal to the lepton processes. As the mass reversal plays an important role in the discrimination between strong and weak interactions for baryon-meson system, it is not expected to be so operative for the lepton process though the similar procedures are very easily applied.*

The selection rule due to mass reversal is almost identical with the even-odd rule, and it gives the instruction about a real process which kind of interaction is operating, but not so effective to determine the physical realization of a certain process.** For this determination, the study of the structure of rectangular matrices T and T' in baryon space will be necessary. The information about them is given mainly with the help of the charge independence hypothesis in the same way as shown by Utiyama.

In conclusion the author would like to express his thanks to K. Senba and M. Yonezawa for their valuable discussions. He also thanks Prof. K. Sakuma and Dr. S. Ogawa for their interests shown to this work. Part of this work has been supported financially by the Scientific Research Expenditure of the Education Ministry extended to the author.

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* See Note added in proof.

** If we express the selection rules by means of Sachs'[11] "attribute" a, they become as (1) the transitions having $\Delta a=0$ (mod. 2) are strong and (2) transitions having $\Delta a=1$ (mod. 2) are weak, where the "attributes" are related to the mass parity ω as $\omega=-(-1)^a$ for bosons and $\omega=(-1)^a$ for fermions.

Note added in proof

As pointed out in the paper cited in 8), the mass reversals such as MR(1) and MR(2) (numbers denote the number of particles to which mass reversal is applied) are in principle possible unlike the other symmetry transformations. By assuming the universal Fermi interactions to be responsible for the lepton processes, we require the invariance of the theory under MR(1), MR(2) and MR(4) simultaneously. This requirement seems reasonable as it produces the consistent results with the recent experiment performed to test the conservation of space parity and the analysis of K-meson done by the members of Hiroshima University (Prog. Theor. Phys. **17** (1957), 89 etc.).

The theory of mass reversal MR(1) and MR(2) should be modified slightly and consistently with the theory of MR(4). Usually mass reversibility requires the sign function ρ_H of interaction Hamiltonian to be $+1$, guaranteeing the relation for S-matrix

$$\langle f;\ \kappa \mid S[\kappa] \mid i;\ \kappa\rangle = \langle f;\ -\kappa \mid S[-\kappa] \mid i;\ -\kappa\rangle. \tag{a}$$

As it seems natural to allow ρ_H to take ± 1 for MR(1) and MR(2), the relation (a) should be replaced by

$$|\langle f;\ \kappa \mid S[\kappa] \mid i;\ \kappa\rangle| = |\langle f;\ -\kappa \mid S[-\kappa] \mid i;\ -\kappa\rangle \tag{b}$$

where $|\cdots|$ means the summation over the different states of spin of the final state and the average over the initial state.

Then from the requirement of invariance under MR(1), we can prove a theorem which states that the lepton processes violate the conservation of space parity and always accompany the pair such as (μ, ν), (e, ν), (μ, μ) and (e, e) [the pair (μ, e) is excluded] where neutrino must have exact zero rest mass and zero self-energy. The conservation of lepton number proposed by Konopinski-Mahmoud (Phys. Rev. **92** (1953) 1045) seems to lose its meaning due to this theorem. From the requirement of MR(2), it is shown that the universal Fermi interactions can have only a linear combination of S, T and P couplings consistently with the results obtained in this paper. Details will be given in the following paper.

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Investigation of the Chew-Low Equation by Iteration

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Chew-Low equation of *P*-wave meson-nucleon scattering is investigated by iteration method to give the recipe to obtain the values of Feynman diagrams without any renormalization procedure. At the same time, the physical meaning of one-meson approximation of Chew-Low equation is made clear.

§ 1. Introduction

During these few years, the static model of $ps(ps)$ meson theory has been studied to explain the experiment of meson-nucleon scattering.[1] But, even in this considerably simplified model, it is very difficult to calculate the scattering phase shift, because, in the case of meson theory, the coupling constant is not so small that the lowest order perturbation is quite meaningless.

Various methods of solution had been presented to overcome this difficulty. First, the application of Tamm-Dancoff method to this problem was studied[2] and the effect of higher order terms in the coupling constant was partially taken into account. In this approximation, however, we could not find any consistent procedure of renormalization.

Then, Chew proposed the consistent method of renormalization in the non-covariant formalism[3]. We can calculate the scattering-matrix of any higher order according to this method. But this method had a weak point that it could be applied only to a perturbation-theoretic calculation. Thus we had come to a deadlock.

It was just at this time that Low presented the method for treating the scattering problem in Heisenberg representation[4] and Wick,[5] Chew and Low[6] applied it to the static model in question.

This theory is very interesting in many respects.

First, the basic equations of the theory can be derived from a few fundamental principles, the conservation of energy, the unitarity of *S*-matrix, etc.

Secondly, on the way of the derivation of basic equations, we use no complicated procedure of renormalization, such as Dyson-Salam renormalization method, but yet the obtained equation does not contain any of unrenormalized quantities.

Thirdly, the form of the equation is seemingly convenient for the approximation procedure. In the equation there is a sum over all possible states of the system. By limiting this sum up to one-meson states, for example, we can easily obtain a simple

integral equation. (Hereafter, according to Chew and Low, we call this approximation "one-meson approximation".) We shall be able to solve this approximate equation, at least numerically. The approximate solution may be in a sense better than that of Tamm-Dancoff theory, for it satisfies the crossing theorem which Tamm-Dancoff solution does not satisfy.

These are main points of interest.

This theory was presented at the time when we had come to a deadlock, as mentioned above. Consequently, the main effort was poured into the use of the method for the explanation of scattering experiments, and the attention was focused specially on the above third point.

Chew and Low tried to obtain the formal solution of the equation in one-meson approximation, and succeeded partially in explaining P-wave phase shifts.[6]

Thus we could fortunately get out of the deadlock. But we may say that at the same time the static model has unfortunately reached the limit of its applicability.

When we came to such a stage of the theory, it should be quite necessary to reconsider the Chew-Low equation from a new view-point. Indeed, it seems that its usefulness for the explanation of experiments was emphasized so much that another point of its excellence has been looked over.

Certainly, in the present stage, it should be more necessary to focus the attention on the renormalization problem in the Chew-Low equation than on the usefulness for the explanation of experiments.

In this paper, we focus our attention on the renormalization problem in the Chew-Low theory, investigate the Chew-Low equation by the iteration method and analyze how the renormalized values of Feynman diagrams are obtained directly from the Chew-Low equation and how the renormalization method are considerably simplified.

Previously, Lehmann analyzed these points in a strictly relativistic model[7]. But the model was so complete but complicated that the simple graphical interpretation was impossible. Here, we use an incomplete but simple model and make clear the correspondence of iterated solutions with Feynman diagrams. These points may be anticipated, to some extent, tacitly from the form of the Chew-Low equation itself, while in this paper they are consciously emphasized and analyzed closely.

In Sec. 2, we examine the iterated solution of the Chew-Low equation and make each term of it correspond to Feynman diagram. In Sec. 3, we explain Chew's method of renormalization briefly, calculating actually the values of Feynman diagrams used in Sec. 2. Comparing them with the results in Sec. 2, we verify the correctness of the preceding correspondence and give a method for obtaining the renormalized T-matrix directly from the Chew-Low equation. In Sec. 4, we investigate closely the relation between the values and forms of diagrams and make clear the physical meaning of the one-meson approximation, etc. In Sec. 5, we collect the results in the preceding sections and present the recipe for calculating the renormalized T-matrix directly from the diagrams.

§ 2. The relation of the iterated solution to Feynman diagrams

In this section, we first try to obtain the iterated solution of the Chew-Low equation

and next correlate it with the Feynman diagrams representing scattering processes.

The Chew-Low equation can be derived either from fundamental principles directly or from the conventional Hamiltonian, as shown in their paper. In order to clarify the relation between Chew's old method of renormalization and the renormalization procedure in the Chew-Low equation, it is more convenient to bear the latter method in mind.

The Hamiltonian of the meson-nucleon system is taken as follows:

$$H=\sum_k{}' \omega_k a_k^+ a_k+\sum_k{}' (V_k^{(0)} a_k+V_k^{(0)+} a_k^+), \tag{1}$$

$$V_k^{(0)}=i\frac{g^{(0)}}{2M}\frac{v(k)}{\sqrt{2\omega_k}}(\sigma \boldsymbol{k})\tau_k. \tag{2}$$

Here, a_k^+ and a_k are, respectively, creation and annihilation operators for single mesons, $\omega_k=(\mu^2+k^2)^{1/2}$, μ is the meson mass, M is the nucleon mass, σ is the nucleon spin vector, τ_k is the k-th component of the isotopic spin operator, $v(k)$ is the cut-off function and $g^{(0)}$ is the unrenormalized coupling constant. (We take $\hbar=c=1$.)

Of course, the following discussions are applicable to other meson theories than the $ps(ps)$ theory by changing the form of $V_k^{(0)}$. Here, we give the above form to $V_k^{(0)}$, as an example.

Wick, Chew and Low showed that, if we define the following quantities

$$T_q(n)=(\Psi_n^{(-)}, V_q^{(0)+}\Psi_0), \tag{3}$$

$T_q(p)$ agrees with the conventional T-matrix[8] when the energies of the initial and final states are equal, and it is given as the solution of the following equation:

$$T_q(p)=-\sum_n{}'\left[\frac{T_p^+(n)\,T_q(n)}{E_n-\omega_p-i\epsilon}+\frac{T_q^+(n)\,T_p(n)}{E_n+\omega_p}\right]. \tag{4}$$

In eq. (3), $\Psi_n^{(-)}$ is the incoming wave solution of Schroedinger equation

$$H\Psi_n^{(-)}=E_n\Psi_n^{(-)} \tag{5}$$

and Ψ_0 is the true vaccum state of the system.

We assume that Hamiltonian H has been normalized by subtracting the nucleon self-energy so that

$$\begin{aligned} &H\Psi_0=0,\\ &H\Psi_k=\omega_k\Psi_k,\\ &H\Psi_{k\,k'}=(\omega_k+\omega_{k'})\Psi_{k,k'}, \end{aligned} \tag{6}$$

etc. The sum in eq. (4) is taken over all states and, through the term of $n=0$, the renormalized coupling constant is introduced, that is, we define the renormalized coupling constant g by the equation

$$T_q(0)=V_q, \tag{7}$$

where V_q is obtained from $V_q^{(0)}$ by replacing $g^{(0)}$ by g. Eq. (6) and eq. (7) are the renormalization procedure in the Chew-Low equation. The former corresponds to the mass renormalization and the latter to the charge renormalization. The equations for other amplitudes, $T_q(k, k')$, etc., are not given in eq. (4). We abbreviate them, as they are easily obtained. (In the Appendix we shall briefly consider the two-meson amplitude $T_q(k, k')$.)

Chew and Low formally solved this basic equation (4), by limiting the sum of intermediate states to the vacuum state and the one-meson states only (one-meson approximation), and by using a skillful mathematical technique (effective range approximation). But, here, we solve this equation elementarily by the successive substitutions without the use of such kinds of approximation.

The solution calculated, for example, to the sixth order in the renormalized coupling constant is as follows:

$$T_q(p)=T_q(p)^{(2)}_{CL-1}+T_q(p)^{(4)}_{CL-1}+T_q(p)^{(6)}_{CL-1}+T_q(p)^{(6)}_{CL-2}+C.G.T., \tag{8}$$

$$T_q(p)^{(2)}_{CL-1}=V_p^+V_q/E, \tag{8\cdot1}$$

$$T_q(p)^{(4)}_{CL-1}=\sum_k (V_p^+V_k-V_kV_p^+)\,(V_k^+V_q-V_qV_k^+)/\omega^2(E-\omega+i\epsilon), \tag{8\cdot2}$$

$$\begin{aligned}T_q(p)^{(6)}_{CL-1}=&\sum_{k,k'}\frac{(V_p^+V_{k'}-V_{k'}V_p^+)\,(V_{k'}^+V_k-V_kV_{k'}^+)\,(V_k^+V_q-V_qV_k^+)}{\omega\omega'^2(\omega-\omega'+i\epsilon)\,(E-\omega+i\epsilon)}\\&-\sum_{k,k'}\frac{(V_p^+V_{k'}-V_{k'}V_p^+)\,(V_{k'}^+V_k-V_kV_{k'}^+)\,(V_k^+V_q-V_qV_k^+)}{\omega^2\omega'(\omega'-\omega+i\epsilon)\,(E-\omega'+i\epsilon)}\\&+\sum_{k,k'}\frac{(V_kV_{k'}-V_{k'}V_k)\,(V_k^+V_p^+-V_p^+V_k^+)\,(V_{k'}^+V_q-V_qV_{k'}^+)}{\omega^2\omega'(\omega+\omega')\,(E-\omega'+i\epsilon)}\\&-\sum_{k,k'}\frac{(V_p^+V_k-V_kV_p^+)\,(V_qV_{k'}-V_{k'}V_q)\,(V_{k'}V_k^+-V_k^+V_{k'})}{\omega\omega'^2(\omega+\omega')\,(E-\omega+i\epsilon)},\end{aligned} \tag{8\cdot3}$$

$$\begin{aligned}T_q(p)^{(6)}_{CL-2}=&\sum_{k,k'}{}'\left[\frac{V_{k'}V_kV_p^+}{\omega'(\omega+\omega')}+\frac{V_pV_{k'}V_k}{\omega(\omega+\omega')}-\frac{V_{k'}V_p^+V_k}{\omega\omega'}\right]\frac{1}{E-\omega-\omega'+i\epsilon}\\&\times\left[\frac{V_qV_kV_{k'}^+}{\omega'(\omega+\omega')}+\frac{V_k^+V_{k'}^+V_q}{\omega(\omega+\omega')}-\frac{V_k^+V_qV_{k'}^+}{\omega\omega'}\right]\\&+\sum_{k,k'}{}'\left[\frac{V_kV_{k'}V_p^+}{\omega(\omega+\omega')}+\frac{V_p^+V_kV_{k'}}{\omega'(\omega+\omega')}-\frac{V_kV_p^+V_{k'}}{\omega\omega'}\right]\frac{1}{E-\omega-\omega'+i\epsilon}\\&\times\left[\frac{V_qV_k^+V_{k'}^+}{\omega'(\omega+\omega')}+\frac{V_k^+V_{k'}^+V_q}{\omega(\omega+\omega')}-\frac{V_k^+V_qV_{k'}^+}{\omega\omega'}\right].\end{aligned} \tag{8\cdot4}$$

Here, $E=\omega_p=\omega_q$, $\omega=\omega_k$ and $\omega'=\omega_{k'}$. *C.G.T.* means the terms which are obtained by interchanging V_q and V_p and replacing E by $-E$ in the preceding terms. Because these terms correspond to the values of Feynman diagrams in which the initial and final meson lines are crossing as shown below, we call them the "crossing graph terms" and abbreviate them as *C.G.T.*

The term $T_q(p)^{(6)}_{CL-2}$ is the contribution due to two-meson amplitudes $T_q(k, k')$ and does not appear in one-meson approximation. The suffix *CL*-2 reflects this fact. (The derivation of this term is very simple, but for completeness it is inserted in Appendix.)

Generally, if some term appears when the sum in eq. (4) is taken up to the n-meson states and does not appear when the sum is taken only up to the $(n-1)$-meson states, we call the term "belonging to *CL*-n" and attach the suffix *CL*-n to it.

The next problem is to relate the terms in eq. (8) with static Feynman diagrams. In the unrenormalized covariant Q.E.D., we could easily connect the perturbation solution with Feynman diagrams by ordering field operators into normal products.[6] It is the present task to obtain the means by which each term in eq. (8) is connected with some Feynman diagram. To do this, we examine fourth-order terms eq. (8·2), as an example. Expanding eq. (8·2), we take up one of terms, for example,

$$\sum_k{}' \frac{V_p V_k V_k V_q}{\omega^2(E-\omega+i\epsilon)} \equiv B(1)^{(4)}_{CL-1}. \tag{9}$$

If we pay our attention to the arrangement of vertex operators $V_p V_k V_k V_q$, we see that the result is the same as that appears when the diagram Fig. 1 (1) is calculated by the conventional perturbation method. Therefore, we make this term correspond to the diagram Fig. 1 (1) temporarily, though we have not known whether such a correspondence is correct or not. In the similar manner, we can connect all terms in eq. (8) with Feynman diagrams by examining the arrangement of vertex operators. For simplicity, we shall use a convenient notation in the following. We call the part of product of vertex operators in some term "the vertex part" and write $\mathcal{V}(X)$. We call the part of remaining energy denominators "the propagator part" and write $\bar{\mathcal{G}}(X)_{CL-n,\beta}$. Here X is the notation of the diagram which is connected to the term through the vertex part and the suffix β is used for the distinction of the terms with the same vertex part. (Usually there are several terms with the same vertex part $\mathcal{V}(X)$.)

For example, we write eq. (9) as follows:

$$\begin{aligned} &B(1)^{(4)}_{CL-1} = \sum_k{}' \mathcal{V}(1)\, \bar{\mathcal{G}}_{CL-1}(1), \\ &\mathcal{V}(1) = V_p V_k V_k V_q, \\ &\bar{\mathcal{G}}(1)_{CL-1} = 1/\omega^2(E-\omega+i\epsilon). \end{aligned} \tag{9'}$$

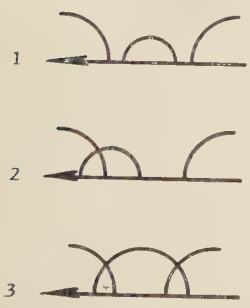

Fig. 1.
Fourth-order diagrams of scattering

In this case the term with $\mathcal{V}(1)$ consists of only one term and the suffix β is unnecessary.

The result obtained by applying this correspondence method to eqs. (8·2–4) is written in Table I. In the columns *CL*-1 and *CL*-2, respectively, the sums of propagator parts belonging to *CL*-1 and *CL*-2 are written. F_0, etc., in the table are the following functions:

Table I. The propagator parts of fourth- and sixth-order diagrams.

	X	$\overline{\mathfrak{S}}(X)_{CL-1}$	$\overline{\mathfrak{S}}(X)_{CL-2}$		X	$\overline{\mathfrak{S}}(X)_{CL-1}$	$\overline{\mathfrak{S}}(X)_{CL-2}$
I	1	F_0	0	II	1,2	0	F_3
	2	$-F_0$	0		3	0	F_4
	3	F_0	0		4	0	F_5
	1,2	F_1	0		5	0	$(-1/2)F_5$
	3	$-F_1$	0	III	1	$-F_1$	F_5
	4,5	$-F_2$	0		2	$-F_1$	F_4
	6,7	F_2	0		3	F_1	$(-1/2)F_5$
					4	$-F_2$	$(1/2)F_5$
					5	F_2	$(-1/2)F_5$
					6	F_2	F_4
					7	$-F_2$	F_3

In the Table, X's correspond to the number attached to each diagram in Fig. 1 and Fig. 2. F's stand for the functions in eq. (10).

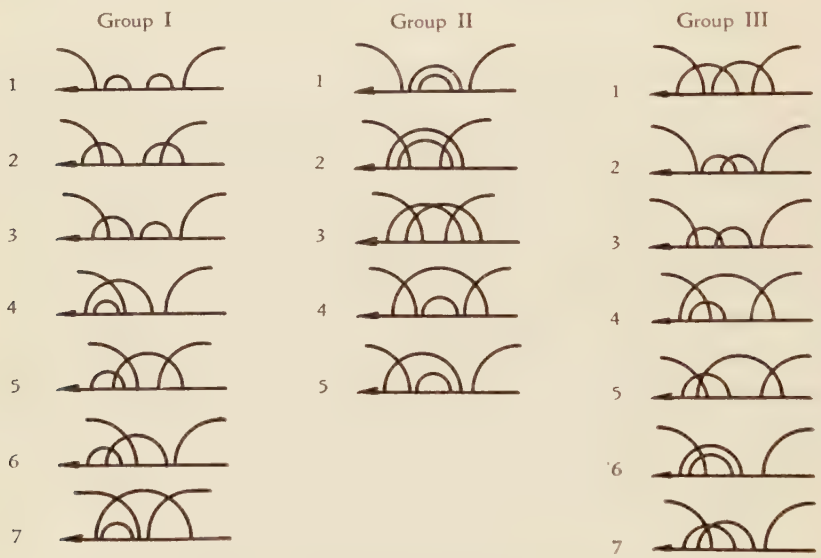

Fig. 2. Sixth-order diagrams of scattering

$$F_0(E) = \frac{1}{\omega^2(E-\omega+i\epsilon)},$$

$$F_1(E) = \frac{E}{(\omega\omega')^2(E-\omega+i\epsilon)\,(E-\omega'+i\epsilon)},$$

$$F_2(E) = \frac{(\omega+\omega')E-2\omega\omega'}{2(\omega\omega')^2(\omega+\omega')\,(E-\omega+i\epsilon)\,(E-\omega'+i\epsilon)}, \tag{10}$$

$$F_3(E) = \frac{\omega^2+\omega'^2}{2(\omega\omega')^2(\omega+\omega')^2(E-\omega-\omega'+i\epsilon)},$$

$$F_4(E) = \frac{1}{(\omega\omega')\,(\omega+\omega')^2(E-\omega-\omega'+i\epsilon)}\,,$$

$$F_5(E) = \frac{1}{(\omega\omega')^2(E-\omega-\omega'+i\epsilon)}\,.$$

The notations of diagrams are shown in Fig. 1 and Fig. 2. Here we abbreviate one of two diagrams which can be obtained through another by overturning the meson-lines like (*a*) and (*b*) in Fig. 3. We also omit the diagrams like Fig. 3 (*c*) in which the initial meson line and final one are crossing, for we can easily see that it is trivially connected with the corresponding *C.G.T.*, that is, the values of them can be easily obtained if we change the sign of E in the propagator parts of the corresponding non-crossing diagram (Fig. 3 (*a*)) and interchange V_p^+ and V_q in the vertex parts.

Now we have thus correlated each term in eq. (8) with some Feynman diagram. But, we have not known the relation of these values to the values correctly renormalized by the actual perturbation calculation.

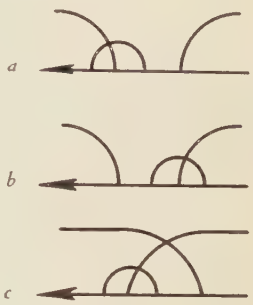

Fig. 3.
An example of abbreviation

In the next section, we shall show that this correspondence method is correct and the sums of *CL*-1 and *CL*-2 in Table I are really equal to the renormalized values of fourth- and sixth- order Feynman diagrams.

It is obvious from the derivation of the Chew-Low equation itself that the grand total of the renormalized values of all Feynman diagrams is equal to the complete solution of the Chew-Low equation, but it is not necessarily obvious that the partial sum of the terms which are obtained by the iteration of the Chew-Low equation exactly corresponds to the renormalized value of a Feynman diagram.

Here, we clarify these points by actually comparing the values in Table I with the values by perturbation calculations.

§ 3. The comparison with Chew's renormalization

In this section, we calculate the values of scattering diagrams to the sixth order with the use of Chew's perturbation method,[3] compare them with the values obtained by the iteration of the Chew-Low equation and verify the correctness of the correspondence method in the preceding section.

Previously, Chew applied Dyson's method of renormalization to the static meson theory and presented the recipe for calculating the probability amplitude of any processes in the model. For completeness, we repeat here his main results briefly.

(i) The T-matrix of the process considered is obtained as follows: we first draw the irreducible static Feynman diagram representing the process, then make the renormalized vertex obtained in (ii) and the renormalized propagator obtained in (iii) correspond to each vertex and each nucleon-line, respectively, and finally integrate it over virtual meson

states. In this case, it is not necessary to consider the propagators corresponding to meson-lines. Also, it is not necessary to consider the diagrams correcting the external lines.

(ii) The renormalized vertex operator can be obtained as follows: Using the renormalized vertices and propagators, we calculate formally the value of the irreducible vertex diagram and write it as

$$V_q L(E_2, E_1),$$

where E_1 is the energy received by the nucleon-line in the past, E_2 is that received in the future and q is the state of the meson which enters the vertex in question. Then the renormalized value of the vertex diagram $V_q(E_2, E_1)$ is calculated by the equation,

$$V_q(E_2, E_1) = V_q(1 + L(E_2, E_1) - L(0, 0)). \tag{11}$$

(iii) The renormalized propagator can be obtained as follows: Using the renormalized vertices and propagators, we calculate formally the value of the irreducible self-energy diagram and write it as $\sum'(E)$. (This is the value between two vertices of the self-energy diagram. E is the energy received by the nucleon-line.) Then the renormalized value of the propagator $S(E)$ is obtained by the equation,

$$\frac{1}{S(E)} = E - \left(\sum'(E) - \sum'(0) - \left(\frac{\partial \sum'(E)}{\partial E} \right)_{E=0} \cdot E \right) \tag{12}$$

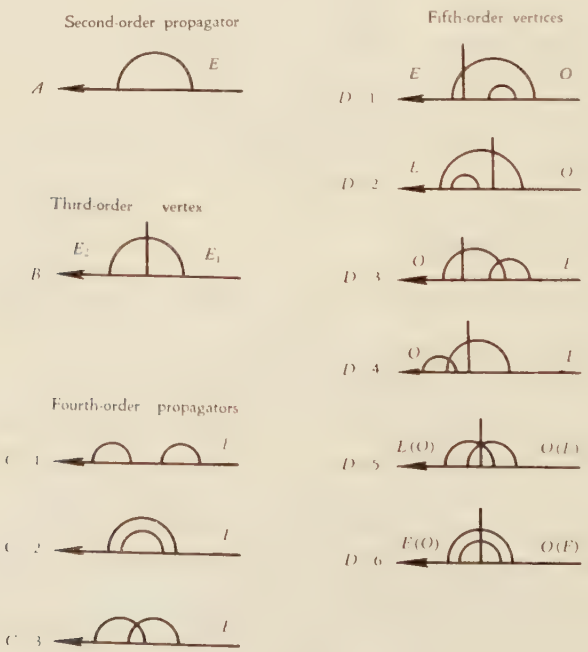

Fig. 4. Diagrams of higher-order propagators and vertices

Eqs. (11) and (12) correspond to eqs. (7) and (6), in a sense. We can calculate the value of any scattering diagram by this method. For convenience, we tabulate the values of propagators and vertices corrected to the fourth order (Fig. 4) in Table II. We shall abbreviate $+i\epsilon$ in the energy denominators in the following.

Table II. Corrected propagators and vertices

Here we write down the propagator parts of the propagators and vertices corrected up to fourth-order. The notation of them is shown in Fig. 4.

diagrams	propagator parts
A	$1/\omega^2(E-\omega)$
B	$[(E_1+E_2)\omega-E_1E_2]/\omega^2(E_2-\omega)(E_1-\omega)$
$C-1$	$E/(\omega\omega')^2(E-\omega)(E-\omega')$
$C-2$	$\frac{\omega^2+\omega'^2}{2(\omega\omega')^2(\omega+\omega')^2(E-\omega-\omega')}$
$C-3$	$\frac{\omega\omega'(E-\omega)(E-\omega')-(\omega+\omega')^2E(E-\omega-\omega')}{(\omega\omega')^2(\omega+\omega')^2(E-\omega)(E-\omega')(E-\omega-\omega')}$
$D-1$	$\frac{-E[E(\omega+\omega')-2\omega\omega']}{2(\omega\omega')^2(\omega+\omega')(E-\omega)(E-\omega')}$
$D-2$	$-E/2(\omega\omega')^2(E-\omega-\omega')$
$D-3$	$\frac{E[(E-\omega)(E-\omega')-2\omega\omega']}{2(\omega\omega')^2(E-\omega)(E-\omega')(E-\omega-\omega')}$
$D-4$	$\frac{E[E(\omega+\omega')-2\omega\omega']}{2(\omega\omega')^2(\omega+\omega')(E-\omega)(E-\omega')}$
$D-5$	$\frac{-E[2(E-\omega)(E-\omega')-(\omega+\omega')(2E-\omega-\omega')]}{2\omega\omega'(\omega+\omega')^2(E-\omega)(E-\omega')(E-\omega-\omega')}$
$D-6$	$\frac{E[\{(\omega+\omega')^2+2\omega\omega'\}E^2-\{(\omega+\omega')^3+4\omega\omega'(\omega+\omega')\}E+2\omega\omega'\{(\omega+\omega')^2+\omega\omega'\}]}{2(\omega\omega')^2(\omega+\omega')(E-\omega)(E-\omega')(E-\omega-\omega')}$

Now we can achieve the aim of this section. First, we calculate the values of all diagrams in Figs. 1 and 2. Next, we compare them with the values in Table I. And we can verify the correctness of the correspondence method in Sec. 2, that is, we can see that the correctly renormalized value of some Feynman diagram X is calculated directly from the Chew-Low equation by taking out the terms with the vertex part $\mathcal{V}(X)$ from the iterated solution of the equation and summing them up all.

For example, the value of diagram III-3 is immediately obtained from Table I as follows:

$$B(III\text{–}3)=\sum\mathcal{V}(III\text{–}3)\left[\frac{E}{(\omega\omega')^2(E-\omega)(E-\omega')}-\frac{1}{2(\omega\omega')^2(E-\omega-\omega')}\right]$$

$$=\sum\mathcal{V}(III\text{–}3)\frac{E^2-(\omega+\omega')E-\omega\omega'}{2(\omega\omega')^2(E-\omega)(E-\omega')(E-\omega-\omega')}. \qquad (13)$$

This is just equal to the value calculated with the use of the renormalized vertex tabulated in Table II.

Here, we verified the correctness of the correspondence method and gave the method for the calculation of diagrams without the use of any explicit renormalization procedure, only with the exemplification. But as it is easily inferred that this holds up to the arbitrarily higher order, we shall avoid unnecessary strictness and proceed to explain the reason for the grouping of diagrams in Fig. 2.

In Table I, the values of fourth-order Feynman diagrams all belong to *CL*-1. On the other hand, in the case of sixth-order, all diagrams of group I belong to *CL*-1, those of group II belong to *CL*-2 and those of group III consist of parts belonging to *CL*-1 and *CL*-2 both.

We shall explain the physical meaning of such a characteristic of each group in the next section.

§ 4. The divisions of diagrams

First, we examine one of the diagrams in group I, for example, I-1. Let us divide this diagram into two real processes at a suitable point (Fig. 5). As a result, we get two modes (*a*) and (*b*) in Fig. 5. The mode (*a*) is interpreted as the incoming meson first undergoes the fourth-order scattering, the scattered meson propagates virtually and lastly the virtual meson goes out as a real meson through the second-order scattering. The mode (*b*) is interpreted as the incoming meson first undergoes the second-order scattering and then the fourth-order one.

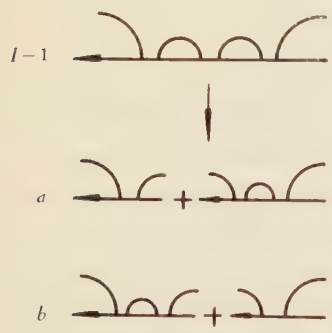

Fig. 5.
The division of diagram I-1

Applying the similar divisions to all diagrams in group I, we can easily see that they are all regarded as the succession of scattering processes.

Next, we divide a diagram in group II, for example, II-1 in the similar manner. In this case, there exist three modes (*a*), (*b*) and (*c*) in Fig. 6. Among them, the mode (*a*) is interpreted as the incoming meson first produces two mesons, the two virtual mesons propagate and then they are annihilated creating a real meson. On the contrary, the mode (*b*) is interpreted as the meson undergoes first the fourth-order scattering and next the second-order scattering, and the mode (*c*) in a reverse way. Fourth-order scatterings in the modes (*b*) and (*c*), however, is corrected in the "external line", so we need not take account of them, according to Chew[2]. Then, eventually, only the mode (*a*) survives in which two meson propagate virtually. Similarly avoiding the divisions which give rise to the corrections of the "external lines", we can see that all diagrams in group II are divided into the modes regarded as the successions of $\pi-2\pi$ processes only.

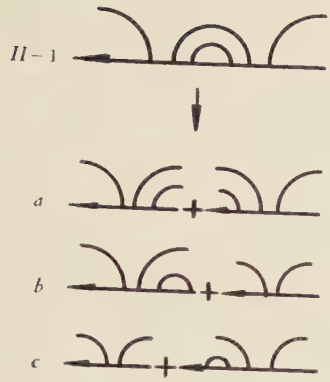

Fig. 6.
The division of diagram II–1

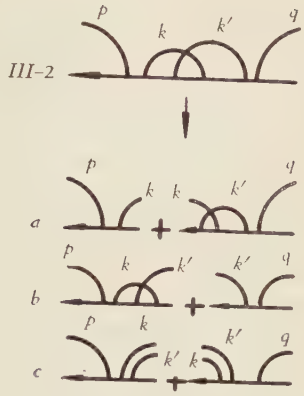

Fig. 7.
The division of diagram III–2

Let us apply the same idea of division to a diagram in group III, for example, III-2 in Fig. 2. There being three modes as Fig. 7, the modes (*a*) and (*b*) are interpreted as the succession of scattering processes, and, on the contrary, the mode (*c*) is interpreted as the succession of $\pi-2\pi$ processes. Examining all other diagrams belonging to this group, we see that two kinds of interpretations can be applied to all of them.

This behavior of the diagrams of each group in regard to the division has direct relations to the characteristic of the group appearing in Table I. That the diagrams of group III belong to both *CL*-1 and *CL*-2 is due to the fact that they can be interpreted into two manners as mentioned above.

In the consequence of this analysis, we have been simultaneously able to make clear the relation between one-meson approximation and Feynman diagrams. In one-meson approximation of Chew-Low equation we take whole account of all diagrams which are interpreted as the succession of scattering processes, neglect completely all diagrams which are interpreted as the succession of $\pi-2\pi$ processes and, in the case of the diagrams which can be interpreted into two manners, take account of the part only which is interpreted as the succession of scattering processes. This is in contrast with Tamm-Dancoff approximation in which some diagrams are taken into account and the other diagrams are completely omitted.

§ 5. Concluding remarks

Collecting the above results, we can obtain the recipe for calculating the arbitrary scattering diagrams in the static model without the use of any renormalization procedure.

(i) We divide the given diagram into two processes each of which is of the form of a real process. Hereby, such divisions are excluded by which one or both of the divided parts get to contain the corrected "external line".

(ii) We construct T-matrix corresponding to each divided part and Chew's propagator is inserted at the dividing point. At this time, we must use the energy value at the dividing point as the argument of T-matrix. Chew's propagator is given by $1/(E-\sum\omega_i)$, where E is the initial meson energy and ω_i are the virtual meson energies at the dividing point.

(iii) Multiplying these in the right order of time, we integrate it over the states of virtual mesons.

(iv) After calculating these integrals for all possible divisions, we sum up these values all together.

By means of this recipe, we can calculate any of arbitrary order non-crossing scattering diagrams, using the amplitudes of the lower-order real processes. In this case, we necessitate no explicit renormalization procedure.

By the way, the value of a crossing diagram is obtained easily from the value of the corresponding non-crossing diagram by changing the sign of the initial meson energy and interchanging the state of incoming meson and that of outgoing meson, as explained in Sec. 2.

In the following, we shall explain this with some examples.

First, we calculate a fourth-order diagram, Fig. 8. There exists only one division shown in the figure. The T-matrices of the future- and past-part are $-V_kV_p^+/\omega$ and $V_k^+V_q/\omega$, respectively, where the energy has been taken equal to the value at the dividing point, and the propagator is $1/(E-\omega)$.

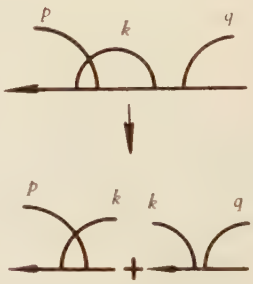

Fig. 8
The division of fourth-order diagram

If we multiply these all together and integrate, we get

$$B(0\text{–}2)=\sum_k'\left(-\frac{V_kV_p^+}{\omega}\right)\frac{1}{E-\omega}\left(\frac{V_k^+V_q}{\omega}\right)=\sum_k'V_kV_p^+V_k^+V_q\frac{-1}{\omega^2(E-\omega)}\,. \tag{14}$$

This coincide with the value in Table I, that is, the renormalized result.

As another example, we calculate a sixth-order diagram, Fig. 7. It is divided into three modes, (a), (b) and (c). In the case of (a), T-matrix of the future-part is V_pV_k/ω, that of the past-part is $\sum_{k'}'-V_{k'}V_kV_{k'}V_q/\omega'^2(\omega-\omega')$ from eq. (14), and the propagator is $1/(E-\omega)$. Therefore the contribution of (a) is

$$\sum_{k,k'}'V_p^+V_kV_{k'}V_k^+V_{k'}^+V_q\frac{-1}{\omega\omega'^2(\omega-\omega')\,(E-\omega)}\,. \tag{15}$$

Similarly, that of (b) is

$$\sum_{k,k'}'V_p^+V_kV_{k'}V_k^+V_{k'}^+V_q\frac{-1}{\omega^2\omega'\,(\omega'-\omega)\,(E-\omega')}\,. \tag{16}$$

In mode (c), T-matrices of the future- and past-part are, respectively, $V_p^+V_kV_{k'}/(\omega+\omega')\,\omega'$

and $V_k^+ V_{k'}^+ V_q/\omega(\omega+\omega')$ and the propagator is $1/(E-\omega-\omega')$. Therefore the contribution of (*c*) is

$$\sum_{k,k'} V_p^+ V_k V_{k'} V_k^+ V_{k'}^+ V_q \frac{1}{\omega\omega'(\omega+\omega')^2(E-\omega-\omega')}. \tag{17}$$

Summing up Eqs. (15), (16) and (17),

$$B(III\text{-}2) = \sum_{k,k'} V_p^- V_k V_{k'} V_k^+ V_{k'}^+ V_q \frac{\omega\omega'(E-\omega)(E-\omega') - (\omega+\omega')^2 E(E-\omega-\omega')}{(\omega\omega')^2(\omega+\omega')^2(E-\omega')(E-\omega-\omega')}. \tag{18}$$

This is equal to the value of the diagram after the subtraction of the overlapping divergence according to Salam.[10]

Thus, the recipe given here is very convenient in the points that we do not need renormalization procedure and that it is easy to form the physical image. Particularly, it is quite effective for the calculation of the complicated diagrams involving overlapping divergences.

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Appendix. The Derivation of two-meson amplitude

Here we explain the derivation of two-meson amplitude $T_p(k, k')$ briefly. By the definition, we have

$$T_p(k, k') = (\Psi_{k,k'}^{(-)} V_p \Psi_0). \tag{A1}$$

Substituting the following expression

$$\Psi_{k\,k'}^{(-)} = a_k^+ a_{k'}^+ \Psi_0 - \frac{1}{H-\omega-\omega'+i\epsilon}(V_k a_{k'}^+ + V_{k'} a_k^+)\Psi_0 \tag{A2}$$

in $\Psi_{k,k}^{(-)}$ in Eq. (A1) and using the relation

$$a_k \Psi_0 = \frac{-1}{H+\omega} V_k^+ \Psi_0, \tag{A3}$$

we obtain

$$T_p(k, k') = \left(\Psi_0 a_{k'} V_p \frac{-1}{H+\omega} V_k^+ \Psi_0\right)$$

$$-\left(\Psi_0 (V_k^+ a_{k'} + V_{k'}^+ a_k) \frac{1}{H-\omega-\omega'+i\epsilon} V_p \Psi_0\right). \tag{A4}$$

Applying the identity

$$a_k \frac{1}{H+x} = \frac{1}{H+x+\omega} a_k + \frac{1}{H+x+\omega}(-V_k^+)\frac{1}{H+x}, \tag{A5}$$

we obtain the result

$$T_p(k, k') = \sum_{n,m} T_p^+(m) \frac{1}{E_m + \omega + \omega'} (\Psi_m^{(-)} V_{k'}^+ \Psi_n^{(-)}) \frac{1}{E_n + \omega} T_k(n)$$
$$+ \sum_{n,m} T_{k'}^+(m) \frac{1}{E_m - \omega' - i\epsilon} (\Psi_m^{(-)} V_k^+ V_n^{(-)}) \frac{1}{E_n - \omega - \omega' - i\epsilon} T_p(n) \qquad \text{(A6)}$$
$$+ \sum_{n,m} T_{k'}^+(m) \frac{1}{E_m - \omega' - i\epsilon} (\Psi_m^{(-)} V_p \Psi_n^{(-)}) \frac{1}{E_n + \omega} T_k(n)$$
$$+ (k \rightleftarrows k')$$

Restricting m and n to the vacuum state only in the result, we can obtain eq. (8·4) at once.

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On the Asymptotic Conditions in Quantum Field Theory

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In this paper the so-called asymptotic conditions are introduced in a different way than usually done, thereby making use of Heisenberg operators alone. These conditions are regarded as the defining equations for a complete set of state vectors, either incoming or outgoing wave states. By closely examining the self-consistency of these defining equations, we get a set of integral equations for the vacuum expectation values of retarded products of operators. The unitarity condition of the S matrix is one of the inevitable consequences of these equations. These integral equations are essentially equivalent to those given by Chew and Low, and also independently by Lehmann, Symanzik and Zimmermann. The advantage of the new integral equations over the older ones of the above mentioned authors consists in that the new ones are manifestly covariant in form and also in that all quantities appearing in the equations are related only to connected Feynman diagrams in contrast to the T products. The possibility to extend the present formalism so as to include the bound states is also discussed. Finally it is shown in the perturbation theory that these integral equations are satisfied only by the renormalized solutions of the renormalizable field theories provided that the microscopic causality condition is imposed as the boundary condition.

§ 1. Introduction

Recently the importance of the asymptotic conditions has been recognized by many authors,[1-5] especially in connection with the use of Heisenberg operators in quantum field theory. In quantum field theory the relation between individual particle states and interacting states are settled through these conditions, and important is that only through them the Heisenberg operators can be related to observable or semi-observable quantities such as the S matrix.

In particle mechanics the asymptotic behaviour of the wave functions can easily be seen by simply dividing the total Hamiltonian into two parts, one corresponding to the kinetic energies of individual particles and the other to the interaction among them, and the latter must be dropped for the investigation of the asymptotic behaviour of the wave functions. In quantum field theory, however, such a separation of the total Hamiltonian can no more be taken for merited, since individual particles interact with their own self-fields even when they are separated at long distances from each other. Consequently the description of the asymptotic behaviour of a system can no more be achieved in the Hamiltonian formulation, but instead one is obliged to stand on the over-all-space-time point of view.[6] The reason in favour of this standpoint of view was intuitively discussed

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by the present author especially in connection with the problem of bound states.[7]

Another important feature of these conditions consists in the fact that they put severe limitations upon the posisble structure of the quantum field theory. Extensive investigations towards this direction were made by Chew and Low,[3] and also by Lehmann, Symanzik and Zimmermann.[5]

In the conventional asymptotic conditions, first the asymptotic behaviour of the matrix elements of Heisenberg operators between two given states was investigated, then after some discussions concerning the limiting procedures the expressions of, say, the S matrix elements were derived in such a manner that the final expressions do no more involve limiting symbols. In order to avoid ambiguous discussions concerning the limiting procedures, we introduce in this paper the asymptotic conditions in a completely different manner than usually done, i. e. we introduce recursion formulas for state vectors as the defining equations of a complete set of state vectors, representing either incoming or outgoing waves. In this way the problem of limiting procedures is replaced by another simpler algebraic one to examine the self-consistency of these defining equations. To write down these equations, we introduce beforehand a new kind of chronological product, retarded product or simply R product, which is closely related to Wick's T product[11] but more convenient for the present purpose. As a result, one can derive a set of integral equations for the vacuum expectation values of R products of Heisenberg operators as the necessary and sufficient condition that the defining equations are self-consistent. These integral equations are very akin and essentially equivalent to those given by Chew and Low and also by Lehmann, Symanzik and Zimmermann. However, the new equations are written in a manifestly covariant form in the sense that they involve no θ-functions. Another advantage over the CL- and LSZ-equations consists in that one can readily write down the complete set of integral equations by making use of the simple algebraic properties of the R-product. (§ 2).

From the structure of these integral equations one can easily see the manner in which the unitarity condition of the S matrix is derived as one of the natural consequences of the asymptotic conditions. Although the asymptotic conditions are more restrictive than the unitarity condition, one might regard the former as the substitute of the latter in quantum field theory so far as not only the S matrix but also the field operators exist. (§ 3).

Once the existence of the unitary S matrix is ascertained, it is not hard to derive the explicit form of the S matrix. (§ 4).

Also it is not hard to verify, at least in principle, that this formalism can be extended so as to include composite particles, but the problem of unstable particles are left without being discussed. (§ 5).

It is also worth while to notice that these equations possess divergence-free solutions just as it was the case for CL- and LSZ-equations. To examine this statement more closely the solutions of these equations are investigated in the perturbation theory. (§ 6).

Since ambiguities inherent in the renormalizable field theory appearing in the evaluation of certain matrix elements arise from the divergent character of the intermediate calcula-

tions, it is hoped that the divergence-free equations might help to remove this difficulty from the current field theory. Taking the microscopic causality condition as the boundary condition, the integral equations are applied in the perturbation theory to quantum electrodynamics to prove that the photon self energy actually vanishes and that the polarization operator is obtained in a gauge invariant form. The method utilized in the perturbation theory is essentially equivalent to that of Stueckelberg[10] who employed the unitarity as well as causality conditions to determine the S matrix. In the e^2-approximation, it is shown that these equations possess really divergence-free solutions of conventional quantum electrodynamics, i.e. they are satisfied by the renormalized functions in quantum electrodynamics. (§ 7).

§ 2. Asymptotic conditions and their self-consistency

We start with introducing a new kind of chronological product, the retarded or R product. Let φ be the field operator of a neutral scalar field, then their retarded products are defined in the following way:

$$R[\varphi(x):\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)] = \sum_{\text{Perm.}}(-i)^n\theta(x-x_1)\,\theta(x_1-x_2)\cdots\theta(x_{n-1}-x_n)[\cdots[\varphi(x),\varphi(x_1)]\varphi(x_2)]\cdots,\varphi(x_n)], \tag{2·1}$$

where the summation should be taken over all possible permutations of variables $x_1, x_2, \ldots, x_n$. From this definition one readily sees the following properties of R products:

(i) If φ is Hermitian, then R is also hermitian.

(ii) R is symmetric with respect to the variables $x_1, x_2, \ldots, x_n$.

(iii) $R=0$, if $x_i > x^{**}$ for any i.

It is not hard to show that the definition (2·1) is equivalent to the following equation (2·2)

$$R[\varphi(x):\varphi(y)\varphi(x_1)\cdots\varphi(x_n)] + i\theta(x-y)\sum_{\text{Comb.}}{}'[R[\varphi(x):\varphi(x_1)\cdots\varphi(x_k)],\ R[\varphi(y):\varphi(x_{k+1})\cdots\varphi(x_n)]]=0, \tag{2·2}$$

where the summation should be taken over all possible divisions of the variables $x_1, x_2, \cdots, x_n$ into two groups. The derivation of eq. (2·2) is given in Appendix I. Furthermore eq. (2·2) is equivalent to the following set of eqs. (2·3a) and (2·3b):

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* This definition of R product is due to W. Zimmermann.[11] The usefulness of R products over T products was shown by many people especially by Goldberger in connection with dispersion relations.[12] $\theta(x)=1$ if $x_0\geqq 0$, and $=0$ if $x_0<0$.

** $x_i > x$ denotes $(x_i)_0 > x_0$.

$$R[\varphi(x):\varphi(y)\varphi(x_1)\cdots\varphi(x_n)]-R[\varphi(y):\varphi(x)\varphi(x_1)\cdots\varphi(x_n)]$$
$$+i\sum_{\text{Comb.}}[R[\varphi(x):\varphi(x_1)\cdots\varphi(x_k)],\ R[\varphi(y):\varphi(x_{k+1})\cdots\varphi(x_n)]]=0, \qquad (2\cdot3a)$$

and

$$R[\varphi(x):\varphi(y)\varphi(x_1)\cdots\varphi(x_n)]=0, \text{ if } x<y. \qquad (2\cdot3b)$$

With the help of R products, we shall introduce a complete set of state vectors corresponding to outgoing waves.

An orthogonal set of state vectors $\mathfrak{S}^{(+)}=\{\Phi_\alpha^{(+)}\}$ is introduced under the conditions:

(i) The vacuum state Ω belongs to $\mathfrak{S}^{(+)}$.

(ii) The stable one particle states* Φ_a's belong to $\mathfrak{S}^{(+)}$.

Then we assume that the field operator $\varphi(x)$ exists and that we know already the following matrix elements:

$$\langle\Omega, R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]\Omega\rangle \text{ or simply } \langle\Omega, R(x:x_1,\cdots,x_n)\Omega\rangle,$$

and $$\langle\Omega,\varphi(x)\Phi_a\rangle \text{ or } \langle\Phi_a,\varphi(x)\Omega\rangle. \qquad (2\cdot4)$$

Especially for the latter quantity, we assume further the normalization condition**

$$\sum_a{}'\langle\Omega,\varphi(x)\Phi_a\rangle\langle\Phi_a,\varphi(y)\Omega\rangle=iJ^{(-)}(x-y)=1/(2\pi)^3\int(dp)\,e^{ip(x-y)}\theta(p_0)\,\delta(p^2+m^2), \qquad (2\cdot5)$$

which holds for any renormalized theory. m is the observed rest mass of the quantum of the field φ. Then it is clear that $\langle\Omega,\varphi(x)\Phi_a\rangle$ or $\langle\Phi_a,\varphi(x)\Omega\rangle$ satisfies the Klein-Gordon equation

$$\langle\Omega,K_x\varphi(x)\Phi_a\rangle=\langle\Phi_a,K_x\varphi(x)\Omega\rangle=0. \qquad (2\cdot6)$$

This is also the consequence of the assumptions that there exists the energy-momentum operator P_μ and that the states belonging to $\mathfrak{S}^{(+)}$ are the eigenvectors of P_μ.

Further we shall assume that any state of $\mathfrak{S}^{(+)}$ allows us to write it in the form

$$\Phi_\alpha^{(+)}=\Phi^{(+)}_{a_1\times a_2\times\cdots\times a_m} \text{ or simply } \Phi^{(+)}_{a_1a_2\cdots a_m}. \qquad (2\cdot7)$$

By the multiplication symbol it is meant that $\Phi_\alpha^{(+)}$ denotes a stationary state formed by incident particles $a_1, a_2, \ldots, a_m$ and satisfying the outgoing wave condition. Therefore $\Phi_\alpha^{(+)}$ is such an eigenvector of P_μ that satisfies

$$P_\mu\Phi_\alpha^{(+)}=(p_\mu^{a_1}+p_\mu^{a_2}+\cdots+p_\mu^{a_m})\Phi_\alpha^{(+)}. \qquad (2\cdot8)$$

* As to the definition of the one particle state, see § 5.

** Throughout this paper we employ the natural units $\hbar=c=1$. Through eq. (2·5), the use of indefinite metric is excluded.

Let $\Phi_\alpha^{(+)}$ and $\Phi_\beta^{(+)}$ be written in the form

$$\Phi_\alpha^{(+)}=\Phi_{a_1 a_2\cdots a_m},\quad \Phi_\beta^{(+)}=\Phi_{b_1 b_2\cdots b_n},$$

then the orthogonality condition is expressed by

$$\langle \Phi_\beta^{(+)},\ \Phi_\alpha^{(+)}\rangle=\delta_{\beta\alpha}=\delta_{mn}\sum_{\text{Perm.}}\delta(a_1 b_{i_1})\,\delta(a_2 b_{i_2})\cdots\delta(a_m b_{i_m}).$$

Up till now, however, the definition of the states belonging to $\mathfrak{S}^{(+)}$ is not explicitly given, and we shall define them through the following recursion formula which we call eventually the asymptotic condition:

(iii a) $\langle \Phi_{\beta b}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_\alpha^{(+)}\rangle$

$$=\langle \Phi_\beta^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_{\alpha/b}^{(+)}\rangle+\int(dz)\langle \Phi_b,\ \varphi(z)\,\Omega\rangle K_z\cdot\langle \Phi_\beta^{(+)},\ R(x:x_1\cdots x_l z)\,\Phi_\alpha^{(+)}\rangle,\tag{2·9a}$$

where K is the Klein-Gordon operator with the rest mass m, i.e. $K=\Box-m^2$, and the division symbol for the states of $\mathfrak{S}^{(+)}$ is defined by

$$\Phi_{\alpha/b}^{(+)}=\sum_{i=1}^{m}\delta(ba_i)\,\Phi^{(+)}_{a_1\cdots a_{i-1}a_{i+1}\cdots a_m}.$$

Of course, a's and b correspond to stable one particle states.

As the conjugate equation to (2·9a), one gets

(iii b) $\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_{\alpha a}^{(+)}\rangle$

$$=\langle \Phi_{\beta/a}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_\alpha^{(+)}\rangle+\int(dy)K_y\langle \Phi_\beta^{(+)},\ R(x:x_1\cdots x_l y)\,\Phi_\alpha^{(+)}\rangle\cdot\langle \Omega,\ \varphi(y)\,\Phi_a\rangle.\tag{2·9b}$$

(iv) Furthermore, as a physical requirement we assume that the orthogonal set of state vectors $\mathfrak{S}^{(+)}$ forms a complete set.*

Through the above conditions, the set $\mathfrak{S}^{(+)}$ is completely settled. By the repeated use of the recursion formula, one gets at once

$$\langle \Omega,\ R(x:x_1\cdots x_l)\,\Phi_\alpha^{(+)}\rangle$$
$$=\int(dy_1)\cdots(dy_m)K_{y_1}\cdots K_{y_m}\langle \Omega,\ R(x:x_1\cdots x_l y_1\cdots y_m)\,\Omega\rangle$$
$$\times\langle \Omega,\ \varphi(y_1)\,\Phi_{a_1}\rangle\langle \Omega,\ \varphi(y_2)\,\Phi_{a_2}\rangle\cdots\langle \Omega,\ \varphi(y_m)\,\Phi_{a_m}\rangle,\tag{2·10a}$$

and

$$\langle \Phi_\beta^{(+)},\ R(x:x_1\cdots x_l)\,\Omega\rangle$$
$$=\int(dz_1)\cdots(dz_n)K_{z_1}\cdots K_{z_n}\langle \Omega,\ R(x:x_1\cdots x_l z_1\cdots z_n)\,\Omega\rangle$$
$$\times\langle \Phi_{b_1},\ \varphi(z_1)\,\Omega\rangle\langle \Phi_{b_2},\ \varphi(z_2)\,\Omega\rangle\cdots\langle \Phi_{b_n},\ \varphi(z_n)\,\Omega\rangle.\tag{2·10b}$$

* In this section it is assumed that there is no bound state.

As we shall see in eq. (2·15), any matrix element of $\varphi(x)$ can be expressed in terms of vacuum expectation values of R and the one particle wave functions.

Now we are in the position to examine the self-consistency of the recursion formulas (2·9a) and (2·9b), and for this purpose we shall first prove the following theorem:

Theorem I. In order that the recursion formulas (2·9a) and (2·9b) be self-consistent, it is necessary that the following set of integral equations hold:

$$\begin{aligned}&\langle \Omega,\ R(x:yx_1\cdots x_n)\Omega\rangle\\&\quad+i\theta(x-y)\sum_{\text{Comb.}}\sum_{l} i^l/l!\cdot\int(du)\,(dv)[K_{u_1}\cdots K_{u_l}\langle \Omega,\ R(x:x_1\cdots x_k u_1\cdots u_l)\Omega\rangle\\&\qquad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\qquad\times K_{v_1}\cdots K_{v_l}\langle \Omega,\ R(y:x_{k+1}\cdots x_n v_1\cdots v_l)\Omega\rangle-(x\rightleftarrows y)]=0,\end{aligned}\tag{2·11}$$

where

$$(du)\,(dv)=(du_1)\cdots(du_l)\,(dv_1)\cdots(dv_l).$$

(proof) By taking the vacuum expectation value of eq. (2·2), we get

$$\begin{aligned}&\langle \Omega,\ R(x:yx_1\cdots x_n)\Omega\rangle\\&+i\theta(x-y)[\sum_{\text{Comb.}}\sum_{\alpha}\langle \Omega,\ R(x:x_1\cdots x_k)\Phi_\alpha^{(+)}\rangle\langle \Phi_\alpha^{(+)},\ R(y:x_{k+1}\cdots x_n)\Omega\rangle-(x\rightleftarrows y)]=0.\end{aligned}$$

Then inserting the expressions (2·10a) and (2·10b) into the above equation and making use of the normalization condition (2·5), one arrives at eq. (2·11).

As we have already mentioned in connection with the recursion formula for the R product, the eq. (2·11) is equivalent to

$$\begin{aligned}&\langle \Omega,\ R(x:yx_1\cdots x_n)\Omega\rangle-\langle \Omega,\ R(y:xx_1\cdots x_n)\Omega\rangle\\&\quad+i\sum_{\text{Comb.}}\sum_{l} i^l/l!\cdot\int(du)\,(dv)[K_{u_1}\cdots K_{u_l}\langle \Omega,\ R(x:x_1\cdots x_k u_1\cdots u_l)\Omega\rangle\\&\qquad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\qquad\times K_{v_1}\cdots K_{v_l}\langle \Omega,\ R(y:x_{k+1}\cdots x_n v_1\cdots v_l)\Omega\rangle-(x\rightleftarrows y)]=0,\end{aligned}\tag{2·12a}$$

with the boundary condition

$$\langle \Omega,\ R(x:yx_1\cdots x_n)\Omega\rangle=0,\ \text{for } y>x.\tag{2·12b}$$

The equation (2·12) has an advantage over the eq. (2·11) in that it does not involve any θ function and hence it is of a manifestly covariant form.

What makes the above equations feasible as the fundamental basis of the field theory is the theorem on the sufficiency of these equations.

Theorem II. In order that the recursion formulas (2·9a) and (2·9b) be self-consistent, it is sufficient that the eq. (2·11) holds.

(proof) Since the matrix element of $\varphi(x)$ between any two of $\mathfrak{S}^{(+)}$, say $\langle \Phi_\beta^{(+)},\ \varphi(x)\ \Phi_\alpha^{(+)}\rangle$ can be expressed in terms of the vacuum expectation values of R products, it is necessary to prove that the expressions such as $\langle \Phi_\beta^{(+)},\ R(x:y)\,\Phi_\alpha^{(+)}\rangle$ calculated from the

matrix elements of $\varphi(x)$ coincides with that derived by directly applying the recursion formulas (2·9a) and (2·9b). In other words, we must prove the self-consistency of the formulas (2·9a) and (2·9b) for various values of l. For this purpose, we first give the following lemma:

Lemma I.

$$\sum_{\tau}\langle \Phi_{\beta}^{(+)}, A\Phi_{\tau\alpha}^{(+)}\rangle\langle \Phi_{\tau}^{(+)}, B\Phi_{\alpha}^{(+)}\rangle = \sum_{\delta}\langle \Phi_{\beta}^{(+)}, A\Phi_{\delta}^{(+)}\rangle\langle \Phi_{\delta/\alpha}^{(+)}, B\Phi_{\alpha}^{(+)}\rangle. \tag{2·13}$$

Though the above lemma can be proved trivially, it is very useful for the following developments.

For the sake of simplicity, we shall put

$$D_z^b \equiv \int (dz)\langle \Phi_b, \varphi(z)\Omega\rangle \cdot K_z \cdots,$$

and then let us calculate the expression

$$D_z^b\langle \Phi_{\beta}^{(+)}, R(x : yx_1\cdots x_l z)\Phi_{\alpha}^{(+)}\rangle$$

in two different ways. First by applying the recursion formula (2·9) directly one finds

$$I = \langle \Phi_{\beta b}^{(+)}, R(x : yx_1\cdots x_l)\Phi_{\alpha}^{(+)}\rangle - \langle \Phi_{\beta}^{(+)}, R(x : yx_1\cdots x_l)\Phi_{\alpha/b}^{(+)}\rangle.$$

On the other hand, if one transforms the above expression first with the help of eq. (2·2), one obtains

$$\begin{aligned} II = -i\theta(x-y)D_z^b\Big[& \sum_{\text{Comb.}\tau}\langle \Phi_{\beta}^{(+)}, R(x : x_1\cdots x_k)\Phi_{\tau}^{(+)}\rangle\langle \Phi_{\tau}^{(+)}, R(y : x_{k+1}\cdots x_l z)\Phi_{\alpha}^{(+)}\rangle \\ & + \langle \Phi_{\beta}^{(+)}, R(x : x_1\cdots x_k z)\Phi_{\tau}^{(+)}\rangle\langle \Phi_{\tau}^{(+)}, R(y : x_{k+1}\cdots x_l)\Phi_{\alpha}^{(+)}\rangle - (x \rightleftarrows y)\Big]. \end{aligned}$$

Then applying the recursion formula (2·9), II reduces to

$$\begin{aligned} II = -i\theta(x-y)\sum_{\text{Comb.}\tau}\Big[& \langle \Phi_{\beta}^{(+)}, R(x : x_1\cdots x_k)\Phi_{\tau}^{(+)}\rangle\{\langle \Phi_{\tau b}^{(+)}, R(y : x_{k+1}\cdots x_l)\Phi_{\alpha}^{(+)}\rangle \\ & - \langle \Phi_{\tau}^{(+)}, R(y : x_{k+1}\cdots x_l)\Phi_{\alpha/b}^{(+)}\rangle\} \\ & + \{\langle \Phi_{\beta b}^{(+)}, R(x : x_1\cdots x_k)\Phi_{\tau}^{(+)}\rangle - \langle \Phi_{\beta}^{(+)}, R(x : x_1\cdots x_k)\Phi_{\tau/b}^{(+)}\rangle\}\langle \Phi_{\tau}^{(+)}, R(y : x_{k+1}\cdots x_l)\Phi_{\alpha}^{(+)}\rangle \\ & - (x \rightleftarrows y)\Big]. \end{aligned}$$

The first term in the first line and the second term in the third line cancels one another due to Lemma I, and there remains

$$II = -\langle \Phi_{\beta}^{(+)}, R(x : yx_1\cdots x_l)\Phi_{\alpha/b}^{(+)}\rangle + \langle \Phi_{\beta b}^{(+)}, R(x : yx_1\cdots x_l)\Phi_{\alpha}^{(+)}\rangle = I.$$

Here, if the recursion formula (2·9) were consistent with the algebraic properties of R for a certain values of l, the consistency continues to larger l. Let us, therefore, prove the statement for the smallest l, i.e. $l=0$.

To discuss this problem, let us first define the matrix elements of $\varphi(x)$ by

$$\langle \Phi_{\beta}^{(+)}, \varphi(x)\Phi_{\alpha}^{(+)}\rangle = \sum_{\lambda} D^{\beta/\lambda}\widetilde{D}^{\alpha/\lambda}\langle \Omega, R(x : \cdots)\Omega\rangle, \tag{2·14}$$

which follows from eqs. (2·9). D^β is defined by

$$D^\beta = D^{b_1} \cdots D^{b_n},$$

and $\widetilde{D}^\alpha$ is defined in a similar way. The operator $\widetilde{D}^\alpha$ for a stable one particle state a is given similar to D^b by

$$\widetilde{D}_y^a \equiv \int (dy) \langle \Omega, \varphi(y) \Phi_a \rangle \cdot K_y \cdots .$$

In eq. (2·14), the arguments on which D and $\widetilde{D}$ operate are spared.

What we are now to do is to prove the equation

$$D_z^b \langle \Phi_\beta^{(+)}, R(x:z) \Phi_\alpha^{(+)} \rangle = \langle \Phi_{\beta b}^{(+)}, \varphi(x) \Phi_\alpha^{(+)} \rangle - \langle \Phi_\beta^{(+)}, \varphi(x) \Phi_{\alpha/b}^{(+)} \rangle$$

with the help of eqs. (2·11) and (2·14).

From the definition of the R product, follows the equation

$$\langle \Phi_\beta^{(+)}, R(x:z) \Phi_\alpha^{(+)} \rangle = -i\theta(x-z) \sum_\gamma \{ \langle \Phi_\beta^{(+)}, \varphi(x) \Phi_\gamma^{(+)} \rangle \langle \Phi_\gamma^{(+)}, \varphi(z) \Phi_\alpha^{(+)} \rangle - (x \rightleftarrows z) \}.$$

The eq. (2·14) is of help to rewrite the above summation as

$$\sum_{\lambda\mu\gamma} D^{\beta/\lambda} \widetilde{D}^{\gamma/\lambda} \langle \Omega, R(x:\cdots) \Omega \rangle D^{\gamma/\mu} \widetilde{D}^{\alpha/\mu} \langle \Omega, R(z:\cdots) \Omega \rangle - (x \rightleftarrows z).$$

(i) For the sake of simplicity, let us first assume that the two states α and β have no common factor state, and write $\gamma/\lambda\mu$ newly as γ, then the above expression is transformed into

$$\sum_{\lambda\mu\gamma} D^{\beta/\lambda} \widetilde{D}^\gamma \widetilde{D}^\mu \langle \Omega, R(x:\cdots) \Omega \rangle D^\gamma D^\lambda \widetilde{D}^{\alpha/\mu} \langle \Omega, R(z:\cdots) \Omega \rangle - (x \rightleftarrows z).$$

Multiplying the above expression by $-i\theta(x-z)$, one gets with the help of eq. (2·11)

$$D^\beta \widetilde{D}^\alpha \langle \Omega, R(x:z\cdots) \Omega \rangle.$$

The derivation of the above result is rather trivial if one only notices that the summations over γ and over λ, μ correspond to those over l and over all possible combinations of variables standing in eq. (2·11).

Hence we have

$$D_z^b \langle \Phi_\beta^{(+)}, R(x:z) \Phi_\alpha^{(+)} \rangle = D^{\beta b} \widetilde{D}^\alpha \langle \Omega, R(x:\cdots) \Omega \rangle. \tag{2·15}$$

From eq. (2·14), on the other hand, follows at once the relation

$$D^\beta \widetilde{D}^\alpha \langle \Omega, R(x:\cdots) \Omega \rangle = \sum_\lambda (-1)^\lambda \langle \Phi_{\beta/\lambda}^{(+)}, \varphi(x) \Phi_{\alpha/\lambda}^{(+)} \rangle, \tag{2·16}$$

where $(-1)^\lambda$ is equal $+1$ or -1 according to whether λ is the product of an even number of factor one particle states or of an odd number of factor states. From eqs. (2·15) and (2·16) and the assumption that α and β have no common factor, one can derive

$$D_z^b\langle \Phi_\beta^{(+)},\ R(x:z)\,\Phi_\alpha^{(+)}\rangle=\langle \Phi_{\beta b}^{(+)},\ \varphi(x)\,\Phi_\alpha^{(+)}\rangle-\langle \Phi_\beta^{(+)},\ \varphi(x)\,\Phi_{\alpha/b}^{(+)}\rangle.$$

(ii) If α and β have common factor states, so can do λ and μ. In this case, let the common factor of λ and μ be denoted by ν, and $\gamma\nu/\lambda\mu$ be denoted newly by γ, then instead of (2·15) we find

$$\langle \Phi_\beta^{(+)},\ R(x:z)\,\Phi_\alpha^{(+)}\rangle=\sum_\nu D^{\beta/\nu}\widetilde{D}^{\alpha/\nu}\langle \Omega,\ R(x:z\cdots)\,\Omega\rangle. \qquad (2\cdot15')$$

Then with the help of (2·16), we get

$$D_z^b\langle \Phi_\beta^{(+)},\ R(x:z)\,\Phi_\alpha^{(+)}\rangle=\sum_\nu D^{\beta b/\nu}\widetilde{D}^{\alpha/\nu}\langle \Omega,\ R(x:\cdots)\,\Omega\rangle$$

$$=\sum_{\lambda,\nu}(-1)^\lambda\langle \Phi_{\beta b/\nu\lambda}^{(+)},\ \varphi(x)\,\Phi_{\alpha/\nu\lambda}^{(+)}\rangle.$$

Although ν cannot be equal to b, λ can be equal to b, and hence

$$=\langle \Phi_{\beta b}^{(+)},\ \varphi(x)\,\Phi_\alpha^{(+)}\rangle-\langle \Phi_\beta^{(+)},\ \varphi(x)\,\Phi_{\alpha/b}^{(+)}\rangle.$$

To complete our proof we need to show

$$\langle \Phi_b,\ \varphi(x)\,\Omega\rangle=D_z^b\langle \Omega,\ R(x:z)\,\Omega\rangle.$$

In the above proof so far given, the operator D_z^b could also be

$$D_z^b=\int(dz)\,f_b(z)\cdot K_z\cdots,$$

but the function $f_b(z)$ is now fixed through the above equation as shown below.

If one requires that $f_b(z)$ satisfies the Klein-Gordon equation, i.e.

$$K_z f_b(z)=0,$$

then we get*

$$\int(dz)\,f_b(z)\cdot K_z\langle \Omega,\ R(x:z)\,\Omega\rangle$$

$$=\int(dz)\,f_b(z)\,(\delta(x-z)+\cdots)=f_b(x)$$

Hence, from the above requirement, we have

$$f_b(x)=\langle \Phi_b,\ \varphi(x)\,\Omega\rangle.$$

Thus we have completed the proof of Theorem II.

Now we can also conclude that one can introduce a complete set of orthogonal states $\mathfrak{S}^{(+)}$ if the equation (2·11) is satisfied.

Based on the Yang-Feldman formalism,[13] Lehmann[14] has derived a set of integral equations for Haag's functions f's,[4] which describe the unitarity condition of the S matrix, i.e.

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* For details of such a derivation one is required to refer to § 5.

$$f(x:yx_1\cdots x_n)-f(y:xx_1\cdots x_n)$$
$$+i\sum_{\text{Comb.}}\sum_{l=0}^{\infty} i^l/l!\cdot\int(du)\,(dv)\,\{f(x:x_1\cdots x_k u_1\cdots u_l)\,\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)$$
$$\times f(y:x_{k+1}\cdots x_n v_1\cdots v_l)-(x\rightleftarrows y)\}=0. \qquad (2\cdot17)$$

Meanwhile, it was shown by Zimmermann[11] that Haag's functions are related to R products through the relation

$$f(x:x_1\cdots x_n)=K_x K_{x_1}\cdots K_{x_n}\langle\Omega,\ R(x:x_1\cdots x_n)\Omega\rangle. \qquad (2\cdot18)$$

If one combines (2·18) with (2·12), eq. (2·17) can be readily derived. In this paper we have derived eq. (2·11) in a different way than Lehmann did in order to avoid the use of Yang-Feldman formalism.

From Lehmann's result that eq. (2·17) represents the unitarity condition, we might suppose that eq. (2·12a) too might closely be related to this condition. It is really the case, and we shall discuss the manner in which this condition is achieved in the next section.

§ 3. Introduction of incoming waves and of the S matrix

To see how the unitarity condition of the S can be concluded from eq. (2·11), we shall first introduce a new product, i.e. advanced product or A product, very akin to the retarded product, through which we introduce a complete set of incoming wave states.

The advanced product is defined by

$$A[\varphi(x):\varphi(x_1)\,\varphi(x_2)\cdots\varphi(x_n)]$$
$$=\sum_{\text{Perm.}}(-i)^n\tilde{\theta}(x-x_1)\,\tilde{\theta}(x_1-x_2)\cdots\tilde{\theta}(x_{n-1}-x_n)[\cdots[\varphi(x),\varphi(x_1)]\varphi(x_2)]\cdots,\varphi(x_n)], \qquad (3\cdot1)$$

where $\tilde{\theta}(x)=\theta(-x)$.

Similarly to the case of R products, A products possess the following properties:

(i) If φ is hermitian, then A is also hermitian.

(ii) A is symmetric with respect to the variables $x_1, x_2,\ldots, x_n$.

(iii) $A=0$, if $x_i<x$ for any i.

The following equation is equivalent to the definition (3·1):

$$A[\varphi(x):\varphi(y)\,\varphi(x_1)\cdots\varphi(x_n)]$$
$$+i\theta(x-y)\sum_{\text{Comb.}}'[A[\varphi(x):\varphi(x_1)\cdots\varphi(x_k)],\ A[\varphi(y):\varphi(x_{k+1})\cdots\varphi(x_n)]]=0. \qquad (3\cdot2)$$

The above equation is further equivalent to the following set of equations:

$$A[\varphi(x):\varphi(y)\,\varphi(x_1)\cdots\varphi(x_n)]-A[\varphi(y):\varphi(x)\,\varphi(x_1)\cdots\varphi(x_n)]$$
$$+i\sum_{\text{Comb.}}'[A[\varphi(x):\varphi(x_1)\cdots\varphi(x_k)],\ A[\varphi(y):\varphi(x_{k+1})\cdots\varphi(x_n)]]=0, \qquad (3\cdot3a)$$

and

$$A[\varphi(x):\varphi(y)\varphi(x_1)\cdots\varphi(x_n)]=0,\ \text{if } x>y. \tag{3·3b}$$

Next we introduce a complete set of incoming wave states $\mathfrak{S}^{(-)}$ in a similar way to the case of $\mathfrak{S}^{(+)}$.

(i) The vacuum state Ω belongs to $\mathfrak{S}^{(-)}$.

(ii) The stable one particle states Φ_α's belong to $\mathfrak{S}^{(-)}$.

(iii a) Other states of $\mathfrak{S}^{(-)}$ than the vacuum and one particle states are defined by the following recursion formula or the asymptotic condition :

$$\begin{aligned}&\langle \Phi_{\beta b}^{(-)}, A(x:x_1\cdots x_l)\Phi_\alpha^{(-)}\rangle\\ &=\langle \Phi_\beta^{(-)}, A(x:x_1\cdots x_l)\Phi_{\alpha/b}^{(-)}\rangle-\int(dz)\langle \Phi_b, \varphi(z)\Omega\rangle\cdot K_z\langle \Phi_\beta^{(-)}, A(x:x_1\cdots x_l z)\Phi_\alpha^{(-)}\rangle.\end{aligned} \tag{3·4a}$$

(iii b) As the conjugate equation to (3·4a), one gets

$$\begin{aligned}&\langle \Phi_\beta^{(-)}, A(x:x_1\cdots x_l)\Phi_{\alpha a}^{(-)}\rangle\\ &=\langle \Phi_{\beta/a}^{(-)}, A(x:x_1\cdots x_l)\Phi_\alpha^{(-)}\rangle-\int(dy)\cdot K_y\langle \Phi_\beta^{(-)}, A(x:x_1\cdots x_l y)\Phi_\alpha^{(-)}\rangle\cdot\langle\Omega, \varphi(y)\Phi_a\rangle\end{aligned} \tag{3·4a}$$

(iv) The orthogonal set of state vectors $\mathfrak{S}^{(-)}$ forms a complete set.

Then in a completely analogous manner to the case of R products, we can prove the following theorem :

Theorem III. The necessary and sufficient condition that a complete set of orthogonal states $\mathfrak{S}^{(-)}$ can be introduced through eqs. (3·4a) and (3·4b) is that the following equation holds for any n :

$$\begin{aligned}&\langle\Omega, A(x:yx_1\cdots x_n)\Omega\rangle\\ &+i\tilde{\theta}(x-y)\sum_{\text{Comb.}}\sum_l i^l/l!\cdot\int(du)(dv)[K_{u_1}\cdots K_{u_l}\langle\Omega, A(x:x_1\cdots x_k u_1\cdots u_l)\Omega\rangle\\ &\quad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\ &\quad\times K_{v_1}\cdots K_{v_l}\langle\Omega, A(y:x_{k+1}\cdots x_n v_1\cdots v_l)\Omega\rangle-(x\rightleftarrows y)]=0.\end{aligned} \tag{3·5}$$

The above equation is equivalent to

$$\begin{aligned}&\langle\Omega, A(x:yx_1\cdots x_n)\Omega\rangle-\langle\Omega, A(y:xx_1\cdots x_n)\Omega\rangle\\ &+i\sum_{\text{Comb.}}\sum_l i^l/l!\int(du)(dv)[K_{u_1}\cdots K_{u_l}\langle\Omega, A(x:x_1\cdots x_k u_1\cdots u_l)\Omega\rangle\\ &\quad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\ &\quad\times K_{v_1}\cdots K_{v_l}\langle\Omega, A(y:x_{k+1}\cdots x_n v_1\cdots v_l)\Omega\rangle-(x\rightleftarrows y)]=0,\end{aligned} \tag{3·6a}$$

and the boundary condition

$$\langle\Omega, A(x:yx_1\cdots x_n)\Omega\rangle=0,\ \text{for } x>y. \tag{3·6b}$$

It is worth while to notice that the eq. (3·6a) is completely of the same form with

(2·12a), and this fact is the key to the whole discussions in this section. In the Hamiltonian formalism, states belonging to either $\mathfrak{S}^{(+)}$ or $\mathfrak{S}^{(-)}$ satisfy the same Schrödinger equation, and we must, also in the present formalism, relate the two sets to one another in a definite way. Otherwise it could be the case that the states belonging to $\mathfrak{S}^{(+)}$ and $\mathfrak{S}^{(-)}$ satisfy different Schrödinger equations. For this purpose we impose an invariance requirement upon the present formalism that the whole theory be invariant against the simultaneous reflections of space and time. Then we have the following theorem:

Theorem IV. The asymptotic conditions for the outgoing waves and those for the incoming waves are equivalent to one another, and if they are satisfied there exists a unitary S matrix.

(proof) Before going into the proof of the above theorem, we must first describe what the invariance against the space-time reflection means. The operation of space-time reflection denoted by asterisk is defined by the following formulas:

$$\begin{aligned}
&\text{(i)} \quad \langle \Psi^*, Q\Phi^* \rangle = \langle \Phi, Q^* \Psi \rangle, \\
&\text{(ii)} \quad (AB)^* = B^* A^*, \text{ (a consequence of (i))} \qquad (3\cdot 7) \\
&\text{(iii)} \quad \Omega^* = \Omega, \\
&\text{(iv)} \quad \varphi(x)^* = \varphi(-x), \text{ (in the scalar meson theory in question).}
\end{aligned}$$

Through the above operation, $\langle \Omega, R(\cdots)\Omega \rangle$ and $\langle \Omega, A(\cdots)\Omega \rangle$ are connected in the following manner:

$$\begin{aligned}
\langle \Omega, R(x : x_1 \cdots x_n)\Omega \rangle &= \sum_{\text{Perm.}} (-i)^n \theta(x-x_1)\,\theta(x_1-x_2)\cdots\theta(x_{n-1}-x_n) \\
&\quad \times \langle \Omega, [\cdots[\varphi(x), \varphi(x_1)]\varphi(x_2)]\cdots\varphi(x_n)]\Omega \rangle \\
&= \sum_{\text{Perm.}} (-i)^n \tilde{\theta}(x_1-x)\,\tilde{\theta}(x_2-x_1)\cdots\tilde{\theta}(x_n-x_{n-1}) \\
&\quad \times \langle \Omega^*, [\cdots[\varphi(x), \varphi(x_1)]\varphi(x_2)]\cdots\varphi(x_n)]\Omega^* \rangle \\
&= (-1)^n \sum_{\text{Perm.}} (-i)^n \tilde{\theta}((-x)-(-x_1))\,\tilde{\theta}((-x_1)-(-x_2))\cdots\tilde{\theta}((-x_{n-1})-(-x_n)) \\
&\quad \times \langle \Omega, [\cdots[\varphi(-x), \varphi(-x_1)]\varphi(-x_2)]\cdots\varphi(-x_n)]\Omega \rangle \\
&= (-1)^n \langle \Omega, A(-x : -x_1, \cdots, -x_n)\Omega \rangle.
\end{aligned}$$

This formula shows that the functions $\langle \Omega, R(\cdots)\Omega \rangle$ and $\langle \Omega, A(\cdots)\Omega \rangle$ are not independent of one another, but they are related to one another by

$$\langle \Omega, R(x : x_1 \cdots x_n)\Omega \rangle = (-1)^n \langle \Omega, A(-x : -x_1, \cdots, -x_n)\Omega \rangle, \qquad (3\cdot 8a)$$

$$\langle \Omega, A(x : x_1 \cdots x_n)\Omega \rangle = (-1)^n \langle \Omega, R(-x : -x_1, \cdots, -x_n)\Omega \rangle. \qquad (3\cdot 8b)$$

If the functions $\langle \Omega, R(\cdots)\Omega \rangle$ satisfy the equations (2·12a) and (2·12b), one can readily see from (3·8a) or (3·8b) that the functions $\langle \Omega, A(\cdots)\Omega \rangle$ also satisfy eqs. (3·6a) and (3·6b), and vice versa. Thus if the equations (2·12a) and (2·12b) have solutions at all, then one can introduce two complete sets of orthogonal states, $\mathfrak{S}^{(+)}$ and $\mathfrak{S}^{(-)}$. Then it is clear that the S matrix defined by

$$S_{\beta\alpha} = \langle \Phi_\beta^{(-)}, \Phi_\alpha^{(+)} \rangle \qquad (3\cdot 9)$$

is unitary. Thus we have completed the proof of Theorem IV. In the above proof it was clarified how the unitarity condition of the S matrix results from eqs. (2·12a) and (2·12b).

It must further be noticed that the assumption on the invariance of the theory against the space-time reflection is not necessarily required to prove Theorem IV, although it serves to simplify the proof. When the invariance does no more hold, one must divide the R functions into even and odd parts according to the transformation properties against reflection, i.e.

$$\langle \Omega,\ R(x:x_1\cdots x_n)\,\Omega\rangle = R_e(x:x_1\cdots x_n) + R_o(x:x_1\cdots x_n) \tag{3·10}$$

and require that if the above expression is inserted into the left-hand side of eq. (2·12a), the even part as well as the odd part of eq. (2·12a) should vanish separately. Then we get two coupled equations for R_e and R_o

$$\begin{aligned}
&R_e(x:yx_1\cdots x_n)\\
&+i\theta(x-y)\sum_{\text{Comb.}}\sum_{l} i_l/l!\cdot\int (du)\ (dv)\,[K_{u_1}\cdots K_{u_l}R_e(x:x_1\cdots x_k u_1\cdots u_l)\\
&\qquad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\
&\qquad\times K_{v_1}\cdots K_{v_l}R_e(y:x_{k+1}\cdots x_n v_1\cdots v_l)-(x\rightleftarrows y)\\
&\qquad+K_{u_1}\cdots K_{u_l}R_o(x:x_1\cdots x_k u_1\cdots u_l)\\
&\qquad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\
&\qquad\times K_{v_1}\cdots K_{v_l}R_o(y:x_{k+1}\cdots x_n v_1\cdots v_l)-(x\rightleftarrows y)\,]=0,
\end{aligned}\tag{3·11a}$$

$$\begin{aligned}
&R_o(x:yx_1\cdots x_n)\\
&+i\theta(x-y)\sum_{\text{Comb.}}\sum_{l} i^l/l!\int (du)\ (dv)\,[K_{u_1}\cdots K_{u_l}R_e(x:x_1\cdots x_k u_1\cdots u_l)\\
&\qquad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\
&\qquad\times K_{v_1}\cdots K_{v_l}R_o(y:x_{k+1}\cdots x_n v_1\cdots v_l)-(x\rightleftarrows y)\\
&\qquad+K_{u_1}\cdots K_{u_l}R_o(x:x_1\cdots x_k u_1\cdots u_l)\\
&\qquad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\
&\qquad\times K_{v_1}\cdots K_{v_l}R_e(y:x_{k+1}\cdots x_n v_1\cdots v_l)-(x\rightleftarrows y)\,]=0,
\end{aligned}\tag{3·11b}$$

$$R_e(x:yx_1\cdots x_n)=R_o(x:yx_1\cdots x_n)=0,\ \text{if } x<y. \tag{3·11c}$$

The relations (3·8a) and (3·8b) are then replaced by

$$\begin{aligned}
R_e(x:x_1\cdots x_n)&=(-1)^n A_e(-x:-x_1,\ \cdots,\ -x_n),\\
R_o(x:x_1\cdots x)&=(-1)^{n+1}A_o(-x:-x_1,\ \cdots,\ -x_n),
\end{aligned}\tag{3·12}$$

and similar expressions hold for A_e and A_0.

It can be easily verified that Theorem IV holds also in this case, although the

invariance against the space-time reflection is no more guaranteed.

To conclude this section, it might perhaps be instructive to remark that equations (3·7) are satisfied automatically if one starts from eqs. (3·8a) and (3·8b) noticing the relations

$$\Phi_\alpha^{(+)*}=\Phi_\alpha^{(-)},\quad \Phi_\alpha^{(-)*}=\Phi_\alpha^{(+)}. \tag{3·13}$$

For vector and spinor fields, the above relation is, of course, more complicated. To avoid inessential complications, however, we shall omit the discussions thereupon.

§ 4. Derivation of the S matrix

In the previous section, it was shown that there exists a unitary S matrix if eqs. (2·11) are satisfied, but the explicit form of the S matrix in terms of R products was not given. In this section we shall try to find the explicit form based upon an additional assumption.

Lemma II. If an operator $O(x_0)$ is given in terms of $\varphi(x)$ and $\dot{\varphi}(x)$ by*

$$O(x_0)=c_0+\int c_1(\vec{x})\,\varphi(x)\,d\vec{x}+\int d_1(\vec{x})\,\dot{\varphi}(x)\,d\vec{x}+\int_{x_0^{\prime}=x_0} c_2(x^{\prime},y^{\prime})\,\varphi(x)\,\varphi(y)\,dx^{\prime}dy^{\prime}+\ldots, \tag{4·1}$$

then the recursion formulas (2·9a) and (2·9b) hold also for $O(x_0)$ substituted in (2·9) for $\varphi(x)$.

(proof) For $O(x_0)=\varphi(x)$, or $\dot{\varphi}(x)$, the above lemma is trivial. If eqs. (2·9) hold for $O_1(x_0)$ and $O_2(x_0)$, it is also the case for $O_1(x_0)+O_2(x_0)$. Let us, therefore, prove that it is also the case for the product $O_1(x_0)\,O_2(x_0)$.

From the definition of the R product, follows the relation

$$\begin{aligned}&R[O_1O_2:x_1x_2\cdots x_l]\\&=\sum_{\text{Comb.}}R[O_1:x_1\cdots x_k]\cdot R[O_2:x_{k+1}\cdots x_l].\end{aligned} \tag{4·2}$$

From the assumption that eqs. (2·9) hold for O_1 and O_2, we have

$$\begin{aligned}&\int(dz)\langle\Phi_b,\varphi(z)\,\Omega\rangle\cdot K_z\langle\Phi_\beta^{(+)},R[O_i:x_1\cdots x_l z]\Phi_\alpha^{(+)}\rangle\\&=\langle\Phi_{\beta b}^{(+)},R[O_i:x_1\cdots x_l]\Phi_\alpha^{(+)}\rangle-\langle\Phi_\beta^{(+)},R[O_i:x_1\cdots x_l]\Phi_{\alpha/b}^{(+)}\rangle.\quad(i=1,2)\end{aligned} \tag{4·3}$$

From eq. (4·2) follows the identity:

$$\begin{aligned}&\langle\Phi_\beta^{(+)},R[O_1O_2:x_1\cdots x_l z]\Phi_\alpha^{(+)}\rangle\\&=\sum_{\substack{\text{Comb.}\\ \gamma}}{}^{\prime}\langle\Phi_\beta^{(+)},R[O_1:x_1\cdots x_k z]|\Phi_\gamma^{(-)}\rangle\langle\Phi_\gamma^{(+)},R[O_2:x_{k+1}\cdots x_l]|\Phi_\alpha^{(-)}\rangle\\&+\langle\Phi_\beta^{(-)},R[O_1:x_1\cdots x_k]\Phi_\gamma^{(+)}\rangle\langle\Phi_\gamma^{(+)},R[O_2:x_{k+1}\cdots x_l z]\Phi_\alpha^{(+)}\rangle.\end{aligned}$$

* $\dot{\varphi}(x)$ is the time derivative of $\varphi(x)$.

Operating D_z^b upon the above identity, and then making use of Lemma I as well as eq. (4·2), one finds that eqs. (2·9) hold also for the product $O_1(x_0)O_2(x_0)$. Thus the recursion formulas (2·9) hold for any element of the algebra generated from $\varphi(x)$ and $\dot{\varphi}(x)$. Thus the proof is completed.

Now let us consider O_R and O_A defined by

$$\begin{aligned} O_R(x_0) &= U(x_0, -\infty)^* O(x_0) U(x_0, -\infty), \\ O_A(x_0) &= U(\infty, x_0) O(x_0) U(\infty, x_0)^*, \end{aligned} \tag{4·4}$$

then they constitute the generating functionals of $R[O(x_0) : \varphi(x_1)\cdots]$ and $A[O(x_0) : \varphi(x_1)\cdots]$ as shown in Appendix I. These two functionals are related to one another through

$$O_A(x_0) = UO_R(x_0)U^*, \tag{4·5}$$

where $U = U(\infty, -\infty)$. It is more convenient to write (4·5) in the form

$$O_A(x_0)U = UO_R(x_0). \tag{4·6}$$

By differentiating the above equation with respect to the external source Q, one gets the following relation connecting R and A products:

$$\begin{aligned} &\sum_{\text{Comb.}} (-1)^m (-i)^{n-m} \langle \Omega, A[O(x_0) : x_1 \cdots x_m] \cdot T[x_{m+1} \cdots x_n] \Omega \rangle \\ &= \sum_{\text{Comb.}} (-i)^l \langle \Omega, T[x_1 \cdots x_l] \cdot R[O(x_0) : x_{l+1} \cdots x_n] \Omega \rangle. \end{aligned} \tag{4·7}$$

Now let us apply the operation $D^3 = \prod_{i=1}^{n} D^{b_i}$ upon the above equation, then from the frequency condition, or the positive energy condition, survives only the term $n=m$ on the left-hand side of the above equation to give $\langle \Phi_\beta^{(-)}, O(x_0)\Omega \rangle$. The right-hand side reduces, if one applies Lemma II, to

$$\begin{aligned} &\sum_{\beta_1 \times \beta_2 = \beta} \sum_{\Upsilon} (-i)^{\beta_1} D^{\beta_1} \langle \Omega, T[\cdots] \Phi_\Upsilon^{(+)} \rangle D^{\beta_2} \langle \Phi_\Upsilon^{(+)}, R[O(x_0) : \cdots] \Omega \rangle \\ &= \sum_{\beta_1 \times \beta_2 = \beta} \sum_{\Upsilon} (-i)^{\beta_1} D^{\beta_1} \langle \Omega, T[\cdots] \Phi_\Upsilon^{(+)} \rangle \langle \Phi_{\Upsilon\beta_2}^{(+)} O(x_0) \Omega \rangle, \end{aligned}$$

where the arguments upon which D's are to operate are spared just as in eq. (2·14).

With the aid of Lemma I, we get

$$\begin{aligned} \langle \Phi_\beta^{(-)}, O(x_0)\Omega \rangle &= \sum_{\substack{\beta_1 \times \beta_2 = \beta \\ \Upsilon}} (-i)^{\beta_1} D^{\beta_1} \langle \Omega, T[\cdots] \Phi_{\Upsilon/\beta_2}^{(+)} \rangle \langle \Phi_\Upsilon^{(+)}, O(x_0)\Omega \rangle \\ &= \sum_{\Upsilon} T_{\beta\Upsilon} \langle \Phi_\Upsilon^{(+)}, O(x_0)\Omega \rangle, \end{aligned} \tag{4·8}$$

where

$$T_{\beta\Upsilon} = \sum_{\beta_1 \times \beta_2 = \beta} (-i)^{\beta_1} D^{\beta_1} \langle \Omega, T[\varphi(x_1), \cdots] \Phi_{\Upsilon/\beta_2}^{(+)} \rangle. \tag{4·9}$$

On the other hand, we know

$$\begin{aligned} \langle \Phi_\beta^{(-)}, O(x_0)\Omega \rangle &= \sum_{\Upsilon} \langle \Phi_\beta^{(-)}, \Phi_\Upsilon^{(+)} \rangle \langle \Phi_\Upsilon^{(+)} O(x_0)\Omega \rangle \\ &= \sum_{\Upsilon} S_{\beta\Upsilon} \langle \Phi_\Upsilon^{(+)}, O(x_0)\Omega \rangle. \end{aligned} \tag{4·10}$$

Combining (4·8) with (4·10), one finds

$$\sum_{\Upsilon}(S_{\beta\Upsilon}-T_{\beta\Upsilon})\langle\Phi_{\Upsilon}^{(+)},\, O(x_0)\,\Omega\rangle=0\,,$$

or

$$(S-T)\,O(x_0)\,\Omega=0\,. \tag{4·11}$$

Now let us assume[7] that *state vectors of the form* $O(x_0)\,\Omega$, *where* $O(x_0)$ *is given by eq.* (4·1), *completely span the whole Hilbert space.* Then from (4·11) follows the equation

$$S=T\,.$$

Thus we have proved:

Theorem V. The S matrix is given by

$$S_{\beta\alpha}=\sum_{\beta_1\times\beta_2=\beta}(-i)^{\beta_1}D^{\beta_1}\langle\Omega,\, T[\varphi(x_1),\,\cdots]\Phi_{\alpha/\beta_2}^{(+)}\rangle. \tag{4·12}$$

If we start, instead of (4·6), with*

$$O_A(x_0)\,(U/\langle\Omega,\, U\Omega\rangle)=(U/\langle\Omega,\, U\Omega\rangle)\,O_R(x_0)\,, \tag{4·13}$$

we get:

Theorem VI. The S matrix is given by

$$S_{\beta\alpha}=\sum_{\beta_1\times\beta_2=\beta}(-i)^{\beta_1}D^{\beta_1}\langle\Omega,\, N[\varphi(x_1),\,\cdots]\Phi_{\alpha/\beta_2}^{(+)}\rangle. \tag{4·14}$$

In order to express the T products in terms of R products, one must utilize the recursion formula

$$(-i)^n T[\varphi(x)\,\varphi(x_1)\cdots\varphi(x_n)]=\sum_{\text{Comb.}}(-i)^m T[\varphi(x_1)\cdots\varphi(x_m)]\cdot R[\varphi(x)\,:\varphi(x_{m+1})\cdots\varphi(x_n)]\,, \tag{4·15}$$

which follows from the functional equation

$$T[U\varphi(x)]=U\cdot\varphi_R(x)\,.$$

To conclude this section, we shall give another representation of the S matrix without giving its proof.

Theorem VII. The S matrix is given by

$$S_{\beta\alpha}=\sum_{\lambda}(-i)^{\alpha}(-i)^{\beta}D^{\alpha/\lambda}D^{\beta/\lambda}\langle\Omega,\, T[\varphi(x_1)\cdots]\Omega\rangle. \tag{4·16}$$

§ 5. Inclusion of bound states

In the foregoing sections we have assumed that there are no bound states, but this assumption is, of course, to strong to cover all possible field theories. The reason why we have avoided the use of Yang-Feldman formalism consists in this point that we would like to keep the possibility to include the bound states in our formalism. In this section

* Notice that $U/\langle\Omega,\, U\Omega\rangle$ is the generating functional of the N product.

we shall show how bound states can be introduced in the present scheme. For this purpose it is necessary and sufficient to extend the asymptotic conditions (2·9) so as to include bound states in a self-consistent manner. As the author has shown in previous works,[7] one needs a certain quantum number to normalize the Feynman amplitudes, and it is also the case in the present formulation. It must be noticed, however, that the Hamilton function is not explicitly given in the present framework theory, and consequently we can make use of only such quantum numbers whose forms do not explicitly depend upon the details of the interaction, as electric charge, nucleon number, isobaric spin and so on. None of them, however, can be employed in the neutral scalar theory. In this sense the extention of the present framework theory to bound states problem is considerably restricted on a formal point of view.

Nevertheless, on a practical point of view the above quantum numbers might be sufficient to cover all known bound states.

In this section we shall first try to extend the neutral scalar theory to include states, but in the last stage we shall shift to charged scalar theory so that we can make use of the total charge of a given system as a suitable quantum number to normalize "wave function".

To begin with we shall first introduce various products which are closely related to R or A products. As given in Appendix I, we define the functional U by

$$U=T\exp\left(-i\int_{-\infty}^{\infty}(dx)\,Q(x)\,\varphi(x)\right).$$

The T product is given by

$$T[\varphi(x_1)\,\varphi(x_2)\cdots\varphi(x_n)]=i^n\left[\frac{\delta^n U}{\delta Q(x_1)\,\delta Q(x_2)\cdots\delta Q(x_n)}\right]_{Q=0}, \tag{5·1}$$

whereas the N product is defined by[16]

$$N[\varphi(x_1)\,\varphi(x_2)\cdots\varphi(x_n)]=i^n\left[\frac{\partial^n}{\delta Q(x_1)\,\delta Q(x_2)\cdots\delta Q(x_n)}\,\frac{U}{\langle\Omega,\,U\Omega\rangle}\right]_{Q=0}. \tag{5·2}$$

We call $\langle\Omega,\,N[\varphi(x_1)\,\varphi(x_2)\cdots\varphi(x_n)]\Phi\rangle$ the Feynman amplitude of the state Φ, then a state Φ has the following Fock representation through Feynman amplitudes:

$$\Phi=\begin{Bmatrix}\langle\Omega,\,\Phi\rangle\\ \langle\Omega,\,\varphi(x_1)\,\Phi\rangle\\ \langle\Omega,\,N[\varphi(x_1)\,\varphi(x_2)]\Phi\rangle\\ \vdots\end{Bmatrix}. \tag{5·3}$$

When the first non-vanishing component is given by

$$\langle\Omega,\,N[\varphi(x_1)\cdots\varphi(x_n)]\Phi\rangle,$$

we call Φ an n-body state for the sake of convenience. All stable elementary particle states are one-body states according to this nomenclature, which we have already used in

the foregoing sections as one particle states.

Without loss of generality we shall discuss two-body bound states in this section. Let s be a two-body bound state, and we introduce the asymptotic condition for this state in rhe following form:

$$\langle \Phi_{\beta s}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_{\alpha}^{(+)}\rangle$$
$$=\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_{\alpha/s}^{(+)}\rangle+\int (dz_1)\,(dz_2)\,f_s^*(z_1z_2)\cdot K_z\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_lz_1z_2)\,\Phi_{\alpha}^{(+)}\rangle, \tag{5·4a}$$

where K_z is the Klein-Gordon operator for the centre of mass coordinates z between z_1 and z_2. As the conjugate equation to (5·4a) we have

$$\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_{\alpha s}^{(+)}\rangle$$
$$=\langle \Phi_{\beta/s}^{(+)},\ R(x:x_1\cdots x_l)\,\Phi_{\alpha}^{(+)}\rangle+\int (dy_1)\,(dy_2)\cdot K_y\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_ly_1y_2)\,\Phi_{\alpha/s}^{(+)}\rangle\cdot f_s(y_1y_2). \tag{5·4b}$$

The function f_s is determined later so as to make the above asymptotic condition self-consistent. For the time being, we give only the following condition which directly results from the translation invariance of the theory:

$$\frac{1}{i}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)_{\mu} f_s(y_1y_2)=p_{\mu}^{\,s}\cdot f_s(y_1y_2),$$
$$\frac{1}{i}\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)_{\mu} f_s^*(z_1z_2)=-p_{\mu}^{\,s}\cdot f_s^*(z_1z_2), \tag{5·5}$$

where $p_{\mu}^{\,s}$ is the energy-momentum four vector belonging to the state s.

Next we shall examine if the proofs for the self-consistency given in the previous sections also hold in the case of bound states.

We define D_z^s by

$$D_z^s=\int (dz_1)\,(dz_2)\,f_s^*(z_1z_2)\cdot K_z\cdots\cdots,$$

and calculate the following quantity in two different ways:

$$D_z^s\langle \Phi_{\beta}^{(+)},\ R(x:yx_1\cdots x_lz_1z_2)\,\Phi_{\alpha}^{(+)}\rangle.$$

From (5·4a) follows at once the following expression:

$$I=\langle \Phi_{\beta s}^{(+)},\ R(x:yx_1\cdots x_l)\,\Phi_{\alpha}^{(+)}\rangle-\langle \Phi_{\beta}^{(+)},\ R(x:yx_1\cdots x_l)\,\Phi_{\alpha/s}^{(+)}\rangle.$$

On the other hand, if one first utilizes the equation for the R products, one gets

$$\begin{aligned}II=&-i\theta(x-y)\,D_z^s\Big[\sum_{\text{Comb.}\,\Upsilon}\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_k)\,\Phi_{\Upsilon}^{(+)}\rangle\langle \Phi_{\Upsilon}^{(+)},\ R(y:x_{k+1}\cdots x_lz_1z_2)\,\Phi_{\alpha}^{(+)}\rangle\\&+\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_kz_1z_2)\,\Phi_{\Upsilon}^{(+)}\rangle\langle \Phi_{\Upsilon}^{(+)},\ R(y:x_{k+1}\cdots x_l)\,\Phi_{\alpha}^{(+)}\rangle-(x\rightleftarrows y)\Big]\\&-i\theta(x-y)\,D_z^s\Big[\sum_{\text{Comb.}\,\Upsilon}\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_kz_1)\,\Phi_{\Upsilon}^{(+)}\rangle\langle \Phi_{\Upsilon}^{(+)},\ R(y:x_{k+1}\cdots x_lz_2)\,\Phi_{\alpha}^{(+)}\rangle\\&+\langle \Phi_{\beta}^{(+)},\ R(x:x_1\cdots x_kz_2)\,\Phi_{\Upsilon}^{(+)}\rangle\langle \Phi_{\Upsilon}^{(+)},\ R(y:x_{k+1}\cdots x_lz_1)\,\Phi_{\alpha}^{(+)}\rangle-(x\rightleftarrows y)\Big].\end{aligned} \tag{5·6}$$

In order to prove the equality $I=II$, one has to prove that the second bracket in the above equation vanishes. This can be done as follows.

First of all we must point out that according to (5·5) and the presence of K_z in the defining equation of D_z^s, only such terms can survive after the integration that involve poles of the form $(P_\mu^2+m_s^2)^{-1}$, where m_s is the rest mass of the bound state s. Hence, what we must do now is to prove the absence of such a pole in the second bracket. We investigate the pole of the second bracket in the momentum space with respect to $P=p_1+p_2$, p_1 and p_2 being the canonical conjugates of z_1 and z_2, respectively. Next one must notice that the pole of the integral

$$\int (dz_1)\exp(-ip_1z_1)\langle \Phi_\beta^{(+)},\ R(x:x_1\cdots x_k z_1)\,\Phi_\gamma^{(+)}\rangle$$

with respect to p_1 results only from the region $x, x_1, \cdots, x_k > z_1$ as seen from

$$\int_{-\infty}^{a}\exp(ip_0x_0)\,dx_0=(-i)/(p_0-i\varepsilon),$$

$$\int_{b}^{a}\exp(ip_0x_0)\,dx_0=(\exp(ip_0a)-\exp(ip_0b))/(ip_0),$$

the latter possesses no pole at $p_0=0$. Therefore, as a whole, the Fourier transform of the second bracket in (5·6) has only such poles as being of the form*

$$1/((p_1)_0-E_1-i\varepsilon)\,((p_2)_0-E_2-i\varepsilon)=1/((P/2+p)_0-E_1-i\varepsilon)\,((P/2-p)_0-E_2-i\varepsilon),$$

where $P=p_1+p_2$, $p=(p_1-p_2)/2$. In order to see the singularity of the expression in the complex P_0 plane, we transform back the above expression to the configuration space with respect to p_0,

$$\int dp_0\frac{\exp(ip_0z_0)}{((P/2+p)_0-E_1-i\varepsilon)\,((P/2-p)_0-E_2-i\varepsilon)}.$$

Since the above integral has no pole in the P_0 plane as readily seen by carrying out the integration, the equality $I=II$ is ascertained.** Although a denominator of the form $(P_0-E_1-E_2-i\varepsilon)$ occurs, it is not a pole since summations over all possible E_1 and E_2 are taken for a fixed value of the total momentum. On the other hand, if both z_1 and z_2 are involved in the same R product, a pole in the P_0 plane results from the Fourier integral of the R product over the region $x, x_1, \cdots > z_1, z_2$, provided that there exist bound states. The above proof corresponds to the former half of Theorem II in this case, and to complete the proof of Theorem II in this case one has to make use of the integral equation for the vacuum expectation values of the R products.

Putting

$$\sum_s f_s(y_1y_2)\,f_s^*(z_1z_2)=i\Delta^{(+)}(y_1y_2:z_1z_2), \qquad (5\cdot7)$$

---

* The space integrations are omitted here since they give only δ functions.

** One can prove in a similar way the formula $\langle \Omega,\ \varphi(x)\,\Omega\rangle=0$.

one can readily derive a set of integral equations analogously to (2·11), thereby employing the contraction function (5·7) for bound states. One needs now to prove only the following formula:

$$D_z^s\langle \Phi_\beta^{(+)},\ R(x:z_1z_2)\,\Phi_\alpha^{(+)}\rangle$$
$$=\langle \Phi_{\beta s}^{(+)},\ \varphi(x)\,\Phi_\alpha^{(+)}\rangle-\langle \Phi_\beta^{(+)},\ \varphi(x)\,\Phi_{\alpha|s}^{(+)}\rangle, \tag{5·8}$$

provided that the matrix element $\langle \Phi_\beta^{(+)},\ \varphi(x)\,\Phi_\alpha^{(-)}\rangle$ is defined by an equation completely analogous to (2·14). If one expresses $R(x:z_1z_2)$ as the product of three R products, then one can prove (5·8) similarly to the latter part of the proof of Theorem II. The trick utilized above is useful also in this case. Finally we shall determine the function $f_s(y_1y_2)$ or $f_s^*(z_1z_2)$. They are determined through

$$\int (dz_1)\,(dz_2)\,f_s^*(z_1z_2)\cdot K_z\langle \Omega,\ R[O(x_0):\varphi(z_1)\,\varphi(z_2)]\,\Omega\rangle=\langle \Phi_s,\ O(x_0)\,\Omega\rangle, \tag{5·9a}$$

or

$$\int (dy_1)\,(dy_2)\,K_y\langle \Omega,\ R[O(x_0):\varphi(y_1)\,\varphi(y_2)]\,\Omega\rangle\cdot f_s(y_1y_2)=\langle \Omega,\ O(x_0)\,\Phi_s\rangle. \tag{5·9b}$$

In the case of one particle state we could choose $O(x_0)=\varphi(x)$, and the function $\langle \Omega,\ \varphi(x)\,\Phi\rangle$ was fixed by (2·5). In the present case, however, the situation is much more complicated. To determine the function f_s from (5·9b), we put

$$(y_1+y_2)/2=y,\ y_1-y_2=\eta,\quad f_s(y_1y_2)=\exp(iP_sy)\,g_s(\eta), \tag{5·10}$$

then we get

$$\int (d\eta)\,g_s(\eta)\int (dy)\,K_y\langle \Omega,\ R[O(x_0):\varphi(y_1)\,\varphi(y_2)]\,\Omega\rangle\cdot\exp(iP_sy)=\langle \Omega,\ O(x_0)\,\Phi_s\rangle. \tag{5·11}$$

As has already been noticed, we must investigate the pole of the Fourier transform of $\langle \Omega,\ R[O(x_0):\varphi(y_1)\,\varphi(y_2)]\,\Omega\rangle$ in the P_0 plane.

If we put

$$\int (dy)\langle \Omega,\ R[O(x_0):\varphi(y_1)\,\varphi(y_2)]\,\Omega\rangle\exp(iPy)=R(x_0:P,\eta), \tag{5·12}$$

the left-hand side of (5·11) can be written by

$$-\int (dP)\,\delta(P-P_s)\,(P^2+M^2)\,R(x_0:P,\eta) \tag{5·13}$$

apart from the integration with respect to η. Since $P_s^2+M^2=0$, only the corresponding pole of $R(x_0:P,\eta)$ contributes to the above integral,[16] and we shall ask for the pole of $R(x_0:P,\eta)$. In an analogous way to the foregoing discussions, one can readily see that only the domain $x_0\cdot y_1, y_2$ gives rise to such a pole in the integral (5·12). In this domain, one has

$$R[O(x_0):\varphi(y_1)\varphi(y_2)]$$
$$=-O(x_0)\cdot T[\varphi(y_1)\varphi(y_2)]-\widetilde{T}[\varphi(y_2)\varphi(y_1)]\cdot O(x_0)$$
$$+\varphi(y_1)O(x_0)\varphi(y_2)+\varphi(y_2)O(x_0)\varphi(y_1).$$

The last two terms of the right-hand side give only continuous poles, i.e. branching points, so that we consider only the first two terms of the above. Furthermore, because of the frequency condition we need to retain only the first term. Thus one obtains

$$R(x_0:P,\eta)=-\int_{-\infty}^{x_0-|\eta|/2}(dy)\sum_\alpha\langle\Omega, O(x_0)\Phi_\alpha^{(+)}\rangle\langle\Phi_\alpha^{(+)}, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle e^{iPy}$$
$$+\cdots(\text{regular terms at } P_0=\sqrt{\vec{P}^2+M^2}),$$
$$=-i\int(d^3y)\sum_\alpha\langle\Omega, O(x_0)\Phi_\alpha^{(+)}\rangle\frac{1}{P_0-P_0^\alpha+i\varepsilon}\langle\Phi_\alpha^{(+)}, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle$$
$$\times\exp(iPy)\,(i(P_0^\alpha-P_0)\,(x_0-y_0'))$$
$$+\cdots\cdots,$$

where $y'=\max(y_1, y_2)$. Inserting the above expression into (5·13), one finds

$$-\int(dP)\,\delta(P-P_s)\,(P^2+M^2)R(x_0:P,\eta)$$
$$=i\int(d^3y)\int(dP)\,\delta(P-P_s)\sum_\alpha\langle\Omega, O(x_0)\Phi_\alpha^{(+)}\rangle\frac{P^2+M^2}{P_0-P_0^\alpha+i\varepsilon}\langle\Phi_\alpha^{(+)}, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle$$
$$\times\exp(iPy)\exp(i(P_0^\alpha-P_0)\,(x_0-y_0'))$$
$$=i\int(d^3y)\int(dP)\,\delta(P-P_s)\sum_\alpha\langle\Omega, O(x_0)\Phi_\alpha^{(-)}\rangle\frac{(\sqrt{M^2+\vec{P}_s^2}+P_0)\,(\sqrt{M^2+\vec{P}_s^2}-P_0)}{\sqrt{M^2+\vec{P}_s^2}-P_0^\alpha}$$
$$\times\langle\Phi_\alpha^{(+)}, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle\exp(iPy)\exp\,(i(P_0^\alpha-P_0)\,(x_0-y_0')).$$

Now it is clear that the only contribution to the above integral arises from the term $P=P^\alpha$, in which case we have

$$=-2i\sum_\alpha P_0^s\delta^3(P_\alpha-P_s)\,(2\pi)^3\langle\Omega, O(x_0)\Phi_\alpha^{(+)}\rangle\langle\Phi_\alpha^{(+)}, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle e^{iP_sy}.$$

In the relativistic normalization, eq. (5·11) takes the form

$$-2(2\pi)^3i\int(d\eta)\,g_s(\eta)\,e^{iP_sy}\langle\Omega, O(x_0)\Phi_s\rangle\langle\Phi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle$$
$$=\langle\Omega, O(x_0)\Phi_s^{(-)}\rangle,$$

or

$$\int(d\eta)\,f_s(y_1y_2)\langle\Phi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle=i/2(2\pi)^3. \qquad (5\cdot14)$$

Thus the relation between the function f_s appearing in the asymptotic condition and the Feynman amplitude is clarified.

The equation (5·14) is the necessary and sufficient condition that the asymptotic condition (5·4) be self-consistent. Since, however, we do not know how to calculate the reciprocal Feynman amplitude $\langle \Psi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle$, this condition does not much help us. In order to complete the treatments of bound states, we must further derive another equation to determine the Feynman amplitude.

If the time reversal operation is given by (3·7), then the reciprocal Feynman amplitude $\langle \Psi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle$ is related to the ordinary Feynman amplitude through

$$\langle \Psi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle = \langle \Omega, T[\varphi(-y_1)\varphi(-y)]\Psi_s\rangle. \tag{5·15}$$

In order to derive another equation to determine the Feynman amplitude for the two-body bound states, let us shift to the charged scalar theory.

First it is trivial to write down the integral equations for the vacuum expectation values of the R products, so we shall not discuss them here.

When a function $f(x)$ is given in the Fourier representation by

$$f(x) = \int (dp)\, e^{ipx} F(p),$$

we write for the sake of simplicity as

$$\frac{d}{dp} f(x) = \int (dp)\, e^{ipx} \frac{dF(p)}{dp}.$$

With this notation, one can express the Ward's identity[17] in quantum electrodynamics in the form

$$\begin{aligned} i\frac{d}{dP_\mu} S_F'(x_1-x_2) &= i\int S_F'(x_1-x_3)\, \Gamma_\mu(x_3x_4:x_0)\, S_F'(x_4-x_2)\,(dx_3)\,(dx_4)\,(dx_0) \\ &= \int \langle \Omega, T[\psi(x_1)\bar{\psi}(x_2)\, j_\mu(x_0)]\Omega\rangle (dx_0), \end{aligned} \tag{5·16}$$

where p is the conjugate momentum to x_1-x_2, j_μ the current four vector divided by the unit electric charge e, and all functions appearing in the above equation are renormalized ones. One can readily extend the Ward's identity to two-body system as seen from

$$\begin{aligned} 2i\frac{d}{dP_\mu}\langle \Omega, T[\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)]\Omega\rangle \\ = \int (dx_0)\langle \Omega, T[\psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\, j_\mu(x_0)]\Omega\rangle, \end{aligned} \tag{5·17}$$

where P is the momentum conjugate to $(x_1+x_2)/2-(x_3+x_4)/2$, where in addition x_1-x_2, and x_3-x_4 are taken to be independent of each other.

In the charged scalar theory, one has

$$\begin{aligned} 2i\frac{d}{dp_\mu}\langle \Omega, T[\varphi(x_1)\varphi(x_2)\varphi^*(x_3)\varphi^*(x_4)]\Omega\rangle \\ = \int (dx_0)\langle \Omega, T[\varphi(x_1)\varphi(x_2)\varphi^*(x_3)\varphi^*(x_4)\, j_\mu(x_0)]\Omega\rangle, \end{aligned} \tag{5·18a}$$

or one can equally well derive

$$2i\frac{d}{dP_\mu}\langle \Omega, T[\varphi^*(x_1)\varphi^*(x_2)\varphi(x_3)\varphi(x_4)]\Omega\rangle$$
$$=-\int (dx_0)\langle \Omega, T[\varphi^*(x_1')\varphi^*(x_2')\varphi(x_3)\varphi(x_4)j_\mu(x_0)]\Omega\rangle. \quad (5\cdot 18\text{b})$$

Next we introduce the following formula:

$$\int (dX')\Delta^{(+)}(X-X')K_{X'}\langle \Omega, T[\varphi^*(x_1)\varphi^*(x_2)\varphi(y_1)\varphi(y_2)]\Omega\rangle$$
$$=\sum_{\substack{\text{bound}\\ \text{states}}}\langle \Omega, T[\varphi^*(x_1)\varphi^*(x_2)]\Phi_s\rangle\langle \Phi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle, \quad (5\cdot 19)$$

where $\Delta^{(+)}$ and K_X refer to the bound state mass M, and $X=(x_1+x_2)/2$, $X'=(x_1'+x_2')$, and $x_1-x_2=x_1'-x'$. The proof of eq. (5·19) is completely analogous to that of (5·14). The summation in (5·19) must be taken over (negatively) double charged states.

Let us put

$$G(x_1x_2y_1y_2)=\langle \Omega, T[\varphi^*(x_1)\varphi^*(x_2)\varphi(y_1)\varphi(y_2)]\Omega\rangle,$$

and define F_μ by

$$-2i\frac{d}{dP_\mu}G(x_1x_2y_1y_2)=\int G(x_1x_2x_1'x_2')F_\mu(x_1'x_2'y_1'y_2')G(y_1'y_2'y_1y_2)(dx_1')(dx_2')(dy_1')(dy_2'). \quad (5\cdot 20)$$

By applying the operation of (5·19) on (5·20) and using (5·18b), we get

$$\int\langle \Phi_s, j_\mu(x_0)\Phi_{s'}\rangle(dx_0)$$
$$=\int (dx_1)(dx_2)(dy_1)(dy_2)\langle \Phi_s, [T[\varphi(x_1)\varphi(x_2)]\Omega\rangle F_\mu(x_1x_2y_1y_2)\langle \Omega, T[\varphi^*(y_1)\varphi^*(y_2)]\Phi_{s'}\rangle. \quad (5\cdot 21)$$

On the other hand, we have from (5·14)

$$\int (dx_1)(dx)f_s(x_1x_2)\langle \Phi_s, T[\varphi(x_1)\varphi(x_2)]\Omega\rangle=i\pi\delta_{ss'}\delta(P_s-P_{s'}),$$

and also we have from $q\Phi_s=-2\Phi_s$, $q=\int j_\mu d\sigma_\mu=\int j_0 d^3x$

$$\int\langle \Phi_s, j_\mu(x_0)\Phi_{s'}\rangle(dx_0)=-4\pi P_\mu^s\delta(P_s-P_{s'})\delta_{ss'},$$

where $\delta_{ss'}$ refers to discrete quantum numbers, and the relativistic normalization is employed. (See (5·14)).

On comparing these equations, one gets

$$4iP_\mu^s f_s(x_1x_2)=\int (dy_1)(dy_2)F_\mu(x_1x_2y_1y_2)\langle \Omega, T[\varphi(y_1)\varphi(y_2)]\Phi_s\rangle. \quad (5\cdot 22)$$

Thus we have a complete set of equations governing five unknown quantities, i.e. $\langle \Omega, R(x:x_1\cdots)\Omega\rangle$, G, f_s, F_μ, and $\langle \Phi_s, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle$. These equations are 1) integral equations for $\langle \Omega, R(x:x_1\cdots)\Omega\rangle$, 2) the recursion formula to reduce the T products into R products, eq. (4·15), 3) eq. (5·20), 4) eq. (5·19), and 5) finally eq. (5·22).

Practically it seems too much complicated to solve such a set of integral equations but in principle all necessary equations to settle a field theory with bound states are already given in this section. Extension of the results in this section so as to include three- or more body bound states is formally trivial. In Appendix II, we shall see that eq. (5·21) is useful also for practical purposes to normalize the Feynman amplitudes for bound states.

§ 6. Solutions of the integral equations in the perturbation theory

In the foregoing sections we have derived a set of integral equations as a necessary and sufficient condition that the asymptotic conditions be satisfied, and in this section we shall discuss to what extent these equation can limit possible field theories. For this purpose, the solutions of these integral equations shall be investigated in the purturbation theory. To carry out this program, we impose, in addition to the asymptotic conditions, the Lorentz invariance as well as the microscopic causality condition upon the formalism. The latter two conditions are expressed by stating that the Lorentz invariant function $\langle \Omega, R(x:x_1\cdots x_n)\Omega\rangle$ vanishes unless $x>x_i$ and $(x-x_i)^2<0$ for every i.

In order to apply the perturbation theory upon eq. (2·12a),* we assume

$$\langle \Omega, R(x:yx_1\cdots x_n)\Omega\rangle \sim O(\varepsilon^n), \tag{6·1}$$

where ε is a small expansion parameter. One can generally expand it as

$$\langle \Omega, R(x:yx_1\cdots x_n)\Omega\rangle = \sum_{m=0}^{\infty} R_{n+2m}(x:yx_1\cdots x_n), \tag{6·2}$$

where

$$R_{n+2m}(x:yx_1\cdots x_n)\sim O(\varepsilon^{n+2m}).$$

In the lowest order in ε, only $\langle \Omega, R(x:y)\Omega\rangle = R_0(x:y)$, and $R_0(x:y)=R(x-y)$ satisfies

$$R(x-y)-R(y-x)=\int (du)\,(dv)\,K_u R(x-u)\cdot \Delta(u-v)\cdot K_v R(y-v). \tag{6·3}$$

By means of Fourier transformation, one finds as the unique non-vanishing retarded solution

$$R(x-y)=-\Delta_{\mathrm{ret}}(x-y), \tag{6·4}$$

where $\Delta_{\mathrm{ret}}(x)$ is defined by

$$\Delta_{\mathrm{ret}}(x)=-\theta(x)\Delta(x),$$

* For the moment we consider only the neutral scalar theory without bound states.

and satisfies

$$K_x \Delta_{\text{ret}}(x) = -\delta(x).$$

Now by inserting the expression (6·2) into (2·12a), one readily finds that in each order of the perturbation theory the integral equation is of the form

$$\begin{aligned} &g(x : y x_1 \cdots x_n) - g(y : x x_1 \cdots x_n) \\ &\quad + \int (du)\,[\Delta(y-u) \cdot K_u g(x : u x_1 \cdots x_n) - \Delta(x-u) \cdot K_u g(y : u x_1 \cdots x_n)] \\ &\quad = \text{inhomogeneous terms}, \end{aligned} \tag{6·5a}$$

with the retarded boundary condition

$$g(x : y x_1 \cdots x_n) = 0, \quad \text{for } x < y. \tag{6·5b}$$

In the above equation, the unknown quantity $g(x : y x_1 \cdots x_n)$ is equal to $R_{n+2m}(x : y x_1 \cdots x_n)$ in question, and the inhomogeneous terms are assumed to be known, i.e. they can be expressed in terms of R functions of lower orders than $n+2m$.

In order to give the general solutions of (6·5), we give beforehand two lemmas.

Lemma III.

$$g(x : y x_1 \cdots x_n) = \int (du)\,(dv)\,\Delta_{\text{ret}}(x-u)\,\Delta_{\text{ret}}(v-y) \cdot K_u K_v g(u : v x_1 \cdots x_n). \tag{6·6}$$

To prove this lemma, let us put

$$K_u K_v g(u : v x_1 \cdots x_n) = h(u : v x_1 \cdots x_n), \tag{6·7}$$

then the general solution of the above equation is given by

$$\begin{aligned} g(x : y x_1 \cdots x_n) = &\int (du)\,(dv)\,\Delta_{\text{ret}}(x-u)\,\Delta_{\text{ret}}(v-y)\,h(u : v x_1 \cdots x_n) \\ &+ G_1(x : y x_1 \cdots x_n) + G_2(x : y x_1 \cdots x_n), \end{aligned}$$

where

$$K_x G_1(x : y x_1 \cdots x_n) = K_y G_2(x : y x_1 \cdots x_n) = 0.$$

Since, however, G's must vanish if $x < y$ for any fixed value of y, we have as the only possible solution under the retarded boundary condition

$$g(x : y x_1 \cdots x_n) = \int (du)\,(dv)\,\Delta_{\text{ret}}(x-u)\,\Delta_{\text{ret}}(v-y)\,h(u : v x_1 \cdots x_n). \tag{6·8}$$

Combining (6·7) with (6·8), the proof of Lemma III is finished.

Lemma IV. The inhomogeneous terms in the right-hand side of (6·5a) can be wtitten in the form

$$\int (du)\,(dv)\,\Delta_{\text{ret}}(x-u)\,\Delta_{\text{ret}}(y-v)\,Q(u;\, v : x_1 \cdots x_n). \tag{6·9}$$

where Q is antisymmetric with respect to u and v.

This lemma is proved trivially if one combines lemma III with eq. (2·12a). With the aid of these lemmas, let us transform the integral equation

$$g(x:yx_1\cdots x_n)-g(y:xx_1\cdots x_n)$$
$$+\int(du)[\Delta(y-u)\cdot K_n g(x:ux_1\cdots x_n)-\Delta(x-u)\cdot K_n g(y:ux_1\cdots x_n)]$$
$$=\int(du)\,(dv)\,\Delta_{\rm ret}(x-u)\,\Delta_{\rm ret}(y-v)\,Q(u,v:x_1\cdots x_n) \qquad (6\cdot10)$$

into an equation for h. Inserting the expression (6·8) into the above equation, one obtains

$$\text{left side of (6·10)}=\int(du)\,(dv)[\Delta_{\rm ret}(x-u)\,\Delta_{\rm ret}(v-y)-\Delta_{\rm ret}(y-u)\,\Delta_{\rm ret}(v-x)]$$
$$\times h(u:vx_1\cdots x_n)$$
$$-\int(du)\,(dv)[\Delta(y-v)\,\Delta_{\rm ret}(x-u)\,h(u:vx_1\cdots x_n)-(x\rightleftarrows y)]$$
$$=\int(du)\,(dv)\,\Delta_{\rm ret}(x-u)[\Delta_{\rm ret}(v-y)+\Delta(v-y)]h(u:vx_1\cdots x_n)-(x\rightleftarrows y).$$
$$=\int(du)\,(dv)\,\Delta_{\rm ret}(x-u)\,\Delta_{\rm ret}(y-v)\,h(u:vx_1\cdots x_n)-(x\rightleftarrows y)$$
$$=\int(du)\,(dv)\,\Delta_{\rm ret}(x-u)\,\Delta_{\rm ret}(y-v)\,Q(u,v:x_1\cdots x_n)$$
$$=\text{right side of (6·10)}.$$

Hence the integral equation (6·10) is reduced to

$$h(u,v)-h(v,u)=Q(u,v), \qquad (6\cdot11)$$

where, for the sake of simplicity, the variables $x_1,\cdots,x_n$ are spared.

According to the boundary condition (6·5), h must also satisfy

$$h(u,v)=0, \quad \text{if } u<v. \qquad (6\cdot12)$$

Hence the general solution of (6·11) is given by

$$h(u,v)=\theta^*(u-v)\,Q(u,v)+C_0\delta(u-v)+C_1\times\text{derivatives of }\delta(u-v), \qquad (6\cdot13)^*$$

where $\delta(u-v)$ and its derivatives are the solutions of (6·11) without the inhomogeneous part, and we have already utilized the Lorentz invariance at this stage. The function θ^* is essentially equivalent to the θ function, but different from θ when Q involves derivatives of some singular functions. Namely the operation θ^* is defined by

$$\theta^*(u-v)\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\nu}\cdots q(u,v)=\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\nu}\cdots\theta(u-v)\,q(u,v), \qquad (6\cdot14)$$

if $q(u,v)$ is sufficiently non-singular so that the value of $\theta(u-v)\;q(u-v)$ does not depend on the choice of a special frame of reference. For instance, the solution of the equation

* The constants of integration C's do no more involve v, but they depend upon $v, x_1, \cdots, x_n$.

$$h(u, v) - h(v, u) = \frac{\partial^2}{\partial u_\mu \partial u_\nu} \Delta(u-v) \tag{6·15}$$

is given by

$$\begin{aligned} h(u, v) &= \theta^*(u-v) \frac{\partial^2}{\partial u_\mu \partial u_\nu} \Delta(u-v) + \cdots \\ &= \frac{\partial^2}{\partial u_\mu \partial u_\nu} \theta(u-v) \Delta(u-v) + \cdots, \end{aligned} \tag{6·16}$$

but not by

$$\theta(u-v) \frac{\partial^2}{\partial u_\mu \partial u_\nu} \Delta(u-v) + \cdots. \tag{6·17}$$

The latter depends upon the choice of a special frame of reference so that it cannot be defined without fixing a special Lorentz system. In this case, the modification of the θ function into θ^* operation is necessary just as in the case of T^* product[18] which is the invariant modification of T product. Here, it must be noted that rhe former choice is required from the Lorentz invariance condition although both (6·16) and (6·17) satisfy the equation (6·15).

With reference to (6·8), eq. (6·13) can be integrated to give $g(x : y x_1 \cdots x_n)$ as

$$\begin{aligned} g(x : y x_1 \cdots x_n) = &\int (du)\,(dv)\, \Delta_{\text{ret}}(x-u)\, \Delta_{\text{ret}}(v-y)\, \theta^*(u-v)\, Q(u, v : x_1 \cdots x_n) \\ &+ \lambda \int (du)\, \Delta_{\text{ret}}(x-u)\, \Delta_{\text{ret}}(u-y)\, \Delta_{\text{ret}}(u-x_1) \cdots \Delta_{\text{ret}}(u-x_n) \\ &+ \text{derivatives of the above term.} \end{aligned} \tag{6·18}$$

In this solution, the dependence of the function $g(x : y x_1 \cdots x_n)$ upon variables $x_1, \cdots, x_n$ are determined based on the symmetry character of the function g. The second term in the right-hand side is the simplest solution of (6·10) without the inhomogeneous term Q. This can readily be verified as follows:

Put

$$g(x : y x_1 \cdots x_n) = \lambda \int (du)\, \Delta_{\text{ret}}(x-u)\, \Delta_{\text{ret}}(u-y)\, \Delta_{\text{ret}}(u-x_1) \cdots \Delta_{\text{ret}}(u-x_n), \tag{6·19}$$

and insert this expression into

$$\int (du) [\Delta(y-u) \cdot K_u g(x : u x_1 \cdots x_n) - \Delta(x-u) \cdot K_u g(y : u x_1 \cdots x_n)],$$

then one finds

$$\begin{aligned} &= -\lambda \int (du) [\Delta(y-u)\, \Delta_{\text{ret}}(x-u)\, \Delta_{\text{ret}}(u-x_1) \cdots \Delta_{\text{ret}}(u-x_n) \\ &\qquad - \Delta(x-u)\, \Delta_{\text{ret}}(y-u)\, \Delta_{\text{ret}}(u-x_1) \cdots \Delta_{\text{ret}}(u-x_n)] \\ &= -\lambda \int (du) [\Delta(y-u)\, \Delta_{\text{ret}}(x-u) - \Delta(x-u)\, \Delta_{\text{ret}}(y-u)] \\ &\qquad \times \Delta_{\text{ret}}(u-x_1) \cdots \Delta_{\text{ret}}(u-x_n), \end{aligned} \tag{6·20}$$

The quantity in the bracket can be transformed in the following form:

$$\begin{aligned}&\Delta(y-u)\,\Delta_{\mathrm{ret}}(x-u)-\Delta(x-u)\,\Delta_{\mathrm{ret}}(y-u)\\&\quad=\Delta_{\mathrm{ret}}(x-u)\,\Delta_{\mathrm{ret}}(u-y)-\Delta_{\mathrm{ret}}(y-u)\,\Delta_{\mathrm{ret}}(u-x).\end{aligned}$$

Inserting this result into (6·20), one finds at once

$$(6\cdot20)=g(y:xx_1\cdots x_n)-g(x:yx_1\cdots x_n).$$

Thus it is proved that (6·19) is really the solution of (6·10) without inhomogeneous part. The solution (6·19) corresponds to the introduction of an interaction of the form φ^{n+2}. The last term of (6·18) corresponds to the introduction of derivative couplings, the general form of which is given by

$$D_{xyx_1\cdots x_n}\int(du)\,\Delta_{\mathrm{ret}}(x-u)\,\Delta_{\mathrm{ret}}(u-y)\,\Delta_{\mathrm{ret}}(u-x_1)\cdots\Delta_{\mathrm{ret}}(u-x_n),\qquad(6\cdot21)$$

where D is generally a Lorentz invariant differential operator with respect to x, y, x_1, $\cdots$, x_n and symmetric with respect to y, x_1, $\cdots$, x_n. A closer examination of the perturbation theory shows us that the differential operation should be at highest linear in each variable, otherwise the iteration procedure brings about non-locality into the formalism in contradiction with the requirement of microscopic causality condition. This condition is already known in the T^* product formulation of quantized field theories, and we shall call this condition the derivative condition which is, of course, closely related to the assumption made in the derivation of the S matrix in § 4.

Under such circumstances, it is already clear that the only possible solutions of the asymptotic conditions or the integral equations are those given by renormalizable field theories if we confine ourselves to the perturbation theory. For unrenormalizable but local theories, the equations have no convergent solution in the perturbation theory. It is of course an unsettled question if the integral equations have other solutions than those given by the perturbation theory. If it were the case, as shown by Castillejo, Dalitz and Dyson[10] for a simplified Low equation, such solutions are equally well accessible as the perturbation theoretic solution, since they satisfy automatically all physically sensible conditions as proved in the foregoing sections.

§ 7. Application to quantum electrodynamics

The integral equations representing the asymptotic conditions have very rich contents in many respects, especially they have solutions which are free from ultra-violet divergences. The most important point is the fact that in the derivation of these equations we have used only such conditions of quantized field theories that seem for the time being to be reliable as the basis of field theories.

The fact that these integral equations have divergence-free solutions is rather clear in comparison with the equations of Chew and Low and of Lehmann, Symanzik and Zimmermann. The benefit of the present equations over older ones consists in their manifestly covariant form and also in the simplicity to write down their formal solutions

in the perturbation theory as was done in the previous section. Although one can expect to get divergence-free solutions without recourse to the perturbation theory since one needs not use the perturbation theory to renormalize the equations, we shall confine ourselves only to the perturbation theory in this paper. Since ambiguities inherent in the conventional renormalizable field theories arise from the divergent character of the integrals occuring in the intermediate stage of the calculation, one might hope that such a divergence-free formulation would be of much help to get rid of ambiguities. Hoping that it would really be the case, the self-energy of the photon as well as the polarization operator for the electromagnetic field were calculated. The results are satisfactory in the sense that the photon self-energy vanishes and the gauge invariance of the theory is really retained.

Before entering into quantum electrodynamics, we must slightly extend the formalism adopted to scalar theory in previous sections.

Definition of R Product for Fermions

So far the R product has been defined only for Boson operators, but in order to discuss quantum electrodynamics it is necessary to extend the definition for Fermion operators too. For this purpose it is instructive to remember the procedure to derive Wick's T product[9] from Dyson's P product.[20]

Let $\eta(x)$ be an external field operator which anti-commutes with all Fermion field operators and also with each other, then the T product for Fermion operators can be defined from P product in the following manner:

$$\begin{aligned} &P[\eta(x_1)\psi(x_1),\ \eta(x_2)\psi(x_2),\ \cdots,\ \eta(x_n)\psi(x_n)] \\ &\quad=\eta(x_n)\cdots\eta(x_2)\eta(x_1)\,T[\psi(x_1)\psi(x_2)\cdots\psi(x_n)], \end{aligned} \tag{7·1}$$

where some of ψ's can be replaced by $\bar{\psi}$'s.*

We can apply this procedure to define the R product for Fermion operators.

(a) If $\varphi(x)$ is of even order in Fermion operators, one can define R by

$$\begin{aligned} &R[\varphi(x):\eta(x_1)\psi(x_1),\ \cdots,\ \eta(x_l)\psi(x_l),\ \eta(y_1)\bar{\psi}(y_1),\ \cdots,\ \eta(y_m)\bar{\psi}(y_m),\ A(z_1)\cdots A(z_n)] \\ &=\eta(y_m)\cdots\eta(y_1)\eta(x_l)\cdots\eta(x_1)\,R[\varphi(x):\psi(x_1)\cdots\psi(x_l)\bar{\psi}(y_1)\cdots\bar{\psi}(y_m)A(z_1)\cdots A(z_n)], \end{aligned} \tag{7·2a}$$

where A is a Boson field operator.

(b) If $\varphi(x)$ is of odd order in Fermion operators, one has

$$\begin{aligned} &R[\eta(x)\varphi(x):\eta(x_1)\psi(x_1),\ \cdots,\ \eta(x_l)\psi(x_l),\ \eta(y_1)\bar{\psi}(y_1),\ \cdots,\ \eta(y_m)\bar{\psi}(y_m), \\ &\qquad A(z_1),\ \cdots,\ A(z_n)] \\ &=\eta(y_m)\cdots\eta(y_1)\eta(x_l)\cdots\eta(x_1)\eta(x)\,R[\varphi(x):\psi(x_1)\cdots\psi(x_l)\bar{\psi}(y_1)\cdots\bar{\psi}(y_m) \\ &\qquad A(z_1)\cdots A(z_n)]. \end{aligned} \tag{7·2b}$$

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* $\bar{\psi}=\psi^*\beta=\psi^*\gamma_4$.

According to the above definition, R products possess the following properties:

(i) R is anti-symmetric against the interchange between two Fermion operators, and symmetric against that between two Boson operators.

(ii) $R=0$, if $x_i>x$ or $y_i>x$ or $z_i>x$ for any i.

(iii) Under hermitian conjugation, R is transfoxmed as

$$R[\varphi(x):\psi(x_1)\cdots]^*=(-1)^{(l+m+1)(l+m)/2}R[\varphi(x)^*:\psi(x_1)^*,\cdots], \tag{7·3a}$$

if $\varphi(x)$ is even, and

$$=(-1)^{(l+m+2)(l+m+1)/2}R[\varphi(x)^*:\psi(x_1)^*,\cdots], \tag{7·3b}$$

if $\varphi(x)$ is odd.

Introduction of State Vectors

In order to introduce state vectors as done in § 2, one must take account of Fermi statistics. Let a's be one Fermion states, then one has

$$\Phi^{(\pm)}_{a_1\times\cdots\times a_i\times\cdots\times a_j\times\cdots\times a_m}=-\Phi^{(\pm)}_{a_1\times\cdots\times a_j\times\cdots\times a_i\times\cdots\times a_m}. \tag{7·4}$$

Namely we must fix the order of multiplication of factor one particle states. Similarly the division of states is defined by

$$\Phi^{(\pm)}_{a/b}=\sum_{i=1}^{m}(-1)^{i-m}\delta(ba_i)\,\Phi_{a_1\times\cdots\times a_{i-1}\times a_{i+1}\times\cdots\times a_m}. \tag{7·5}$$

Asymptotic Conditions

(1) For a one-photon state a or b, we have

$$\langle\Phi^{(+)}_{\beta b},R[\cdots:\cdots]\Phi^{(+)}_{\alpha}\rangle$$
$$=\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots]\Phi^{(+)}_{\alpha/b}\rangle+\int(dz)\langle\Phi_b,A_\mu(z)\,\Omega\rangle\cdot\Box_z\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots A_\mu(z)]\Phi^{(+)}_{\alpha}\rangle, \tag{7·6a}$$

and its conjugate equation

$$\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots]\Phi^{(+)}_{\alpha a}\rangle$$
$$=\langle\Phi^{(+)}_{\beta/b},R[\cdots:\cdots]\Phi^{(+)}_{\alpha}\rangle+\int(dy)\,\Box_y\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots A_\mu(y)]\Phi^{(+)}_{\alpha}\rangle\langle\Omega,A_\mu(y)\,\Phi_a\rangle. \tag{7·6b}$$

(2) For a one-electron state a or b, we have

$$\langle\Phi^{(+)}_{\beta b},R[\cdots:\cdots]\Phi^{(+)}_{\alpha}\rangle$$
$$=\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots]\Phi^{(+)}_{\alpha/b}\rangle+\int(dz)\langle\Phi_b,\bar{\psi}(z)\,\Omega\rangle\cdot D_z\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots\psi(z)]\Phi^{(+)}_{\alpha}\rangle, \tag{7·6c}$$

$$\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots]\Phi^{(+)}_{\alpha a}\rangle$$
$$=\langle\Phi^{(+)}_{\beta/a},R[\cdots:\cdots]\Phi^{(+)}_{\alpha}\rangle+\int(dy)\,\widetilde{D}_y\langle\Phi^{(+)}_{\beta},R[\cdots:\cdots\bar{\psi}(y)]\Phi^{(+)}_{\alpha}\rangle\cdot\langle\Omega,\psi(y)\,\Phi_a\rangle, \tag{7·6d}$$

where D and $\widetilde{D}$ are defined by

$$D=\gamma\cdot\partial+m,\quad \widetilde{D}=\gamma^{T}\cdot\partial-m.$$

(3) For a one-positron state a or b, we have

$$\begin{aligned}&\langle \Phi^{(+)}_{\beta b},\, R[\cdots:\cdots]\Phi^{(+)}_{\alpha}\rangle\\ &=\langle \Phi^{(+)}_{\beta},\, R[\cdots:\cdots]\Phi^{(+)}_{\alpha/b}\rangle+\int (dz)\langle \Phi_b,\,\psi(z)\,\Omega\rangle\cdot\widetilde{D}_z\langle \Phi^{(+)}_{\beta},\, R[\cdots:\cdots\bar{\psi}(z)]\Phi^{(+)}_{\alpha}\rangle,\end{aligned}\tag{7·6e}$$

$$\begin{aligned}&\langle \Phi^{(+)}_{\beta},\, R[\cdots:\cdots]\Phi^{(+)}_{\alpha a}\rangle\\ &=\langle \Phi^{(+)}_{\beta/a},\, R[\cdots:\cdots]\Phi^{(+)}_{\alpha}\rangle+\int (dy)\,\widetilde{D}_y\langle \Phi^{(+)}_{\beta},\, R[\cdots:\cdots\bar{\psi}(y)]\Phi^{(+)}_{\alpha}\rangle\langle \Omega,\,\psi(y)\,\Phi_a\rangle.\end{aligned}\tag{7·6f}$$

The contraction of one-particle wave functions is given by*

$$\sum_a\langle \Omega,\,\psi_\alpha(x)\,\Phi_a\rangle\langle \Phi_a,\,\bar{\psi}_\beta(y)\,\Omega\rangle=-iS^{(+)}_{\alpha\beta}(x-y)=-i(\gamma\cdot\partial_x-m)_{\alpha\beta}\Delta^{(+)}(x-y)$$

for electron, (7·7a)

$$\sum_b\langle \Omega,\,\bar{\psi}_\alpha(x)\,\Phi_b\rangle\langle \Phi_b,\,\psi_\beta(y)\,\Omega\rangle=-i\widetilde{S}^{(+)}_{\alpha\beta}(x-y)=-i(\gamma^{T}\cdot\partial_x+m)_{\alpha\beta}\Delta^{(+)}(x-y)$$

for positron, (7·7b)

$$\sum\langle \Omega,\,A_\mu(x)\,\Phi_c\rangle\langle \Phi_c,\,A_\nu(y)\,\Omega\rangle=i\delta_{\mu\nu}D^{(+)}(x-y)\qquad\text{for photon.}\tag{7·7c}$$

After these preliminaries, let us apply the formalism to calculate the photon propagation function up to the order e^2. The result is of course already well known, but it would be still instructive to show an unambiguous method to evaluate this function.

The retarded propagation function satisfies the equation

$$\begin{aligned}&\langle \Omega,\, R[A_\mu(x):A_\nu(y)]\,\Omega\rangle-\langle \Omega,\, R[A_\nu(y):A_\mu(x)]\,\Omega\rangle\\ &\quad+i\sum_\alpha[\langle \Omega,\,A_\mu(x)\,\Phi^{(+)}_\alpha\rangle\langle \Phi^{(+)}_\alpha,\,A_\nu(y)\,\Omega\rangle\\ &\quad-\langle \Omega,\,A_\nu(y)\,\Phi^{(+)}_\alpha\rangle\langle \Phi^{(+)}_\alpha,\,A_\mu(x)\,\Omega\rangle]=0.\end{aligned}\tag{7·8}$$

In the lowest order approximation, if we take only one photon states as the intermediate states, one finds at once

$$\langle \Omega,\, R[A_\mu(x):A_\nu(y)]\,\Omega\rangle=-\delta_{\mu\nu}D_{\rm ret}(x-y)=\delta_{\mu\nu}\cdot\theta(x-y)\,D(x-y).\tag{7·9}$$

In virtue of the gauge invariance, it is sometimes convenient to rewrite it as

$$g^{(0)}_{\mu\nu}(x-y)=-\delta_{\mu\nu}\cdot D_{\rm ret}(x-y)\to-(\delta_{\mu\nu}-\partial_\mu\partial_\nu/\Box)\,D_{\rm ret}(x-y),\tag{7·10}$$

where the function g is defined by

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* The relations between Δ functions are given by

$$\Delta(x)=\Delta^{(+)}(x)+\Delta^{(-)}(x),\quad \Delta^{(+)}(-x)=-\Delta^{(\mp)}(x).$$

The following functions are also useful for later purposes

$$S_{\rm ret}(x)=(\gamma\cdot\partial-m)\,\Delta_{\rm ret}(x),\quad \widetilde{S}_{\rm ret}(x)=(\gamma^{T}\cdot\partial+m)\,\Delta_{\rm ret}(x).$$

$$\langle \Omega,\, R[A_\mu(x) : A_\nu(y)]\Omega\rangle = g^{(0)}_{\mu\nu}(x-y) + g^{(2)}_{\mu\nu}(x-y) + \cdots$$

In the next order, one must solve the equation

$$\begin{aligned}
&\langle \Omega,\, R[A_\mu(x) : A_\nu(y)]\Omega\rangle - \langle \Omega,\, R[A_\nu(y) : A_\mu(x)]\Omega\rangle \\
&+ i\sum_{abc}[\langle \Omega,\, A_\mu(x)\,\Phi_c\rangle\langle \Phi_c,\, A_\nu(y)\,\Omega\rangle + \langle \Omega,\, A_\mu(x)\,\Phi^{(+)}_{ab}\rangle\langle \Phi^{(-)}_{ab},\, A_\nu(y)\,\Omega\rangle \\
&\qquad - (x \rightleftarrows y,\ \mu \rightleftarrows \nu)] = 0\,,
\end{aligned} \tag{7·11}$$

where a, b, c are one electron, one-positron and one-photon states, respectively. Now making use of the asymptotic conditions and of the contraction functions, one can rewrite eq. (7·11) in the form

$$\begin{aligned}
&g^{(2)}_{\mu\nu}(x-y) - g^{(2)}_{\mu\nu}(y-x) \\
&+ \int (du)\,(dv)\,[\Box_u D_{\mathrm{ret}}(x-u)\cdot D(u-v)\cdot \Box_v g^{(2)}_{\mu\nu}(y-v) - (x \rightleftarrows y,\ \mu \rightleftarrows \nu)] \\
&= \int (dy_1)\,(dy_2)\,(dz_1)\,(dz_2)\, D_{y_1}\widetilde{D}_{y_2}\langle \Omega,\, R[A_\mu(x) : \psi(y_1)\,\bar{\psi}(y_2)]\Omega\rangle \\
&\qquad \times \widetilde{D}_{z_1} D_{z_2}\langle \Omega,\, R[A_\nu(y) : \bar{\psi}(z_1)\,\psi(z_2)]\Omega \\
&\qquad \times \widetilde{S}^{(+)}(y_1-z_1)\,S^{(+)}(y_2-z_2) - (x \rightleftarrows y,\ \mu \rightleftarrows \nu)\,,
\end{aligned} \tag{7·12}$$

where $g^{(2)}_{\mu\nu}(x-y)$ is the second term of the power series expansion of $g_{\mu\nu}(x-y)$, i.e.

$$g_{\mu\nu}(x-y) = g^{(0)}_{\mu\nu}(x-y) + g^{(2)}_{\mu\nu}(x-y) + \cdots. \tag{7·13}$$

There are other equations governing other quantities such as $\langle \Omega,\, R[A : \psi, \bar{\psi}]\Omega\rangle$ so that we can settle them in the perturbation theory. In the present treatment, however, we shall pick up out of many solutions only such ones that correspond to quantum electrodynamics. For this purpose we shall make use of the unrenormalized field equations in quantum electrodynamics which is enough to calculate the lowest order contribution,

$$\begin{aligned}
&\Box A_\mu(x) = -j_\mu(x) = -ie\bar{\psi}(x)\,\gamma_\mu\psi(x)\,, \\
&(\gamma_\mu(\partial_\mu - ieA_\mu(x)) + m)\,\psi(x) = 0\,, \\
&(\gamma_\mu(\partial_\mu + ieA_\mu(x)) - m)\,\bar{\psi}(x) = 0\,,
\end{aligned} \tag{7·14}$$

then one finds in the e^1 approximation the result

$$\begin{aligned}
&\langle \Omega,\, R[A_\mu(x) : \psi_\alpha(y),\, \bar{\psi}_\beta(z)]\Omega\rangle = ie\int (du)\, D_{\mathrm{ret}}(x-u)\,[\widetilde{S}_{\mathrm{ret}}(u-y)\,\gamma_\mu S_{\mathrm{ret}}(u-z)]_{\alpha\beta}\,, \\
&\langle \Omega,\, R[A_\mu(x) : \bar{\psi}_\alpha(y),\, \psi_\beta(z)]\Omega\rangle = -ie\int (du)\, D_{\mathrm{ret}}(x-u)\,[S^T_{\mathrm{ret}}(u-y)\,\gamma_\mu^T S^T_{\mathrm{ret}}(u-z)]_{\alpha\beta}\,.
\end{aligned} \tag{7·15}$$

Now eq. (7·12) is written by

$$\begin{aligned}
&g^{(2)}_{\mu\nu}(x-y) - g^{(2)}_{\mu\nu}(y-x) \\
&= \int (du)\,[D(x-u)\cdot\Box_u g^{(2)}_{\mu\nu}(y-u) + \Box_u g^{(2)}_{\mu\nu}(x-u)\cdot D(u-y)] \\
&- i\int (du)\,(dv)\, D_{\mathrm{ret}}(x-u)\, D_{\mathrm{ret}}(y-v)\, Q_{\mu\nu}(u,v)\,,
\end{aligned} \tag{7·16}$$

where

$$Q_{\mu\nu}(u, v) = e^2 Sp[\gamma_\mu \cdot (\gamma\partial - m) \Delta^{(+)}(u-v) \cdot \gamma_\nu (\gamma\partial + m) \Delta^{(+)}(u-v)] - (u \rightleftarrows v). \tag{7·17}$$

In what follows, we shall solve this integral equation in two different ways.

Method I.

To solve eq. (7·16), one can utilize the formal solution (6·18) derived in the previous section. To our regret, there occur divergences in the intermediate stage of calculation, but in turn this method can be utilized to evaluate self-energies and other divergent quantities. Especially one finds in this way that the photon self-energy is equal to zero.

The formal solution of eq. (7·16) is given by

$$g^{(2)}_{\mu\nu}(x-y) = -i \int (du)\,(dv)\, D_{\text{ret}}(x-u)\, D_{\text{ret}}(v-y)\, \theta^*(u-v)\, Q_{\mu\nu}(u, v) + \lambda_{\mu\nu} \int (du)\, D_{\text{ret}}(x-u)\, D_{\text{ret}}(u-y), \tag{7·18}$$

where $\lambda_{\mu\nu}$ is a differential operator at highest linear in x and in y.

Let us first evaluate $Q_{\mu\nu}(u, v)$ in the momentum space.

$$\begin{aligned}
&Sp[\gamma_\mu \cdot (\gamma\partial - m) \Delta^{(+)}(x) \cdot \gamma_\nu \cdot (\gamma\partial + m) \Delta^{(+)}(x)] - (x \to -x) \\
&= (-i)^2/(2\pi)^6 \int (dp)\,(dq)\, e^{i(p+q)x} Sp[\gamma_\mu (ip\gamma - m)\, \gamma_\nu (iq\gamma + m)]\, \theta(p_0)\, \theta(q_0) \\
&\qquad \times \delta(p^2+m^2)\, \delta(q^2+m^2) - (x \to -x) \\
&= -1/(2\pi)^6 \int (dk)\, e^{ikx} \int (dp)\,(dq)\, \delta(p+q-k)\, \theta(p_0)\, \theta(q_0)\, \delta(p^2+m^2)\, \delta(q^2+m^2) \\
&\qquad \times Sp[\gamma_\mu (ip\gamma - m)\, \gamma_\nu (iq\gamma + m)] - (x \to -x) \\
&= -1/(2\pi)^6 \int (dk)\, e^{ikx} \frac{\pi}{\sqrt{-k^2}} (\delta_{\mu\nu} k^2 - k_\mu k_\nu) \sqrt{-m^2 - k^2/4} \\
&\qquad \times \left(2 + \frac{8}{3k^2}(-m^2 - k^2/4)\right) \varepsilon(k_0)\, \theta(-m^2 - k^2/4) \\
&= \int_{4m^2}^{\infty} d(\kappa^2)\, (\pi/3\kappa) \cdot \sqrt{\kappa^2 - 4m^2}\, (2 + 4m^2/\kappa^2) \\
&\qquad \times -1/(2\pi)^6 \int (dk)\, (\delta_{\mu\nu} k^2 - k_\mu k_\nu)\, e^{ikx}\, \varepsilon(k_0)\, \delta(k^2 + \kappa^2).
\end{aligned}$$

If we define the polarization operator $\Pi_{\mu\nu}$ by

$$(7·18) = \int (du)\,(dv)\, D_{\text{ret}}(x-u)\, \Pi_{\mu\nu}(u, v)\, D_{\text{ret}}(v-y),$$

it is given by

$$\begin{aligned} \Pi_{\mu\nu}(x) &= -i\theta^*(x)\, Q_{\mu\nu}(x) + \lambda_{\mu\nu}\cdot\delta(x) \\ &= e^2 \int_{4m^2}^{\infty} d(\kappa^2)\, (2\pi/3)\, \sqrt{\kappa^2-4m^2}\,(\kappa^2+2m^2)/\kappa^3 \\ &\times (2\pi)^{-7} \int (dk)\, (\delta_{\mu\nu}k^2 - k_\mu k_\nu)\, e^{ikx}/(k^2+\kappa^2-i\eta\,\varepsilon(k_0)) + \lambda_{\mu\nu}\delta(x), \end{aligned} \tag{7·19}$$

where η is an infinitesimal positive quantity to select the correct path of integration. Due to the derivative condition, $\lambda_{\mu\nu}$ is a quadratic function of k, and choosing

$$\lambda_{\mu\nu} = (\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu) \times \text{const.}, \tag{7·20}$$

one can make $\Pi_{\mu\nu}(x)$ convergent as given by

$$\begin{aligned} \Pi_{\mu\nu}(x) &= e^2/3 \int_{4m^2}^{\infty} d(\kappa^2)\, \sqrt{\kappa^2-4m^2}\,(\kappa^2+2m^2)/\kappa^5 \\ &\times 1/(2\pi)^6 \int (\delta_{\mu\nu}k^2 - k_\mu k_\nu)\, (-k^2)/(k^2+\kappa^2-i\eta\,\varepsilon(k_0))\, e^{ikx}(dk). \end{aligned} \tag{7·21}$$

The introduction of $\lambda_{\mu\nu}$ of the form (7·20) corresponds to the introduction of a counter term of the form

$$(-1/4 F_{\mu\nu}F_{\mu\nu}) \times \text{const.} \tag{7·22}$$

which clearly expresses the Z_3 renormalization.

From the above result one can make three remarks. First, one need not introduce a counter term of the form $\lambda_{\mu\nu} = \delta_{\mu\nu} \times$ const. i.e. the self-energy of the photon vanishes according to the present method of calculation. Secondly the resulting expressing (7·21) is really gauge invariant as it should be. In this respect, the present method of calculating the propagation function has benefit over the older ones. Finally it is worth while to notice that although divergences appeared in the course of calculation the final result (7·21) which is the solution of (7·16) is really divergence-free, i.e. eq. (7·16) possesses a divergence-free solution. This suggests us that one might hope to be able to integrate eq. (7·16) without encountering any divergence in the course of calculation. This is really the case and we shall show another method in what follows.

Method II.

Based on the general properties of field theories, Källen,[21] Lehmann[22] and others were able to show that the exact propagation function can be expressed as the sum of free propagators with different rest masses. Accordingly, let us put

$$\Delta^{(2)}_{\mu\nu}(x-y) = \int d(\kappa^2)\, \rho(\kappa^2)\, 1/(2\pi)^4 \int (dk)\, (\delta_{\mu\nu} - k_\mu k_\nu/k^2)\, e^{ikx}/(k^2+\kappa^2), \tag{7·23}$$

where the path of integration at poles must be chosen as in (7·21). Then one finds that the first term in the right-hand side of eq. (7·16) vanishes when the expression (7·23) is inserted, and that $\rho(\kappa^2)$ is given by

$$\rho(\kappa^2) = -e^2/3\,(2\pi)^2 \cdot \sqrt{\kappa^2-4m^2}\,(\kappa^2+2m^2)/\kappa^5 \cdot \theta(\kappa^2-4m^2). \tag{7·24}$$

In the course of this integration no divergence occurs.

If we apply the transformation of variables

$$\kappa^2 = m^2/x(1-x)$$

in (7·21), we get

$$\Pi^{*}_{\mu\nu}(y) = -(e^2/2\pi^2)(1/2\pi)^4 \int dx \cdot x(1-x) \int (dk) \log\left((k^2 x(1-x) + m^2)/m^2\right) \times e^{iky}(\partial_{\mu\nu} k^2 - k_\mu k_\nu) \tag{7·25}$$

in conformity with the well-known expression. Namely, eq. (7·16) is satisfied by the renormalized propagation function in quantum electrodynamics.

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The author would like to express his sincere thanks to Prof. W. Heisenberg for the kind hospitality extended to him in the Max-Planck-Institut für Physik and for the interest taken in this work. Thanks are also due to Prof. H. Lehmann and Dr. W. Zimmermann for informing him of their results in advance of publication.

Appendix I

Proof of Equation (2·2)

Let us consider a Schrödinger equation

$$i\frac{\partial}{\partial x_0} U(x_0, -\infty) = \int d^3x H^{\mathrm{ext}}(x)\, U(x_0, -\infty), \tag{A·1}$$

where

$$H^{\mathrm{ext}}(x) = Q(x)\,\varphi(x).$$

$\varphi(x)$ is a quantized field operator and $Q(x)$ a c-number external source. Then let us define $\varphi_Q(x)$ by

$$\varphi_Q(x) = U(x_0, -\infty)^{*}\varphi(x)\, U(x_0, -\infty). \tag{A·2}$$

By differentiating $\varphi_Q(x)$ with respect to $Q(x)$, we get

$$\frac{\delta\varphi_Q(x)}{\delta Q(y)} = -i\theta(x-y)[\varphi_Q(x), \varphi_Q(y)]. \tag{A·3}$$

By repeated differentiations, one arrives at

$$\frac{\partial^n \varphi_Q(x)}{\partial Q(y_1)\cdots\partial Q(y_n)} = (-i)^n \theta(x-y_1)\,\theta(y_1-y_2)\cdots\theta(y_{n-1}-y) \times [\cdots[\varphi_Q(x), \varphi_Q(y_1)]\varphi_Q(y_2)]\cdots\varphi_Q(y_n)], \quad \text{if} \quad y_1 > y_2 > \cdots > y_n. \tag{A·4}$$

Then by summing up over all possibie permutations of operators, one finds

$$\frac{\delta^n \varphi_Q(x)}{\delta Q(y_1)\cdots\delta Q(y_n)} = R[\varphi_Q(x) : \varphi_Q(y_1)\cdots\varphi_Q(y_n)]. \tag{A·5}$$

If one differentiates both sides of (A·3) and then applies (A·5) on the both sides, one finds after putting $Q=0$ the equation (2·2).

If one starts instead of (A·2) with

$$\varphi_Q'(x) = U(\infty, x_0)\,\varphi(x)\,U(\infty, x_0)^*, \tag{A·6}$$

one finds eq. (3·2).

Appedix II

Application of Equation (5·21) *to normalize Feynman Amplitudes*

Let us consider the problem of bound states consisting of two charged scalar particles and interacting through massless scalar field.

Let G be the Green function for the two-particle system, i.e.

$$G(x_1x_2y_1y_2) = \langle \Omega,\ T[\varphi(x_1)\,\varphi(x_2)\,\varphi^*(y_1)\,\varphi^*(y_2)]\,\Omega\rangle. \tag{B·1}$$

In the lowest ladder approximation, G satisfies the following Bethe-Salpeter equation[2,3]

$$(K_{x_1}K_{x_2} - f^2 D_F(x_1-x_2))\,G(x_1x_2y_1y_2) = -[\delta(x_1-y_1)\,\delta(x_2-y_2) + (x_1 \rightleftarrows x_2)], \tag{B·2}$$

where K_x is the Klein-Gordon operator for the charged scalar field, and D_F the Feynman propagator for the massless scalar field.

The equation for a bound state s is given by

$$(K_{x_1}K_{x_2} - f^2 D_F(x_1-x_2))\,\langle \Omega,\ T[\varphi(x_1)\,\varphi(x_2)]\,\Phi_s\rangle = 0.$$

Wick succeeded to solve this equation,[24] and the present author[25] gave a method to normalize the Feynman amplitude $\langle \Omega,\ T[\varphi(x_1)\,\varphi(x_2)\,|\Phi_s\rangle$, but we intend to derive the normalization condition without recourse to the original field equations The new method is useful in the sense that the normalization condition can be obtained if one only knows the form of the Bethe-Salpeter equation, even if it is of a phenomenological one. By differentiating (B·1) with respect to P_μ, one gets

$$\begin{aligned} 2i\frac{d}{dP_\mu}&\langle \Omega,\ T[\varphi(x_1)\,\varphi(x_2)\,\varphi^*(y_1)\,\varphi^*(y_2)]\,\Omega\rangle \\ &= 2i\frac{d}{dP_\mu}G(x_1x_2y_1y_2) \\ &= \int (dx_0)\,\langle \Omega,\ T[\varphi(x_1)\,\varphi(x_2)\,\varphi^*(y_1)\,\varphi^*(y_2)\,j_\mu(x_0)]\,\Omega\rangle. \end{aligned} \tag{5·18a}$$

On the other hand, one gets from (B·2)

$$i\frac{d}{dP_\mu}G(x_1x_2y_1y_2) = 1/2\int (dx')\,(dy')\,G(x_1x_2x_1'x_2')\,i\frac{d}{dP_\mu}\,D(x_1'x_2'y_1'y_2')\,G(y_1'y_2'y_1y_2), \tag{B·3}$$

where

$$D(x_1x_2y_1y_2)=(K_{x_1}K_{x_2}-f^2D_F(x_1-x_2))\,\delta(x_1-y_1)\,\delta(x_2-y_2).$$

Combining (5·18a) with (B·3), one finds

$$\int(dx_0)\langle\Phi_s,\,j_\mu(x_0)\,\Phi_{s'}\rangle=\int(dx)\,(dy)\langle\Phi_s,\,T[\varphi^*(x_1)\,\varphi^*(x_2)]\Omega\rangle\,i\,\frac{d}{dP_\mu}D(x_1x_2y_1y_2)$$
$$\times\langle\Omega,\,T[\varphi(y_1)\,\varphi(y_2)]\Phi_{s'}\rangle. \tag{B·4}$$

Let us assume that a function $F(x_1x_2)$ satisfies

$$i(\partial/\partial x_1+\partial/\partial x_2)_\mu F(x_1x_2)=P_\mu F(x_1x_2),$$

then one can write in the momentum space

$$DF(p)=(p^2+pP+\mu^2+P^2/4)\,(p^2-pP+\mu^2+P^2/4)\,F(p)$$
$$+if^2/(2\pi)^4\int(dq)\,F(q)/((p-q)^2-i\varepsilon),$$

where p and q are the relative momentum. Then we readily get

$$i\frac{d}{dP_\mu}D=i(p_{1\mu}(p_2^2+\mu^2)+p_{2\mu}(p_1^2+\mu^2))$$
$$=-(\partial_\mu^{(1)}K_{x_1}+\partial_\mu^{(2)}K_{x_2}),$$

and we can write (B·4) as

$$\int(dx_0)\langle\Phi_s,\,j_\mu(x_0)\,\Phi_{s'}\rangle=-\int(dx_1)\,(dx_2)\langle\Phi_s,\,T[\varphi^*(x_1)\,\varphi^*(x_2)]\Omega\rangle$$
$$\times(\partial_\mu^{(1)}K_{x_2}+\partial_\mu^{(2)}K_{x_1})\langle\Omega,\,T[\varphi(x_1)\,\varphi(x_2)]\Phi_{s'}\rangle, \tag{B·5}$$

or after separating the center of mass coordinates from relative coordinates, one gets

$$\langle\Phi_s,\,j_\mu(x_0)\,\Phi_s\rangle=-2\int(dx)\langle\Phi_s,\,T[\varphi^*(x_1)\,\varphi^*(x_2)]\Omega\rangle$$
$$\times\partial_\mu^{(1)}K_{x_2}\langle\Omega,\,T[\varphi(x_1)\,\varphi(x_2)]\Phi_s\rangle, \tag{B·6}$$

where $x=x_1-x_2$.

This is the desired normalization condition derived basing upon the Ward's identity. It is not hard to see that eq. (B·6) gives the same result with that given previously, i.e. eq. (4·19) of reference 7), III.

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Mass Differences of Σ^+, Σ^- and Σ^0 Hyperons*

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It is shown that there is a possibility to interpret the observed mass differences of Σ^+, Σ^- and Σ^0 hyperons in terms of electromagnetic self-energies, if the Σ hyperons are assumed to be Dirac particles with anomalous magnetic moments of a few nuclear magnetons.

The mass difference of $\Sigma^- - \Sigma^+$ hyperons ($\sim 15\, m_e$) has recently been observed by several persons.[1),2),3)] Furthermore, the existence of the Σ^0, which is lighter than the Σ^- (by at least $\sim 8m_e$), has been found in the latest bubble chamber experiments.[4),5)] These experiments show that the most probable value of Σ^0 mass is somewhere between the masses of Σ^- and Σ^+.

We shall examine the possibility that these observed mass differences of Σ hyperons can be explained as the electromagnetic self-energies just as in the case of the neutron-proton mass difference.[6),7)]

If we assume, for the sake of simplicity, that the Σ hyperons are Dirac particles[8)] and have anomalous magnetic moments owing to the interactions $\Sigma N\bar{K}$, $\Sigma\Xi K$, $\Sigma\Sigma\pi$, etc., we may obtain the following mass shifts as the second order electromagnetic self-energies of the charged and neutral Σ hyperons, using our result calculated for the n-p mass difference,[7)]

$$\Delta M(\Sigma^{\pm}) = \frac{\alpha}{\pi} M\left[F_0(\lambda) \mp \mu_{\pm}\left(\frac{M}{M_0}\right)F_1(\lambda) - \mu_{\pm}^2\left(\frac{M}{M_0}\right)^2 F_2(\lambda)\right] \tag{1}$$

$$\Delta M(\Sigma^0) = -\frac{\alpha}{\pi} M\mu_0^2\left(\frac{M}{M_0}\right)^2 F_2(\lambda) \tag{2}$$

where $\mu_{\pm}$ and μ_0 are the anomalous magnetic moments of $\Sigma^{\pm}$ and Σ^0, respectively,

* After the completion of part of this note, we found the article by Sudershan and Marshak on the $\Sigma^- - \Sigma^+$ mass difference (Phys. Rev. **104** (1956), 267). Our basic idea is the same as theirs, but the results are generally different owing to an unfortunate wrong sign of the third term in their eq. (2). (In the special case that $\mu_+ = \mu_- > 0$, our expression for the $\Sigma^- - \Sigma^+$ mass difference agrees with theirs.)

We at first estimated the $\Sigma^{\pm} - \Lambda^0$ mass difference instead of the $\Sigma^{\pm} - \Sigma^0$, before the evidence for the existence of the Σ^0 became conclusive. We wish to thank Dr. Iloff for informing us of that evidence.

We also wish to thank Professor Marshak and Dr. Sudershan for replying about the above disagreement and sending us their kind comment on our note.

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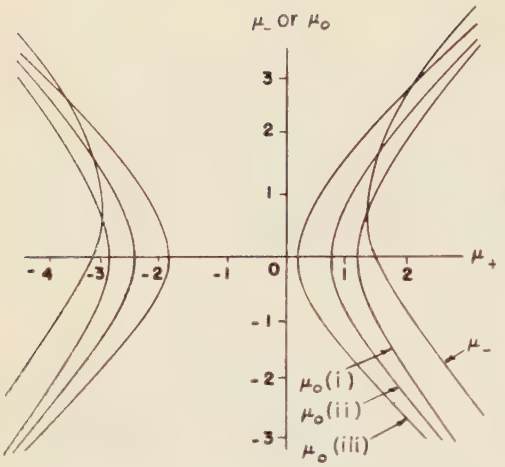

Fig. 1. $\lambda=1.0$

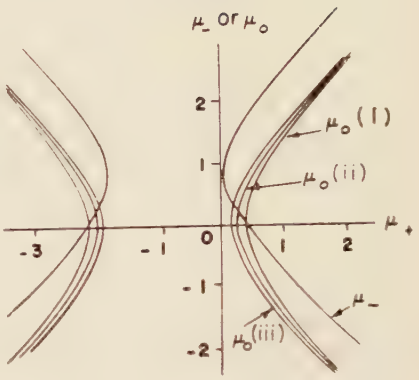

Fig. 2. $\lambda=2.0$

and $F_0(\lambda)$ is the contribution from the charge-charge self interaction part, $F_1(\lambda)$ from the charge-moment part, and $F_2(\lambda)$ from the moment-moment part. Since F_0 is logarithmically divergent, and F_1 and F_2 are quadratically divergent in $\lambda=K\,M$, where M is the unperturbed $\sum$ hyperon mass level and K is the cutoff momentum, we have used a straight cutoff for simplicity. F_0, F_1, and F_2 are always positive for any real value of λ. As we express $\mu_\pm$ and μ_0 in units of nuclear magnetons, a factor (M/M_0) appears in the above equations, where M_0 is the nucleon mass. In the following we shall take the unperturbed mass $M=2335m_e$, about which we have not any reliable knowledge, but the choice of this numerical detail does not significantly affect the final result for the mass difference.

We shall get the relationships between μ_+, μ_-, and μ_0 so that the mass differences calculated from eqs. (1) and (2) just give the observed values From (1) and (2), we have

$$(\mu_+ + a)^2 - (\mu_- - a)^2 = b \tag{3}$$

$$(\mu_+ + a)^2 - \mu_0^2 = b' + c, \tag{4}$$

where

$$a = F_1(\lambda) \Big/ 2\left(\frac{M}{M_0}\right) F_2(\lambda)$$

$$b = [\,\Delta M(\Sigma^+) - \Delta M(\Sigma^-)\,]_{\text{obs}} \Big/ \frac{\alpha}{\pi} M \left(\frac{M}{M_0}\right)^2 F_2(\lambda)$$

$$b' = [\,\Delta M(\Sigma^0) - \Delta M(\Sigma^+)\,]_{\text{obs}} \Big/ \frac{\alpha}{\pi} M \left(\frac{M}{M_0}\right)^2 F_2(\lambda)$$

$$c = a^2 + F_0(\lambda) \Big/ \left(\frac{M}{M_0}\right)^2 F_2(\lambda)\ . \tag{5}$$

From the experimental data, we shall take the following values and examine the results,

$$[\Delta M(\Sigma^-) - \Delta M(\Sigma^+)]_{\text{obs.}} = 15 m_e\,, \tag{6}$$

$$[\Delta M(\Sigma^0) - \Delta M(\Sigma^+)]_{\text{obs.}} = \begin{cases} \text{(i)} & +5m_e \\ \text{(ii)} & 0 \\ \text{(iii)} & -5m_e\,. \end{cases} \tag{7}$$

The value of a is not sensitive to the choice of λ, but the values of b, b' and c are sensitive to some extent (Table 1).

Table 1.

λ		0.5	1.0	1.5	2.0	2.5	∞
a		0.81	0.86	0.88	0.89	0.89	0.90
$\sqrt{b}$		5.50	2.16	1.30	0.92	0.71	0
$\sqrt{b'+c}$	(i)	4.02	2.05	1.51	1.29	1.17	0.90
	(ii)	2.62	1.62	1.31	1.17	1.10	0.90
	(iii)	(1.79i)	1.03	1.08	1.05	1.02	0.90

Equation (3) indicates that a pair of values for μ_+ and μ_- always represents a point on a hyperbola with foci on a line parallel with the μ_+ axis, and so μ_+ must satisfy the inequalities $\mu_+ < -\sqrt{b} - a$ or $\mu_+ > \sqrt{b} - a$, while μ_- has no restriction. If we take the cutoff $\lambda = 0.5$, we get unreasonably large absolute values for μ_+, i.e., $\mu_+ < -6.31$ or $\mu_+ > 4.69$. Then we are tempted to take $\lambda \gtrsim 1.0$ which gives reasonably small absolute values for μ_+ as well as μ_-.

Similarly, eq. (4) indicates that a pair of values for μ_+ and μ_0 represents a point on a hyperbola. In this case, however, if we take the choice of (7-iii) and if we take a rather small cutoff (e.g. $\lambda = 0.5$), the hyperbola will have its foci on the μ_0 axis, and μ_0 must satisfy $|\mu_0| > \sqrt{|b'+c|}$ (~ 1.79 for $\lambda = 0.5$). If we take the cutoff $\lambda \gtrsim 1.0$, the hyperbola will have its foci on the μ_+ axis and μ_+ has to satisfy $\mu_+ < -\sqrt{b'+c} - a$ or $\mu_+ > \sqrt{b'+c} - a$.

Combining the above two arguments about $\mu_+ - \mu_-$ and $\mu_+ - \mu_0$, for $\lambda \gtrsim 1.0$, we have the inequalities, $\mu_+ < \text{Min.}\,(-\sqrt{b} - a, -\sqrt{b'+c} - a)$ or $\mu_+ > \text{Max.}\,(\sqrt{b} - a, \sqrt{b'+c} - a)$. Since we can get the reasonable small values for μ_+, μ_- and μ_0 only for $\lambda \gtrsim 1.0$, we may again recognize that the situation favors taking the cutoff $\lambda \gtrsim 1.0$. Figures 1 and 2 show these hyperbolas for $\lambda = 1.0$ and $\lambda = 2.0$. For example, if we take $\lambda = 1.0$ and the choice of (7-i) and if we assume $\mu_- = 1.00$, we have $\mu_+ = 1.31$ or -3.03, and $\mu_0 = 0.72$ or -0.72 from eqs. (3) and (4), and $\Delta M(\Sigma^-) = 8.3 m_e$, $\Delta M(\Sigma^+) = -6.7 m_e$ and $\Delta M(\Sigma^0) = -1.7 m_e$ from eqs. (1) and (2).

Needless to say, these numerical details should not be taken very seriously, not only in view of the introduction of the static anomalous magnetic moments of the Σ hyperons instead of the electromagnetic interaction through rather complicated meson clouds, but also in view of the ambiguous treatment cutting off the divergent integrals. We might say, however, that if μ_+, μ_- and μ_0 have reasonable absolute values of a few nuclear

magnetons, and if they satisfy the eqs. (3) and (4), there is a possibility to explain the observed mass differences of the Σ^{+}, Σ^{-} and Σ^{0} in terms of the electromagnetic self-energy.

For the cutoff momentum in the above calculation, we took values without requiring them to reproduce the observed nucleon mass difference, because the structure of the meson cloud associated with the hyperon might be different from that of the nucleon.

We wish to thank Professor B. C. Carlson for his valuable discussions and critical reading of the manuscript. We also wish to express our appreciation to Professor Gerald W. Fox, Head of the Physics Department of Iowa State College, for his kind hospitality.

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Note added in proof

After this report was written, we obtained the mirror theorem for the anomalous magnetic moments of baryons according to the theory of d'Espagnat and Prentki and showed the relation $\mu_0 = 1/2 \cdot (\mu_+ + \mu_-)$ for the Σ hyperons without use of perturbation theory. Combining this relation and eqs. (3) and (4), we determined a set of values μ_+, μ_- and μ_0. (A similar argument has been given independently by Marshak and others, Phys. Rev. **106** (1957), 599.) A comparison of the thus obtained numerical values of μ_+, μ_- and μ_0 with the perturbation results and a note on the mirror theorem will be reported in a separate note.

In the present report we omitted the effect of the virtual dissociation of the Σ^0 into a Λ^0 and a photon as well as the electromagnetic interaction between the virtually dissociated charged baryon and boson.

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A Remark on High Energy Phenomena

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A possible model which discriminates π-N, K-N and N-N collisions in the cosmic-ray energy region is proposed. The difference in dependence of the mutiplicities on the energy of incident particles in π-N, K-N and N-N collisions is discussed, which is due to the difference of the kinds of the incident particles. Experimental data obtained with nuclear emulsions are analysed by using our model. If our model is justified, the mean energy of the secondary particles of the π-primary jets must be greater (about twice) than that of the N-primary one, both in nucleon target and nucleus target cases.

§ 1. Introduction

In many works, high energy jets and extensive air showers events have been analysed on the assumption that in the cosmic ray energy region π- and K-mesons colliding with nucleons produce the same jets as those produced by nucleons with the same energy. This assumption, which has of course no direct experimental evidence, rests only upon a speculation that the difference of the rest masses could be ineffective in these high energy collisions.

The idea of "families" of elementary particles, which we believe to be of fundamental importance, suggests that elementary particles have internal structures and we may believe the internal structures of strongly interacting particles to play an important role in the high energy events. The difference of the rest masses might be neglectable but that of the internal structures is not so. The fact that π-mesons, K-mesons and nucleons belong to different families is enough to make the above assumption not at all self-evident.

This assumption must be examined thoroughly, since its examination would not only give us some information about the internal structure of these strongly interacting particles but affect the method for the analyses of the experimental data on high energy jets and extensive air showers.

In this work the multiplicity in jet phenomena (which can be most unambiguously observed experimentally) is discussed from the view-point mentioned above. Fig. 1 shows the experimental results on the energy-dependence of this multiplicity of high energy jets observed in nuclear emulsion with less than one black or grey track (except Schein's star). These data do not necessarily tell us that higher energies of the incident particles promise higher multiplicities. The multiplicity obtained theoretically is proportional to $\sqrt{W}$,[1] where W is the energy of the primary particle in the center of mass system.

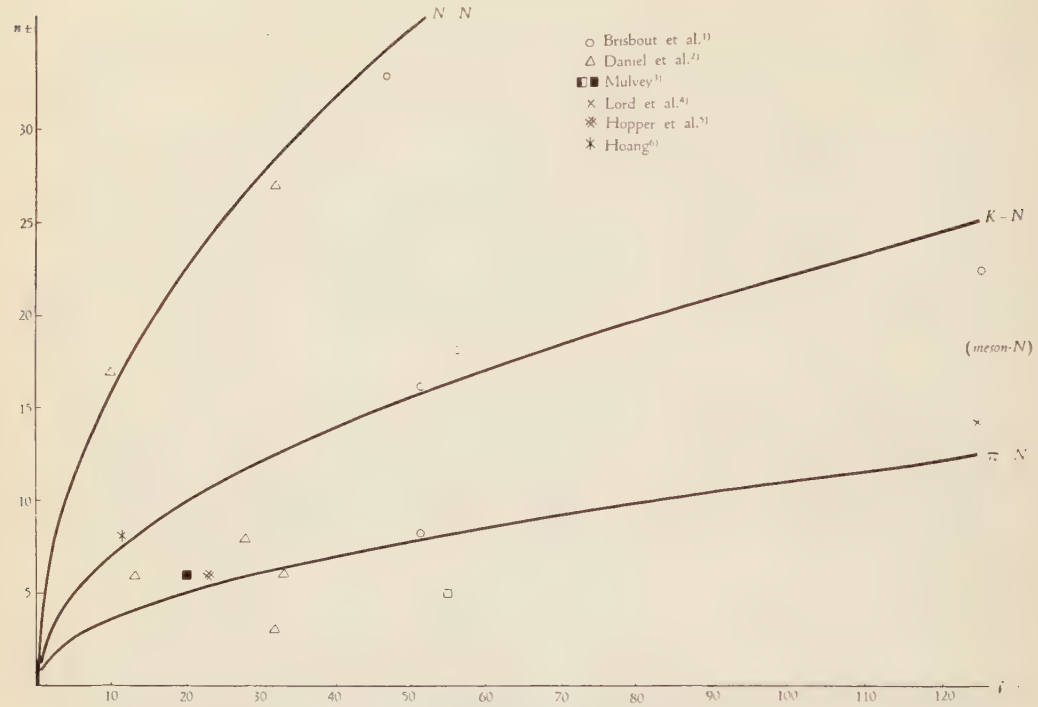

Fig. 1. Experimental data with $\gamma_{1/2} \gtrsim 10$

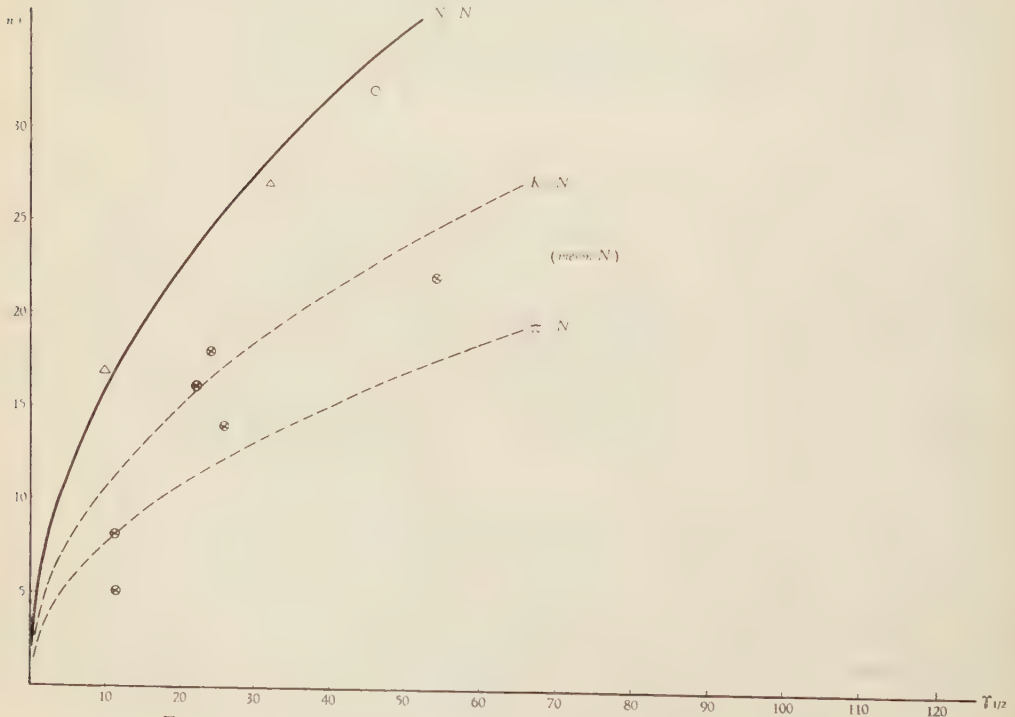

Fig. 2. Experimental data with $\gamma_0 \gtrsim 10$ adjusted according to our model

This discrepancy between the experimental data and the theoretical results might be explained by introducing the impact parameter dependence of the inelasticity, but here we propose another possible explanation[8] in which the difference between the π-meson and/or K-meson-nucleon (π-N; K-N) collisions and a nucleon-nucleon (N-N) collisions is taken into account. We will separate the experimental points into three groups and introduce three different $\sqrt{W}$-curves which correspond to the three groups. Now, we shall regard the three $\sqrt{W}$-curves in Fig. 1 as corresponding to the three groups of collisions, π-N, K-N and N-N respectively.

The assignment of the three $\sqrt{W}$-curves made above in Fig. 1 seems at first sight to lead to a curious conclusion that some of the K- and π-mesons have energies much higher than that of the nucleon. We shall discuss in § 2 a possible model which explains the above separation and is free of this curious conclusion.

§ 2. A possible model of high energy π-N, K-N and N-N collisions

Limiting our present aim chiefly to discussion of the energy dependence of the multiplicity of secondary particles in the high energy collision process, we can start from the following simple assumptions.

The collision process takes place in the centre of mass system (C.M.S.) as follows:

I. In the N-N collision, an incident nucleon produces an energy lump at rest in the C.M.S. after the collision.

II. In the π-N and K-N collision, an incident meson with momentum $\boldsymbol{P}$ produces an energy lump with momentum $(1-\alpha)\boldsymbol{P}$, where α is a parameter which will be determined from the experimental data. The target nucleon flies away with momentum $-(1-\alpha)\boldsymbol{P}$.*[9]

III. The multiplicity of the secondary particles is proportional to the root of the energy of the lump in its rest system, i.e. to the root of the mass of the lump, with a proportionality factor, k, which is independent of the kind of the incident particle.

IV. The secondary particles are produced symmetrically in the forward and backward directions in the rest system of the lump.

From I to IV, we obtain the following results.

1) The case in which the target is a nucleon

For the meson-primary jets, we obtain Table 1. From Table 1 $\gamma_{1/2}$, which is determined from the half-angle method, is expressed as

$$\gamma_{1/2}\equiv 1/\sqrt{1-v_L^2}=1/\sqrt{\alpha}\cdot W/M\equiv 1/\sqrt{\alpha}\cdot\gamma_C=\sqrt{E/2\alpha M}\,,$$

therefore the multiplicity N is given by

$$N\equiv k(M^*/M)^{1/2}=k(2\sqrt{\alpha}\,W/M)^{1/2}=k\alpha^{1/4}\sqrt{2\gamma_C}$$
$$=k\sqrt{2\alpha\gamma_{1/2}}=k(2\alpha E/M)^{1/4}\,.$$

*) Here it is assumed that the mass of the target nucleon does not change. Even if the target nucleon is excited to some extent so as to be able to emit a few mesons, the following discussions are not seriously affected.

Table 1

		in C.M.S.		in L.S.	
		target nucleon	incident meson	target nucleon	incident meson
before collision	energy	$W(\simeq\|\boldsymbol{P}\|)$	$W'(\simeq W)$	M	$E(\simeq 2W'^2/M)$
	momentum	$-\boldsymbol{P}$	$\boldsymbol{P}$	0	$\boldsymbol{P}_L(\simeq E)$
		target nucleon	lump	target nucleon	lump
after collision	energy	$(1-\alpha)W$	$(1+\alpha)W$	$\frac{1}{2}\frac{2-2\alpha+\alpha^2}{1-\alpha}M$	$\sim E$
	momentum	$-(1-\alpha)\boldsymbol{P}$	$(1-\alpha)\boldsymbol{P}$	$\frac{1}{2}\frac{\alpha(2-\alpha)}{1-\alpha}M$	$\sim E$
	mass	M	$M^*\equiv 2\sqrt{\alpha}W$	M	$M^*\equiv\sqrt{2\alpha ME}$
velocity of lump		$v_c\equiv\frac{1-\alpha}{1+\alpha}\left[1-\frac{1}{2}(M/W)^2\right]$		$v_L\equiv 1-\frac{\alpha}{2}(M/W)^2$	

Here it is assumed $(1-\alpha)^2\gg(M/W)^2$.

For the nucleon-primary jet, the mass of the lump is obtained by putting $\alpha=1$ in Table 1. The multiplicity, N, is given by

$$N=k\sqrt{2\gamma_C}=k\sqrt{2\gamma_{1/2}}=k(2E/M)^{1/4}.$$

2) The case in which the target is a nucleus

In this case the collision process would take place as follows: The primary meson collides with the first nucleon in the target nucleus and produces the energy lump with the momentum $(1-\alpha)\boldsymbol{P}$ in the C.M.S. of the meson and the first nucleon. The lump then collides with the second nucleon and, in the C.M.S. of the lump and the second nucleon, loses momentum by the same fraction as in the first collision and so forth.

The results shown in Table 2 are obtained by a straightforward calculation. Furthermore

$$\gamma_{1/2}\equiv 1/\sqrt{1-v_L^2}=(1/\sqrt{n\alpha})\cdot W/M\equiv 1/\sqrt{n\alpha}\cdot\gamma_{\mathrm{e.v.}}=\sqrt{E/2n\alpha M}\,,$$

$$N=k(2\sqrt{n\alpha}W/M)^{1/2}=k(n\alpha)^{1/4}\sqrt{2\gamma_{\mathrm{e.v.}}}=k\sqrt{2n\alpha\gamma_{1/2}}=k(2n\alpha E/M)^{1/4}.$$

It should be noted that nucleons which have collided with the primary meson or the lump do not collide with each other owing to the fact that they all have the same momentum, as will be seen from Table 2.

When the primary particle is a nucleon, only one energy lump will appear after the collision. The mass of the lump, etc., are obtained by putting $\alpha=1$ in Table 2.

$$\gamma_{1/2}=1/\sqrt{n}\cdot W/M=1/\sqrt{n}\cdot\gamma_{\mathrm{e.v.}}=\sqrt{E/2nM}\,,$$

$$N=k(2\sqrt{n}W/M)^{1/2}=kn^{1/4}\sqrt{2\gamma_{\mathrm{e.v.}}}=k\sqrt{2n\gamma_{1/2}}=k(2nE/M)^{1/4}.$$

Table 2

		in E.V.S.		in L.S.	
		each nucleon in target nucleus	incident meson	each nucleon in target nucleus	incident meson
before collision	energy	W	W	M	E
	momentum	$-\boldsymbol{P}$	$\boldsymbol{P}$	0	$\boldsymbol{P}_L$
		1st nucleon	lump	1st nucleon	lump
after 1st collision	energy	$(1-\alpha)W$	$(1+\alpha)W$	$\frac{1}{2}\frac{2-2\alpha+\alpha^2}{1-\alpha}M$	$\sim E$
	momentum	$-(1-\alpha)\boldsymbol{P}$	$(1-\alpha)\boldsymbol{P}$	$\frac{1}{2}\frac{\alpha(2-\alpha)}{1-\alpha}M$	$\sim E$
	mass	M	$2\sqrt{\alpha}W$	M	$\sqrt{2\alpha ME}$
		nth nucleon	lump	nth nucleon	lump
after nth collision	energy	$(1-\alpha)W$	$(1+n\alpha)W$	$\frac{1}{2}\frac{2-2\alpha+\alpha^2}{1-\alpha}M$	$\sim E$
	momentum	$-(1-\alpha)\boldsymbol{P}$	$(1-n\alpha)\boldsymbol{P}$	$\frac{1}{2}\frac{\alpha(2-\alpha)}{1-\alpha}M$	$\sim E$
	mass	M	$2\sqrt{n\alpha}W$	M	$\sqrt{2n\alpha ME}$
velocity of lump after nth collision		$v_{e\cdot v.}\equiv\frac{1-n\alpha}{1+n\alpha}\left[1-\frac{1}{2}(M/W)^2\right]$		$v_L\equiv 1-\frac{n\alpha}{2}(M/W)^2$	

Here it is assumed $(1-\alpha)^2\gg(M/W)^2$. E.V.S. means the equi-velocity system.*

§ 3. Summary and discussions

The three multiplicity curves in Fig. 1 can be obtained if we put

$$k\simeq 3.6,\quad \alpha_\pi\simeq 0.047\quad\text{and}\quad \alpha_K\simeq 0.19.$$

These curves are plotted against $\gamma_{1/2}$. When we plot them against γ_0 using the above α-values, we obtain the dotted curves in Fig. 2 for π- and K-primary jets, and the experimental points go over to the corresponding points on these curves (marked with ⊗). The true values of the energies of the incident π- and K-mesons are not extremely high and it might not be unreasonable that π- and K-mesons with those energies are observed at balloon altitudes of 20,000∼30,000 m.

The inelasticity is almost 1 for any collision in the sense that the incident particle

* Rigorously speaking, the system used is the centre of mass system of the incident meson and the 1st nucleon in the target nucleus, which coincide with the equi-velocity system when the incident particle is a nucleon.

plays no special role among the secondary particles. But if we understand inelasticity to be the fraction of the energy carried away by π-mesons, as is done in the analysis of extensive air showers, it cannot be determined without knowledge of the composition of the secondary particles, an analysis of which is beyond the scope of the present work.

Because of the lower multiplicity and the almost equal inelasticity, the mean energy of the secondary particles of the π-primary jets is about twice greater, than that of the nucleon-primary ones. The fact that the value of α_π is very small means that the π-primary gives little of its energy to the nucleons in the target nucleus.

As was mentioned in § 1, the random distribution of the multiplicity shown in Fig. 1 may be due to the impact-parameter dependence of the inelasticity. We are in need of much more experimental data to clarify this point.

If it is established experimentally that the neutral primary jets do not appear in the π-primary region the present explanation will become quite promising.

We have given two $\sqrt{W}$-curves each corresponding to π-primary jets and K-primary jets, but if the difference of family is to play an important role, it might be more plausible not to discriminate between the π-N and the K-N collisions, taking the value of the parameter α between 0.04 and 0.2. In that case the region between the lower two curves in Figs. 1 and 2 would correspond to the meson-nucleon collisions.

We would emphasize the possibility that the nature of the jets whose primary particles are of different kinds, especially of different families, might be different from each other.

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Letters to the Editor

Effect of Long-Range Correlations on Free Electron Diamagnetism

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In a recent paper by one of us (H.K.)[1], the diamagnetism of a free-electron gas has been studied by means of the collective description of the Coulomb interactions developed by Bohm and Pines[2] (BP). An earlier attempt to deal with this problem had been made by two of the authors (B.D. and N.H.M.)[3] and the present note represents an attempt to assess briefly the relation between the two treatments, to comment on the small numerical differences between the susceptibilities thus obtained, and to mention some relevant further work.

The earlier method (DM)[3] utilised the relations between energy and wave vector, in the absence of a magnetic field, developed by BP (see particularly Pines[4,5]), these being inserted in the Landau-Peierls-Wilson formula. This is in contrast to the more recent treatment of Kanazawa (K)[1], in which the collective description of electrons in a magnetic field was examined, and the current density calculated to first order in the magnetic field by a perturbation method. If we introduce the difference $\Delta\chi$ between the modified diamagnetic susceptibility and the Landau value χ_0 for perfectly free electrons, then for the case of sodium it turns out from the method of K that $\Delta\chi/\chi_0=0.05$, whereas the method of D M gives 0.13. Thus both treatments lead to a relatively small departure from the Landau value.

While the numerical agreement thus obtained might be regarded as reasonably satisfactory, it should be pointed out that differences between the two treatments will arise from the fact that exchange has been neglected in the method of K, whereas it is included in the work of DM. One of us (H.K.) is now investigating the introduction of exchange in the method of K and it is anticipated that this will increase the ratio $\Delta\chi/\chi_0$ somewhat. In this connection, it is relevant to remark that Matsudaira (unpublished work) has calculated the specific heat of the alkali metals, taking into account both the long-range correlations and the exchange energy, and his results suggest that quantitative differences may occur between the results of Pines[4] and those obtained by the method of K.

Nevertheless, a proper conclusion would appear to be that simultaneous inclusion of exchange and correlation effects tends to lead to results for both the diamagnetism and the specific heat which are in fair agreement with the simple free-electron theory.

Defects in the mathematical basis of the BP work (see for example, Adams[6] and Kuper[7]) have, of course, not been considered here. It seems fairly well agreed, however, that the physical basis for the BP theory is sound, and we do not expect that improvements in the foundations of the theory will markedly alter the general conclusions of this note.

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Acceleration of Cosmic Rays by Hydromagnetic Wave

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Since Fermi proposed an acceleration mechanism of cosmic rays,[1] many specific theories have been developed.[2)3] Here we give some remarks on the acceleration of charged particles by a hydromagnetic wave.

We consider a hydromagnetic wave in ideal incompressible fluid with infinite conductivity, with constant density, ρ, and with unit magnetic permeability. We assume that the primary magnetic field, $\boldsymbol{H}_0$ is homogeneous and the amplitude of the hydromagnetic wave is smaller than $|\boldsymbol{H}_0|$. In the incompressible fluid only transverse waves occur. The wave velocity, $\boldsymbol{V}$, is larger than the fluid velocity, $\boldsymbol{v}$, and they are related with $\boldsymbol{H}_0$ and the induced field, $\boldsymbol{h}$, respectively, as[4]

$$\boldsymbol{V}=(4\pi\rho)^{-1/2}\boldsymbol{H}_0,\quad \boldsymbol{v}=-(4\pi\rho)^{-1/2}\boldsymbol{h}. \qquad (1)$$

The simplest form of such wave, the plane wave, is found, for example, when $\boldsymbol{v}$ and $\boldsymbol{h}$ are both parallel to the y-axis, while $\boldsymbol{H}_0$ is parallel to the z-axis. The induced electric field, $\boldsymbol{E}$, has only the component parallel to the x-direction.

When a charged particle encounters with such wave, it results in the energy change, which is easily estimated by the procedure adopted by Fermi[1] as follows. In the system moving with the wave velocity, $\boldsymbol{V}$, the magnetic field is static and the electric field does not appear, then the charged particle is scattered elastically by the increase of the magnetic field. The particle will inverse its direction of motion when the reflexion condition is fulfilled. Transforming back to the rest system, the reversal of the motion of the particle results in the increase of the momentum component parallel to $\boldsymbol{H}_0$ in a head-on collsion and in the decrease in an overtaking collision. Therefore the fractional energy change, $\delta w/w$, in one collision is proportional to the wave velocity as $\delta w/w \simeq \pm (V/c)$.

If the angular distribution of particles is isotropic, the net energy gain per collision is

$$\delta w/w \simeq (V/c)^2. \qquad (2)$$

It should be noted that it is, in this case, the wave velocity and not the fluid velocity which is responsible for the acceleration.

Our result differs from that obtained by Parker, who asserted that the fluid velocity is responsible for the acceleration by hydromagnetic wave.[5] In the rest system the resultant energy change mentioned above is described as the *cumulative* effect of the induced electric field. In Parker's calculation, the time during which the acceleration occurs was taken to be the period of gyration. But it must be taken to be the period of the hydromagnetic wave.

We make an additional remark. The reflexion condition of the charged particle colliding with the transverse wave is given in the moving system, if the period of the gyration of the particle is short compared with that of the wave, by

$$\cot\theta \lesssim h_0/H_0\,, \qquad (3)$$

where θ is the angle between the velocity vector of the incident particle and $\boldsymbol{H}_0$, and h_0 is the amplitude of the wave at maximum. This relation is different from the usual one, $\cot^2\theta \lesssim \delta H_0/H_0$, being due to the nature of the transverse wave.

In compressible medium a longitudinal hydromagnetic wave can occur and the similar argument as above will be applied as far as the amplitude of the wave is small. If the amplitude of this type of wave is sufficiently large, a hydromagnetic shock wave may develope and complicate the acceleration mechanism.

The efficiency of the acceleration rate required by the analysis of the observed cosmic rays is much higher than that expected from the astrophysical evidence.[6] We see that the wave velocity exceeds the fluid velocity in the spiral arm of the Galaxy and also in the Crab Nebula which might be a powerful source of cosmic rays.[7][8] In the spiral arm of the galaxy, V is of the order of 10^7cm/sec., with $H_0 \simeq 10^{-5}$ gauss and $\rho \simeq 10^{-24}$gcm^{-3}, while $v \simeq 1.4 \times 10^6$cm/sec.[9] In the Crab Nebula we may assign from the available experimental data $V \simeq 10^8$cm/sec and $v \simeq 3 \times 10^7$cm/sec.[10] If we use the formula (2), approximating the medium of these places by ideal incompressible fluid, the expected efficiency increases almost by one order than that of the usual formula $\delta w/w \simeq (v/c)^2$, but still lower than the required efficiency. Especially in the Crab Nebula, the acceleration by the magnetic shock wave might be possible[11] and the high efficiency might be expected. This problem will be discussed in the following paper.

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Interpretation on the Ratio of the Emission Probabilities for Protons and Neutrons

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It has been pointed out by Cohen and Newman[1] that the ratios $\sigma(p\,;\,p\,n)/\sigma(p\,;\,2n)$, measured for medium weight nuclei bombarded with 21.5 Mev proton beam, are unexpectedly large. This result seems to be strange, when interpreted by means of the statistical theory of nuclear reactions, since the emission of neutrons is usually predominant over that of protons. Further, from the experiment of $(n\,;\,p)$ reactions induced by fast neutrons, Blosser and Handley[3] have emphasized that $(n\,;\,np)$ process is quite substantial at these energies. One may expect that the large yield of protons is perhaps due to the direct interaction of the incident proton with a nucleon in the target nucleus.

For free collisions, the cross section of *n-p* scattering is about four times larger than that of *p-p* or *n-n* scattering. Assuming the half exchange property for nuclear force and neglecting the effect of the Coulomb barrier, a frivolous consideration leads to the following results :

$$\left.\begin{aligned}\sigma(p:p) &\propto Z\bar{\sigma}_{pp}+(N/2)\bar{\sigma}_{np}\simeq(3/5)\sigma_t\\ \sigma(p\,;n) &\propto (N/2)\,\bar{\sigma}_{np} \qquad\simeq(3/5)\sigma_t\end{aligned}\right\} \quad (1)$$

where $\bar{\sigma}$ stands for the collision cross section in the nuclear matter and σ_t for total cross section. These predict that the proton emission is favourable in proton induced reactions, in agreement with experimental evidences. The cross section $\sigma(p\,;\,pn)$ is closely related with the details of energy spectra of particles directly knocked out. We have estimated $\sigma(p\,;\,pn)$ on the basis of the Fermi gas model, taking account of the penetrability at the nuclear surface and the absorption in the nuclear matter and regardless of the interaction between the incident proton and the tails of nucleons spreading outside the target nucleus. The energy spectrum of nucleons directly scattered is calculated by a similar method as in the previous work.[2] After lengthy numerical calculations $\sigma(p\,;\,p\,n)$ obtained is at most 5% of the inelastic total cross section. Although our rough estimation is not quite reliable subject to the oversimplification of the process, yet we may conclude that *the direct collision occurring within the nucleus (not at the surface) never interpret the large* $\sigma(p\,;p\,n)$ *values.*

In order to account for the large contribution from $(p\,;\,pn)$ process, several attempts may be proposed, which we summarize as follows.

(i) Surface interaction processes which arises from the interaction between the incident nucleon and the tails of wave functions of nucleons in the target nucleus. Though these have been already investigated by a number of authors, most of these are concerned with angular distributions of reaction products. It seems to be of first importance to calculate total cross sections. It should be noted that the relation (1) may be reserved for the processes due to surface interactions.

(ii) Effect of the diffuse nuclear surface. Recent analyses on nucleon elastic scattering have determined the phenomenological shape of nuclear potential with a diffuse nuclear surface. The diffuseness makes the transmission of charged particles through potential barrier more easier than that considered conventionally.

(iii) Direct deuteron pick-up reactions. $(p\,;\,d)$ reaction is closely related with $(p\,;\,p\,n)$ reaction, in the sense that the former is involved in the latter by means of the activation method. In fact the deuteron production plays an important role at high energy phenomena, while few investigations, both theoretically and experimentally, have been done in the intermediate energy range. It may be of interest to examine the cross section of so-called deuteron "pick-up" process, for which, as in (i), it is desirable to know the total cross section.

(iv) Indirect deuteron pick-up processes. It may frequently occur that a neutron picks up any proton in the neighbourhood of nuclear surface, when the neutron evaporates from the compound nucleus. In both (iii) and (iv), if the deuteron is produced into ^1S state, these processes should be observed as $(p\,;\,pn)$ type reactions.

As to above possibilities, calculations are now being made on which we hope to report in this journal in near future.

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Some Remarks on the Charged Mesic Decay of $^3H_\Lambda$

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A hypertriton $^3H_\Lambda$ is usually regarded as a composite system of proton, neutron and Λ^0 particle.

We have already known various modes of the charged mesic decay of the hypertriton $^3H_\Lambda$:

$$^3H_\Lambda \rightarrow {}^3He+\pi^- \qquad (1)$$

$$\rightarrow d+p+\pi^- \qquad (2)$$

$$\rightarrow p+p+n+\pi^- . \qquad (3)$$

Assuming that the decay mode of Λ^0 is $\Lambda^0 \rightarrow p+\pi^-$, and the decay interaction is invariant under space inversion, the branching ratio of the above competing processes is calculated. Although we do not have enough experimental data for the mesic decay of $^3H_\Lambda$ at the present, it is expected that in future the above calculations will have not a little contribution to determine the spin parity of $^3H_\Lambda$ and Λ^0 by comparing the branching ratios with experimental ones, and further to clarify whether the conservation of parity is also valid for this decay interaction.

For this purpose, the decay probabilities per unit time for the processes (1), (2) and (3) will be calculated. Assuming the spin parity of Λ^0 to be $1/2^{\pm}$, we have the two possible types of the interaction Hamiltonian density :

$$H=i\frac{f}{\mu}\nabla\varphi_\pi(\psi_p^*\sigma\psi_\Lambda), \tag{4}$$

$$H=g\varphi_\pi(\psi_p^*\psi_\Lambda). \tag{5}$$

The decay probabilities per unit time for (1) and (2) can be estimated by the ordinary perturbation method, assuming the following trial functions for the initial state ${}^3H_\Lambda$:

$$\Psi_i=N_i\exp\left[-\frac{\lambda}{2}r_{23}-\frac{\nu}{2}(r_{12}+r_{13})\right]\cdot Y_i(1,2,3) \tag{6}$$

where $\lambda\sim\nu$: $1.00\times10^{13}\mathrm{cm}^{-1}$ [1],

$Y_i(1,2,3)$: spin function for three particles 1 (Λ^0), 2 (proton) and 3 (neutron), and for the final states ${}^3He+\pi^-$ and $d+p+\pi^-$:

$$\Psi_f=N_f\exp\left(i\boldsymbol{p}\frac{\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3}{3}\right)\exp\left[-\frac{\kappa}{2}(r_{12}+r_{13}+r_{23})\right]Y_f(1,2,3) \tag{7}$$

and

$$\Psi_f=\frac{1}{\sqrt{2}}[\psi_D(2,3)\psi_p(1)Y_f(23,1)-\psi_D(1,3)\psi_p(2)Y_f(13,2)], \tag{8}$$

$$\psi_D(2,3)=N_D\exp\left(i\boldsymbol{q}\frac{\boldsymbol{r}_2+\boldsymbol{r}_3}{2}\right)\exp\left(-\frac{s}{2}r_{23}\right)$$

respectively, where $\boldsymbol{p}$: momentum of ${}^3He$, $\boldsymbol{q}$: momentum of the deuteron, s: $1.0\times10^{13}\mathrm{cm}^{-1}$ [2], κ: $0.64\times10^{13}\mathrm{cm}^{-1}$ [3]. The ψ_p, ψ_Λ and φ_π are assumed to be the plane wave. On the other hand, the probability for (3) is obtained by subtracting the net results of (1) and (2) from the total decay rate which is easily estimated by the closure approximation.

Thus, for the parity conserving case, the branching ratios of two, three and four body decays (1), (2) and (3) are given in Table 1, where the neutron and the proton in ${}^3H_\Lambda$ are assumed to be in the ${}^1S$ state or in the ${}^3S$ state. As is instantly seen from this table, these ratios depend appreciably on the parity of Λ^0, except for the case of the neutron and the proton in ${}^3H_\Lambda$ being in ${}^1S$ state. Hence it will be useful to check the spin parity of ${}^3H_\Lambda$ and Λ^0 by comparing the experimental branching ratio with Table 1. Needless to say, these ratios have been obtained by the rough estimation. However, if Λ^0 is a particle of spin 1/2, it may be checked whether the parity conservation for the decay process is valid or not, because, if not conserved, we have to take a linear combination of (4) and (5) for the decay interaction, then the branching ratios will be different from those for the parity conserving case.

Table 1 The branching ratios for ${}^3H_\Lambda$-decay, when the parity conservation is valid in decay interaction.

Spin angular momentum of ${}^3H_\Lambda$	$J=1/2$				$J=3/2$	
	n and p in ${}^3H_\Lambda$ are in ${}^1S$ state					
Spin parity of Λ^0	$1/2^+$	$1/2^-$	$1/2^+$	$1/2^-$	$1/2^+$	$1/2^-$
Decay interactions	(4)	(5)	(4)	(5)	(4)	(5)
${}^3He+\pi^-$	17(%)	17(%)	6(%)	51(%)	22(%)	0(%)
${}^3H_\Lambda\to d+p+\pi^-$	27	27	1	9	33	74
$p+p+n+\pi^-$	56	56	93	40	45	26

1). S. Iwao and Y. Yamaguchi, Soryushiron Kenkyu (Mimeographed Circular in Japanese) **12** (1956), 574.
2) S is determined by the "size" of the deuteron.
3) J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics*, 199.

An Attempt of Generalizing the Invariance of the Theory under the Charge Conjugation

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Recently the experiment on the scattering of high energy pion, nucleon and electron by nucleon and the annihilation of antiproton by nucleon became possible and several new facts have been found about the structure of the nucleon or the pion-nucleon interaction.[1] Among them the most striking is that the core of the nucleon seems to extend to about one-half of the Compton wave length of the pion. It is very difficult to account for this fact by the usual theory. Also we know that the cut-off length of the pion-nucleon interaction must be taken to be of the order of 10^{-13}cm, which is considerably large.

These situations are very different from those encountered in the case of the the electromagnetic interaction, and it seems that there is a qualitative difference between the pion-nucleon interaction and the electromagnetic interaction. Hitherto, however, the interactions have been stipulated by the principle of Lorentz invariance, hermiticity, etc., and so far such a difference could not have been revealed. This fact seems to suggest that the principles of stipulating the interaction we have had hitherto are not sufficient and have to be supplemented by some new ones. The purpose of this letter is to propose a possible example of such principles by generalizing the invariance of the theory under the charge conjugation.

In a previous paper[2], we proposed a scheme in which the freedom of particle and antiparticle was related to the third component of an angular momentum vector in the charge space. There the gauge transformation was identified with the rotation in the charge space around its third axis, and the expression for the electric charge took the form

$$Q=e(I_3+K_3),$$

$$K_3=\begin{cases} 1/2 & \text{for particle,} \\ -1/2 & \text{for antiparticle,} \end{cases}$$

$$|\boldsymbol{K}|=1/2,$$

where I_3 and K_3 are the third components of the isotopic spin $\boldsymbol{I}$ and the new vector $\boldsymbol{K}$ introduced above.

Under this picture we can show that the charge conjugation corresponds to rhe rotation of π in the I- and K-spaces around their first (or second) axes. The invariance

of the theory under the rotation in the I-space is known as the charge independence. Therefore, the requirement of the invariance of the theory under the charge conjugation is equivalent to the requirement of the invariance of the theory under the rotation of π in the K-space around its first (or second) axis. We assume here that the pion-nucleon interaction must be invariant under the full rotation in the K-space. This generalization is entirely analogous to the one where the charge symmetry was generalized to the charge independence. As for the electromagnetic interaction, it is assumed as the third component of a vector in the charge space.

We can show that if this is assumed, the pion-nucleon interaction, especially the part responsible for the creation and the annihilation of nucleon pairs, cannot be local. (This will be shown in the next letter.) Therefore the pion-nucleon interaction becomes non-local in general. Such a situation does not occur in the case of the electromagnetic interaction. Thus we see that a qualitative difference of these interactions can be revealed by generalizing the invariance of the theory under the charge conjugation.

Details will be reported in the later issue of this journal.

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L. W. Alvarez *et al*, preprint.
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Pion-Nucleon Interaction according to the Previous Theory

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We assumed in the preceding letter that the pion-nucleon interaction must be invariant under the rotation both in the I- and K-spaces. In this letter will be given an example of an interaction which will satisfy such requirements.

Consider, for example, the usual pseudoscalar pion-nucleon interaction,

$$H(x) = i g \bar{\psi}(x) \gamma_5 \tau_i \psi(x) \varphi_i(x) .$$

Expanding $\bar{\psi}(x)$ and $\psi(x)$ into Fourier series, $\bar{\psi}(x)\gamma_5\tau_i\psi(x)$ can be written as

$$\bar{\psi}(x)\gamma_5\tau_i\psi(x) = (1/2V)\sum_{kk'}\sum_{\sigma\sigma'} \times \{\bar{u}(k,\sigma)\gamma_5 u(k',\sigma') e^{i(-k+k')x} \chi^*(k,\sigma) \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix} \chi(k',\sigma') + \bar{v}(k,\sigma)\gamma_5 v(k',\sigma') e^{i(k-k')x} \chi'^*(k,\sigma)$$

$$\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}\chi'(k', \sigma')$$

$$+\bar{u}(k, \sigma)\gamma_5 v(k', \sigma') e^{i(-k-k')x}\chi^*(k, \sigma)$$

$$\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}\chi'(k', \sigma')$$

$$+\bar{v}(k, \sigma)\gamma_5 u(k', \sigma') e^{i(k+k')x}\chi'^*(k, \sigma)$$

$$\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}\chi(k', \sigma')\}, \qquad (1)$$

where $u(k, \sigma)$ and $v(k, \sigma)$ are four solutions of the Dirac equation, and $\chi(k, \sigma)$ and $\chi'(k, \sigma)$ are given by

$$\chi(k, \sigma) = \begin{matrix} p(k, \sigma) \\ n(k, \sigma) \\ -\tilde{n}(k, \sigma) \\ \tilde{p}(k, \sigma) \end{matrix},$$

$$\chi'(k, \sigma) = \begin{matrix} \tilde{p}^*(k, \sigma) \\ \tilde{n}^*(k, \sigma) \\ -n^*(k, \sigma) \\ p^*(k, \sigma) \end{matrix}, \qquad (2)$$

where $p(k, \sigma)(n(k, \sigma))$ and $\tilde{p}(k, \sigma)(\tilde{n}(k, \sigma))$ denote the annihilation operators for proton (neutron) and antiproton (antineutron) respectively.*

In (1), terms like $\chi^*(k, \sigma)\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}\chi(k', \sigma')$ can be written as

$$\chi^*(k, \sigma)\kappa_3\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}\chi(k, \sigma),$$

where κ_3 is one of the Pauli matrices in the K-space. Therefore, this is not a scalar but the third component of a vector in the K-space. To make this invariant, it is necessary to replace κ_3 by the unit matrix. If this modification is made, however, (1) vanishes. The minimum modification necessary to avoid this difficulty is to reverse the sign before the second term and to make the third and the fourth term non-local.** The form of the form factors appearing there is restricted by the condition of hermiticity, and the modified expression for (1) is given by

$$(1/2V)\sum_{kk'}\sum_{\sigma\sigma'}\{\bar{u}(k, \sigma)\gamma_5 u(k', \sigma') e^{i(-k+k')x}$$

$$\times\chi^*(k, \sigma)\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}\chi(k', \sigma')$$

$$-\bar{v}(k, \sigma)\gamma_5 v(k', \sigma') e^{i(k-k')x}\chi'^*(k, \sigma)$$

$$\times\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}\chi'(k', \sigma') - i\iint dydz\, f(y-z)$$

$$\times[\bar{u}(k, \sigma)\gamma_5 v(k', \sigma') e^{i(-ky-k'z)}\chi^*(k, \sigma)$$

$$\times\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}\chi'(k', \sigma')$$

$$+\bar{v}(k, \sigma)\gamma_5 u(k', \sigma') e^{i(ky+k'z)}\chi'^*(k, \sigma)$$

$$\times\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}\chi(k', \sigma')]\}.$$

Here the essential is that $f(y-z)$ must be an odd function of $y-z$. The modified interaction hamiltonian thus obtained is given by

$$H(x) = g\{i[\overline{\psi^{(+)}}(x)\gamma_5\tau_i\psi^{(+)}(x)$$

$$-\overline{\psi^{(-)}}(x)\gamma_5\tau_i\psi^{(-)}(x)]$$

$$+\iint dydz\, f(y-z)[\overline{\psi^{(+)}}(y)\gamma_5\tau_i\psi^{(-)}(z)$$

$$+\psi^{(-)}(y)\gamma_5\tau_i\psi^{(+)}(z)]\}\phi_i(x),$$

where $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$ are positive and negative frequency parts of $\psi(x)$ respectively.

* A similar representation was used by Lee and Yang and by Nakai[1].

** Exactly the same situation occurs in the case of the pseudovector interaction, too.

It is expected that the non-locality here appeared is responsible for the anomalously large annihilation cross section of antiproton and the suppression of virtual pairs, etc.

The other characteristic of this interaction is that the vertex responsible for the emission and the absorption of pion by antinucleon becomes to take the sign opposite to that in the ordinary interaction, as the result of replacing κ_3 by the unit matrix in (1). Therefore the nucleon-antinucleon potential resulting from the exchange of odd number of pions becomes to take the sign opposite to that in the ordinary theory. Therefore the validity of this theory may be checked from the analysis of the angular distribution of antinucleon scattered by nucleon.

Details will be reported in the later issue of this journal.

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The Annihilation of Antinucleon according to the Previous Theory

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We proposed in the preceding letter an attempt of generalizing the invariance of the theory under the charge conjugation. It contains, however, aspects considerably different from the usual theory, and it would be important to check the theory experimentally. As the resulting interaction is a very complicated non-local one, however, it is desirable to do this by the process where the answer can be given from the kinematical consideration alone which does not depend upon the details of the interaction.

An example of such a process is the annihilation of antinucleon by nucleon. Consider, for example, the system composed of a nucleon and an antinucleon. Its particle number is zero. According to our preceding letter (I), however, there can be two states corresponding to $K=0$ and 1 respectively. The former can annihilate into pions as usual, but for the latter this is impossible from the conservation law of $\boldsymbol{K}$, and it can annihilate only by emitting γ-ray. (The selection rules for the γ-transition are given by $\Delta I=\pm 1,\ 0$; $\Delta I_3=0$, and $\Delta K=\pm 1,\ 0$; $\Delta K_3=0$, since the electromagnetic interaction is the third component of a vector in the charge space.) Therefore, according to (I), we must expect that the neutral annihilation product will become more abundant than would be expected from usual theory.

It is difficult to calculate the fraction of the energy carried by the γ-ray thus appears. In the following will be described a very rough estimate based on the statistical theory.[1)]

When the annihilation takes place, the total energy is released to the meson cloud. In the usual theory, it is assumed that a statistical equilibrium is reached in this meson cloud. According to (I) however,

somewhat different situation occurs in the state corresponding to $K=1$. In this state the most rapid process is forbidden, and therefore the lifetime is lengthened. We assume that during this time the modes for the γ-ray are also excited, and reach a statistical equilibrium. If this is assumed, the fraction of the energy carried by the γ-ray can be estimated from the simple statistical formula. The result shows that about 40% of the energy will be carried by the γ-ray.* Since the states corresponding to $K=0$ and 1 occur equally, the average percentage of the energy carried by the γ-ray becomes about 20%. This is considerably larger than would be expected from the usual theory. Therefore the validity of our new theory may be checked by examining whether the γ-ray expected as above are observed.**

Details will be published in a later issue of this journal.

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L. D. Landau, Izv. Akad. Nauk USSR **17** (1953), 51.
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2) E. Segrè, Proceedings of the Sixth Rochester Conference, 1956. L. W. Alvarez *et al*, preprint.

* It should be noted this value should be considered as the upper limit, since the assumption that the γ-ray also reaches the statistical equilibrium presupposes the strong interaction between the γ-ray and the meson cloud.

** The average number of the pion produced in the annihilation of antinucleon is believed to be 5.5 ± 0.5.[2] This value seems to be somewhat larger than would be expected from the statistical theory. According to (I) about 20% of the total energy is carried by the direct γ-ray. This changes the interpretation of the experiment, and it is found that it is reduced to about 4.5 ± 0.5.

The Production Ratio of K^+ to K^- Mesons

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February 8, 1957

In the production of K mesons the positive excess is qualitatively explained by the concept of the strangeness.[1] Recently, Fry et al.[2] found the ratio of K^+ to K^- mesons ($\equiv r$) to be about 156 by exposing the Tantalum target to the proton beam with the bombarding energy 6.2 Gev, and by observing K mesons at 90° to the beam for an average momentum 240Mev/c in the laboratory system. While Schein et al.[3] obtained $1<r<2$ in the emulsion plates exposed on the π-beam with the incident energies 4.6 and 3.0 Gev.

In this note the r values for these cases are quantitatively calculated under the following three assumptions:

1. Both the strangeness and the isotopic spin assigned by Nishijima and Gell-Mann are conserved.
2. The statistical theory[4),5)] is used. In practice, the phase volumes for the final states composed of two or three particles are accurately given by the formula satisfying the conservation of momentum, and in the case of more than three particles system they are given by the formula of Leport and Stuart,[6] in which K mesons and baryons are treated classically and π-mesons far relativistically.
3. The interaction volume is taken to be

Table 1

Initial State	$r=K^+/K^-$		$\Sigma^0\Lambda^0/\Sigma^\pm$	
	4.6 Gev	3.0 Gev	4.6 Gev	3.0 Gev
$\pi^-+p\rightarrow$	1.86	1.97	1.51	1.69
$\pi^-+emul.\rightarrow$	1.21	1.24	1.25	1.24
$\pi^++emul.\rightarrow$	8.92	12.20	1.29	1.30
	6.2 Gev.			
$p+p\rightarrow$	23.7		1.23	
$p+emul.\rightarrow$	13.9		1.48	
$p+\mathrm{Ta}\rightarrow$	13.2		1.51	
$p+\mathrm{Ta}\rightarrow(90^\circ$ 240 Mev/$c)$	89.3			

$$\Omega=(4\pi/3)\,(\hbar/\mu c)^3\,(Mc^2/W)$$

for $\pi+N\rightarrow$

$$\Omega=(4\pi/3)\,(\hbar/\mu c)^3\,(2Mc^2/W)$$

for $N+N\rightarrow$

where M and μ are the mass of the nucleon and meson respectively, and W is the total energy in the center of mass system.

Under these assumptions the following reactions are available :

(i) $\pi+N\rightarrow(K\Lambda)$, $(K\Sigma)$, (KKN), $(K\Lambda\pi)$, $(K\Sigma\pi)$, $(KK\Xi)$, $(KKN\pi)$, $(K\Lambda\pi\pi)$, $(K\Sigma\pi\pi)$;

(ii) $N+N\rightarrow(K\Lambda N)$, $(K\Sigma N)$, $(KKNN)$, $(K\Lambda N\pi)$, $(K\Sigma N\pi)$, $(KK\Lambda\Lambda)$, $(KK\Sigma\Sigma)$, $(KK\Sigma\Lambda)$, $(KK\Xi N)$, $(KKNN\pi)$, $(K\Lambda N\pi\pi)$, $(K\Sigma N\pi\pi)$.

The results are exhibited in the next table.

In this table the ratios of $\Sigma^0+\Lambda^0$ to $\Sigma^++\Sigma^-$ are also presented, and these are substantially the ratios of Λ^0 to $\Sigma^++\Sigma^-$ because the reaction $\Sigma^0\rightarrow\Lambda^0+\nu$ seems to be very fast.

The r values calculated for the π-beam $1.2\sim1.3$ are in good agreement with experimental values $1<r<2$. For the π^+ beam r is obtained to be about ten times the value for the π^- beam, this means that r is strongly dependent on the charge of the incident particles. Furthermore, in these energy region r for π^- beam may be near the minimum value owing to the comparatively strong contribution of the reaction (KKN). In the lower energy region the two body reactions, $(K\Lambda)$ and $(K\Sigma)$, are dominant and r becomes larger, and similarly r increases again in the higher energy region due to the effective contribution of four and five body reactions.

To compare with the experiment by Fry et al., we evaluate the production probability of K meson with corresponding momentum from the momentum distribution in each reaction, by assuming the angular distribution in the center of mass system to be isotropic. After multiplying total fraction of each reaction by this probability we obtain the r value with the definite momentum of K meson. The result thus obtained is presented in the last rank of

the table, and this value seems to be in fairly good agreement with the experimental value, 156.

The ratio of $\sum^0+\Lambda^0$ to $\sum^+ +\sum^-$ is fairy independent on the energy and the kind of incident particles. This is reasonable from a consideration that in any initial state the fraction of produced $\sum^0$ is about a half of $(\sum^+ +\sum^-)$'s and Λ^0 is produced with a fraction of about one half to all $\sum$'s because of their weight of charge states and of the approximate equality of their phase volume. This ratio becomes roughly 1.25 from the estimation that the relative weight of $\Lambda^0+\sum^0$ and $\sum^+ +\sum^-$ are $1+2/3$, $4/3$, respectively.

Finally, we examine the dependence of r on the size of interaction volume. The less interaction volumes are taken, the larger the r values generally becomes; in $p+$Ta reaction the relation of both is approximately inversely proportional, but in $\pi^- +$*emulsion* reaction the r value increases just a little by the decrease of the interaction volume. The ratio $\sum^0+\Lambda^0$ to $\sum^+ +\sum^-$ in both reactions is affected very little by the change of the size of interaction volumes.

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Errata

On the Energy Matrix of the Tensor Forces

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(Prog. Theor. Phys. **16** (1956), 256 (L))

Eq. (5) is insufficiently written. Some terms lack. The author wishes to express his sincere thanks to Professor H. Horie and Mr. A. Arima for pointing out this mistake.

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CONTENTS

Page

Shinjirō YASUMI: Nuclear Reactions induced by the 14 MeV Neutrons 443
Senzo TOKUNAGA: Meson Showers by Heavy Primary Particles 454
Umejiro YOSHIDA and Akira SUZUKI: The Surface Potential of Selenium and its Relation to the Rectification and the Photovoltaic Effects 459
Wataru KINASE and Hidetosi TAKAHASI: On the 180°-type Domain Wall of $BaTiO_3$ Crystal 464
Terutaro NAKAMURA: After Effect of the Domain Motion in Rochelle Salt Crystal 477
Sigemaro NAGAKURA: Study of Metallic Carbides by Electron Diffraction Part I. Formation and Decomposition of Nickel Carbide 482
Tadashi INO: Studies on the Radial Distribution Analysis in Diffraction Methods I. Errors due to the Approximation of Integration by Summation 495
Tadao HORIE, Takayuki NAGURA and Masamoto OTSUKA: Radiative Collisions between Electron and Molecular Beams II. Angular Momentum Distribution of OH^* Radicals Splitting from H_2O_2 Molecules 500
Teruo YUKITOSHI, Hirosi SUGA, Syûzô SEKI and Junkichi ITOH: Nuclear Magnetic Resonance Experiment on Solid Hexamethyldisilane 506
Hiroshi WATANABE: Magnetic Properties of Perovskites Containing Strontium I. Strontium-Rich Ferrites and Cobaltites 515
Yoshika MASUDA: Nuclear Magnetic Resonance Experiment on Metal Cadmium 523
Yoshio SHIBAOKA: On the Buckling of an Elliptic Plate with Clamped Edge II 529
Yasujiro KOBASHI: Measurements of Pressure Fluctuation in the Wake of Cylinder 533
Fumio TAMAKI: Experimental Studies on the Stability of the Transonic Flow past Airfoils 544
Fumio TAMAKI and KIM, Chul-Soo: Studies on the Hypersonic Flow Using a Double-Diaphragm Shock Tube 550

SHORT NOTES

Yukito TANABE and Satoru SUGANO: The Absorption Spectra of Ruby 556
Masakazu MURAKAMI and Toshiro MAEDA: The Absolute Intensities of Cosmic Rays at 2840 Meters and Geomagnetic Latitude 25°N 557
Koichi SHIMODA: Precise Frequency of the 3,3 Inversion Line of Ammonia 558
Tadasi NUMATA: Technique of Obtaining a Minute Pinhole in a Metal Plate 559